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ISOVARIANT HOMOTOPY THEORY AND DIFFERENTIABLE GROUP ACTIONS

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Introduction

This is an expanded version of notes for lectures I had planned to give at the Korea Advanced Institute of Science and Technology during the Seventh KAIST Mathematics Workshop in Algebra and Topology at the Korea Advanced Institute of Science and Technology in Taejon, Korea, from August 11 to August 14, 1992.

There are three parts, with each corresponding to one of the planned lectures. The first two discuss joint work with G. Dula on the foundations of isovariant homotopy theory, the applications of this work to classification problems for smooth manifolds with (smooth) group actions and its relation to work on equivariant surgery over the past two decades; some of the results in the second part have been obtained independently by M. Dawson. The third part discusses joint work with S. Kwasik on a somewhat different but related topic; namely, differentiable actions of finite groups on homology 3-spheres. One common theme relating the second and third parts is the problem of adapting equivariant surgery to cases where a standard technical condition (the *Gap Hypothesis*) does not hold. A second relationship is that the 3-dimensional questions exhibit some basic features of higher-dimensional problems with certain technical simplifications. Finally, advances in the geometrization theory of 3-manifolds over the past two decades have suggested that suitably pseudo-geometric manifolds in higher dimensions form an especially promising subject for future research in geometric topology and transformation groups; perhaps ideas resembling those of Part III will lead to progress in this direction.

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Outline of Contents

PART I

INTRODUCTION TO ISOVARIANT HOMOTOPY THEORY

Background references and notation

Although we shall include a few remarks on the basic facts and concepts of differentiable transformation groups, a detailed account of the subject's foundations is beyond the scope of this article. The following references contain most if not all of the relevant background material:

(1) Chapters I–II and Sections VI.1–2 of Bredon, *Introduction to Compact Transformation Groups* (:= [Bre3]).

(2) Chapter I and Sections II.1–2 of tom Dieck, *Transformation Groups* (:= [tD2]).

(3) Chapters I–II and the Summary of Dovermann–Schultz, *Equivariant Surgery Theories and Their Periodicity Properties* (:= [DoS2]).

Most of the algebraic topology that we use can be found in the standard books by Spanier [Sp] and Milnor and Stasheff [MS].

We shall generally use standard notational conventions in transformation groups including M^G for the fixed point set of a group G acting on a space M, G_x for the isotropy subgroup of G at x, and $M_{(H)}$ for the set of all points whose isotropy subgroups are conjugate to H, and $|G|$ for the order of a finite group G. These (and many others) can be found in the references listed above.

As indicated by the title of this article, we shall deal mainly with smooth group actions. However, for purposes of comparison we shall occasionally most important property of bounded smooth G-manifolds is the existence of *collar neighborhoods* for the boundary.

PROPOSITION 1.3. *Let M be a smooth G-manifold with boundary ∂M. Then ∂M has a neighborhood that is equivariantly diffeomorphic to $\partial M \times [0,1)$ – with trivial action on the second coordinate – such that ∂M corresponds to $\partial M \times \{0\}$. If c_j are two smooth equivariant collar neighborhoods (where $j = 0, 1$), then $c_0|\partial M \times [0, \frac{1}{2}]$ and $c_1|\partial M \times [0, \frac{1}{2}]$ are ambient isotopic.*

This result follows from the same sort of argument that appears in M. Hirsch's differential topology textbook [Hi, Thm. 4.6.1, pp. 113–114]. □

As noted in [Hi, Ch. 4, Sec. 6], the analog of the Tubular Neighborhood Theorem for bounded manifolds requires a suitably refined notion of embedding for manifolds with boundary. Specifically, one needs to work with *neat embeddings* of $(M, \partial M)$ in $(N, \partial N)$ such that boundary goes to boundary and interior to interior (*i.e.*, a proper embedding of manifolds with boundary) and some neighborhood of ∂M in M meets ∂N orthogonally along the boundary (say with respect to some collar neighborhoods). We then have the following equivariant analog of the results for bounded manifolds in [Hi, Thms. 4.6.3 and 4.6.5, pp. 114–116].

