SOULS AND MODULI SPACES OF NONNEGATIVELY CURVED MANIFOLDS

(mostly) joint with
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<u>PROBLEM.</u> Which smooth manifolds support complete riemannian metrics with everywhere nonnegative sectional curvature? If so, then how many "substantially different" such metrics are there (say up to deformation)?

(Emphasis is on the 2nd part here)

<u>CHEEGER – GROMOLL SOUL THEOREM.</u> ***If a noncompact, connected manifold admits such a metric, then it is diffeomorphic to the total space of a vector bundle over a compact manifold which admits such a metric.***

The compact manifold is totally geodesic and is called a <u>SOUL</u> of the original riemannian manifold. The fiber dimension is called the soul's <u>codimension</u>.

<u>HOW UNIQUE IS THE SOUL?</u>

A given metric can have many souls. Consider $S^n \times \mathbb{R}^k$ with the product metric (constant positive curvature on the first factor, flat on the second). Then every first factor slice of the form $S^n \times \{point\}$ is a soul.

<u>However</u>, a result of V. Sharafutdinov implies uniqueness up to ambient isotopy.

But <u>*different*</u> metrics can have non – diffeomorphic souls. For example, there are many pairs of non – diffeomorphic lens spaces L_1 and L_2 such that $L_1 \times \mathbb{R}^3$ and $L_2 \times \mathbb{R}^3$ are diffeomorphic (*cf.* Milnor's first counterexamples to the *Hauptvermutung*). — Incidentally, a result of S.K. + R.S. implies that one cannot replace $\mathbb{R}^3$ by $\mathbb{R}^2$.

<u>*Note.*</u> Milnor's examples were 3 – dimensional, but one can use results of J. Ewing, S. Moolgavkar, L. Smith and R. E. Stong to produce similar examples of lens space pairs in all (odd) dimensions (except 1).

DEFAULT ASSUMPTION. To simplify the discussion, restrict to the simply connected case.

[There are also some results, particularly for small fundamental groups. Some are in the final remarks.]

In fact, there are also simply connected examples with non – diffeomorphic souls. Several exotic 7 – spheres Σ^7 admit metrics with nonnegative sectional curvature, and it is well – known that $\Sigma^7 \times \mathbb{R}^3$ and $S^7 \times \mathbb{R}^3$ are diffeomorphic for all exotic 7 – spheres Σ^7. As before, one cannot replace $\mathbb{R}^3$ by $\mathbb{R}^2$ (this actually goes back to the nineteen sixties).

ENUMERATION PROBLEM. How many distinct diffeomorphism types of souls can be realized by different (nonnegatively curved) metrics on the same manifold?

Easy case. For *simply connected* souls of codimension 1 and dimension at least 5, the h – cobordism Theorem implies that only one diffeomorphism type can be realized.

However, in some cases with larger codimensions, infinitely

many types can be realized (I.B.).

Subsequently, Kapovitch – Petrunin – Tuschmann gave other such examples with better geometric properties and also showed that sufficiently close (nonnegatively curved) metrics have diffeomorphic souls.

<u>THEOREM 1.</u> ***In fact, sufficiently close metrics have (smoothly) ambiently isotopic souls.***

<u>REFINED QUESTION.</u> **How does the answer to the enumeration problem vary with the dimension and codimension of the soul?**

Restrict attention further to souls of dimension at least 5.

<u>THEOREM 2.</u> ***Let E be a riemannian manifold with a complete metric of nonnegative sectional curvature, and suppose that the codimension of the soul is ≤ 3. Then there are only finitely many diffeomorphism classes of smooth***

manifolds which can be souls of complete, nonnegatively curved metrics on E.

The codimension hypothesis in this result is best possible.

<u>THEOREM 3.</u> ***For every $n > 6$, there are compact nonnegatively curved n – manifolds M^n such that $M^n \times \mathbb{R}^4$ supports infinitely many complete nonnegatively curved metrics with pairwise nondiffeomorphic souls.***

By crossing with $\mathbb{R}$ (equipped with the usual flat metric), *<u>we can also obtain examples in all higher codimensions</u>*. The examples of both Belegradek and Kapovitch – Petrunin – Tuschmann are in fact products of the examples in the theorem with sufficiently many copies of $\mathbb{R}$.

<u>IDEAS OF PROOFS.</u> ***Low codimensions*: Codimension 1 was already mentioned.**

In codimension 2, the bundle is determined by its Euler class, which is a fiber homotopy invariant, and this implies that all possible souls are tangentially homotopy equivalent

(the stable tangent bundle pulls back under the homotopy equivalence). By surgery theory, there are only finitely many diffeomorphism classes of closed simply connected manifolds in a given tangential homotopy type.

Codimension 3 can be handled similarly using the fact that *<u>the first rational Pontryagin class is a fiber homotopy invariant for</u>* 3 – *<u>plane bundles</u>* (but <u>NOT</u> for higher fiber dimensions!!). Since there are only finitely many 3 – plane bundles with a given first rational Pontryagin class, one can proceed much as in the codimension 2 case.

***Codimension* 4: The earlier infinite families of examples were constructed by showing that the candidates for souls could be smoothly embedded in certain products of spheres crossed with $\mathbb{R}^k$ for suitable values of *k*. We use an embedding theorem of Browder – Casson – Haefliger – Sullivan – Wall to show that in the earlier examples one can always take *k* equal to 4.**

Here is another type of exotic example in codimension 4:

<u>THEOREM 4.</u> ***There is a complete, nonnegatively curved metric on*** **$S^7 \times \mathbb{R}^4$** ***such that the soul is*** **S^7** ***but the normal bundle of the soul is nontrivial.***

By Theorem 1, this metric and the usual product metric must belong to separate components in the moduli space of complete nonnegatively curved metrics on $S^7 \times \mathbb{R}^4$.

This result is part topological and part geometric. Results of J. Levine and A. Haefliger that S^7 embeds smoothly in $\mathbb{R}^{11}$ with a nontrivial normal bundle, and it follows that the latter's total space is diffeomorphic to $S^7 \times \mathbb{R}^4$. On the other hand, K. Grove and W. Ziller show that the total space of this vector bundle has a complete nonnegatively curved metric whose soul is the nontrivial bundle's zero section.

<u>A REMAINING QUESTION.</u> In the low codimension cases, is it possible to find examples of metric pairs for which the souls are not diffeomorphic? By previous remarks, the only possibilities are codimensions 2 and 3 (and as usual a

positive answer in the first case implies the a positive answer in the second).

<u>Candidates for souls.</u> **Consider the manifolds** $\Sigma^7 \times \mathbb{CP}^{2k}$ **where the first factor is either the standard 7 – sphere or an exotic 7 – sphere. Results of Grove and Ziller imply that many such product manifolds admit nonnegatively curved metrics, and in particular this holds if** Σ^7 **generates the Kervaire – Milnor group** Θ_7 **of homotopy 7 – spheres, which is cyclic of order 28.**

Classifying such products up to diffeomorphism is "an interesting problem" with a long history.

<u>NOTE.</u> **We restrict to even – dimensional complex projective spaces because** $\Sigma^7 \times \mathbb{CP}^{2k+1}$ **and** $S^7 \times \mathbb{CP}^{2k+1}$ **are always diffeomorphic (this was already well – known in the 1960s; in particular, it follows quickly from Sullivan's product formulas for surgery obstructions).**

<u>OBSERVATION.</u> **If** ξ **is a 2 – plane bundle over** $\mathbb{CP}^q$

then $E(\xi)$ has a complete positively curved metric whose soul is the zero section. — The same will hold for many (and for most q, all) products $\Sigma^7 \times E(\xi)$ [more on this later].

Examples of 2 – plane bundles with nondiffeomorphic souls follow from the next two results.

<u>THEOREM 5.</u> ***If the 2 – plane bundle ξ is <u>nontrivial</u>, then $\Sigma^7 \times E(\xi)$ and $S^7 \times E(\xi)$ are always diffeomorphic.***

Note that the conclusion is <u>FALSE</u> if ξ is trivial! In fact, if M and N are closed smooth simply connected manifolds of dimension at least 5 such that $M \times \mathbb{R}^2$ and $N \times \mathbb{R}^2$ are diffeomorphic, then M and N are diffeomorphic by a fairly standard argument using the s – cobordism Theorem.

<u>THEOREM 6.</u> ***If Σ^7 generates the Kervaire – Milnor group of homotopy 7 – spheres, then for each $k > 0$ the manifolds $\Sigma^7 \times \mathbb{CP}^{2k}$ and $S^7 \times \mathbb{CP}^{2k}$ are <u>not</u> diffeomorphic.***

PROOFS. **Both involve surgery theory. The first is a fairly straightforward application of the Sullivan – Wall exact surgery sequence for manifolds with boundary. The second requires a much deeper study of exact surgery sequences, and it has several parts. First, one needs some insight into the (homotopy classes of) homotopy self – equivalences of $S^7 \times \mathbb{CP}^{2k}$. Next, it is necessary to determine which of these maps are homotopic to diffeomorphisms. It turns out that one can answer this question in a complete and useful manner by a variety of homotopy and surgery techniques; in fact, a homotopy self – equivalence will be homotopic to a diffeomorphism if and only if it is normally cobordant to the identity. Finally, one needs a result on the action of Wall group actions on structure sets in the surgery sequence. In principle, the key result was originally due to L. Taylor (some necessary embellishments of this go back to a 1980s paper by R.S. in the Michigan Mathematical Journal).**

THE SPECIAL CASE WHERE $k = 1$. **The product manifolds $\Sigma^7 \times \mathbb{CP}^2$ yield exactly three diffeomorphism**

classes, and in fact every smooth structure on $S^7 \times \mathbb{CP}^2$ is diffeomorphic to one of these three products (so every smooth structure on the latter supports a nonnegatively curved metric).

<u>Higher values of k.</u> For the overwhelming majority of cases, the product manifolds $\Sigma^7 \times \mathbb{CP}^{2k}$ also fall into exactly three diffeomorphism classes. If k is not divisible by 3 then part of this essentially goes back to the 1960s, and it reflects the fact that the Kervaire – Milnor group bP_{4k+8} (homotopy $4k+7$ – spheres which bound parallelizable manifolds) has order prime to 7, which one plays off against the fact that bP_8 has order 28 = 7 × 4 (see W. Browder, 1967 Tulane Conference Proceedings). A second important piece of input is a refinement of the RS proof of Taylor's result to the cases of interest here — namely, $\Sigma^7 \times \mathbb{CP}^{2k}$. One key step here involves characteristic class computations for certain vector bundles over the eightfold suspension of $\mathbb{CP}^{2k}$, and this can be done using a variety of results from algebraic topology.

One can also combine this with results of J. F. Adams on splitting real K – theory at odd primes to show that one obtains three diffeomorphism classes provided k is not congruent to 3 mod 21. On the other hand, simple characteristic class computations for $k = 3$ show that there are a (maximal) number of 15 different diffeomorphism classes of manifolds of the form $\Sigma^7 \times \mathbb{CP}^6$. It follows in this case that one has least 10 different possible diffeomorphism classes of souls; this is because Grove – Ziller shows that 10 out of the 15 unoriented diffeomorphism types of homotopy 7 – spheres have riemannian metrics with nonnegative sectional curvature.

Note. **There always are at least three diffeomorphism types which have such metrics.**

The first unresolved case is to determine the number of diffeomorphism types determined by products of the form $\Sigma^7 \times \mathbb{CP}^{48}$.

FINAL REMARKS

Some open questions, some additional work in progress.

Relation to questions about positive scalar curvature.

The preceding results yield lower bounds on numbers of components in moduli spaces of complete metrics with nonnegative sectional curvature. It is natural to ask what happens if we look instead at moduli spaces for metrics with nonnegative Ricci or scalar curvature. The answers may depend upon the choices of topologies for the moduli spaces.

Non – simply connected examples with large numbers of diffeomorphism types of codimension two souls.

The simplest question here is whether one can ever have infinitely many diffeomorphism types of such souls. For the examples constructed here, the number is fairly small. On the other hand, it is possible to construct examples with an arbitrarily large finite number of diffeomorphism types of codimension two souls, and this can be done in every odd

dimension. Specifically, one can do this by taking suitable classes of lens spaces and crossing them with complex line bundles over even – dimensional complex projective spaces or products of such spaces. This uses the previously cited results of E – M – S – S together with some basic facts about product formulas for surgery obstructions, all in the context of surgery on bounded manifolds.

Fixed point sets of smooth semifree circle actions on homotopy spheres.

One original motivation for interest in some key points of the study of products of exotic spheres with complex projective spaces. Such actions have proven to be very useful "toy models" for more general classification problems (this parallels the transition from Kervaire and Milnor's work to that of Browder and Novikov and …). The fixed point sets of such actions are integral homology spheres, and topologically every integral homology sphere of dimension at least 5 can be realized as the fixed point set of such an action. However, separate results of Becker – RS

and Kh. Knapp yield some nontrivial restrictions on the diffeomorphism types of such fixed point sets. In fact, if their codimensions are sufficiently large then the fixed point sets must bound parallelizable manifolds by consequences of the Segal Conjecture (if G is finite, the stable cohomotopy of BG is the completed Burnside ring of G). The results of the work in this talk also yield some additional restrictions, and it would be enlightening to understand the situation in greater depth. This is also related to some earlier joint work with M. Masuda on smooth semifree actions of S^3 (Osaka J. Math., 1994).

Link for these notes:

http://math.ucr.edu/~res/miscpapers/Muenstertalk.pdf