THEOREM 1.4. *(i) Let $(M, \partial M)$ and $(N, \partial N)$ be smooth (finite dimensional) G-manifolds with boundary, and let $f : (M, \partial M) \to (N, \partial N)$ be a smooth equivariant embedding satisfying the neatness condition described above (with respect to some invariant collar neighborhoods). Then there is an equivariant vector bundle $E \downarrow M$ and a neat smooth embedding $F : E \to N$ whose restriction to the zero section is essentially f.*

(ii) Suppose that $f_j : E_j \to N$ (where $j = 0, 1$) satisfy (i), and suppose we are given invariant riemannian metrics on E_j with unit disk bundles $D(E_j)$. Then there is a metric preserving G-vector bundle isomorphism $\varphi : E_0 \to E_1$ covering the identity on M and a smooth equivariant ambient isotopy $H : N \times [0,1] \to N$ such that $H_1 \circ f_o = f_1 \circ \varphi$. □

The equivariant Tubular Neighborhood Theorems imply that certain invariant subsets of a smooth G-manifold are also smoothly embedded submanifolds or unions of submanifolds (where the dimension may vary from one component to the next). For example, this holds for the fixed point set M^G and the constant orbit type set $M_{(H)}$. Since M is the pairwise disjoint union of these subsets $M_{(H)}$, where (H) runs through the conjugacy classes of subgroups of G, and the induced G-action on $M_{(H)}$ is completely determined by fiber bundle considerations, we therefore have a decomposition of M into G-invariant pieces that can be studied effectively by standard topological methods. In order to understand things more thoroughly it is necessary to determine how these pieces fit together; a fairly complete account of this can be found in work of M. Davis [Dav]. As indicated in [DoS2, Sec. II.4], the process can be viewed as a special case of the Thom-Mather theory of smoothly stratified sets.

Convention on abuse of language. If we are given an invariant closed subset $N \subset M$ of a smooth G-manifold M such that N is a disjoint union of smoothly embedded submanifolds N_j, and the maps $\varphi_j : E_j \to N_j$ define pairwise disjoint tubular (*i.e.*, vector bundle) neighborhoods as in the preceding results, we shall often say that the disjoint union

$$\coprod \varphi_j : \coprod E_j \to \coprod N_j \ (\approx N)$$

is an invariant tubular neighborhood of N in M; if α is the disjoint union of the associated vector bundles α_j (whose fibers may have different dimensions!), then $D(\alpha)$ and $S(\alpha)$ will denote the unions of the associated disk and sphere bundles $\coprod D(\alpha_j)$ and $\coprod S(\alpha_j)$ respectively.

2. Equivariant homotopy theory

One of the main themes in algebraic topology is the use of cohomology groups to analyze the homotopy classes of maps from one space to another. The classical approach to this general question involves *obstruction theory* (*cf.* [Wh1]; for historical background see [EIL] and [Wh2]). Although several new and powerful techniques for studying homotopy classes have emerged over the past four decades, in many cases the obstruction-theoretic approach is still the most useful or illuminating. The applicability of obstruction theory to equivariant homotopy was already understood by the mid nineteen fifties (*cf.* [Dg]), especially when the group action is FREE (*i.e.*, all isotropy subgroups are trivial). Systematic investigations of equivariant homotopy theory began in the nineteen sixties. A historical summary appears in the first part of [Sc10, Sec. 1]; we shall merely sketch the basic mathematical points here.

In principle, classical obstruction theory yields an algebraic setting for describing homotopy classes of maps provided one has the following data:

(*i*) Cellular decompositions for the source spaces.

(*ii*) Suitably defined cohomology groups for the source spaces, in general with twisted coefficients, that can be computed from small cochain complexes reflecting the cell structure. The coefficients are determined by the homotopy type of the target space and some information involving fundamental groups.

In [Bre2] G. Bredon described general and usable equivariant analogs of (*i*) and (*ii*). The objects in (*i*) were forerunners of *$G - CW$ complexes* (*e.g.*, see [tD2, Sec. II.1]), and the cohomology functors in (*ii*) evolved into the so-called *Bredon (equivariant) cohomology groups*; the latter were first defined by Bredon [Bre2], and a close relation of these to ordinary singular cohomology follows from an alternative definition due to Th. Bröcker [Brö] (in this connection also see [IL1]). Equivariant coefficient systems in Bredon cohomology are more complicated than ordinary coefficient groups and require families of abelian groups indexed by the components of each fixed point set Y^H (where H is a subgroup of G), but one still has small cochain complexes for computing the Bredon cohomology groups ${}_{BR}H^*(X, \mathcal{A})$.

The following results reflect the strong analogies between ordinary and equivariant homotopy theory.

PROPERTY 2.0. (G-homotopy extension property) *Let G be a compact Lie group, let X be a $G - CW$ complex, let $A \subset X$ be a subcomplex, let $f : X \to Y$ be a continuous equivariant map into some G-space Y, and* *let $h_t : A \to Y$ be an equivariant homotopy such that $h_0 = f|A$. Then h_t extends to an equivariant homotopy $H_t : X \to Y$ such that $H_0 = f$.* □

PROPERTY 2.1. (Extension theorem) *Let G, A, X, Y be as above, and let $h : A \to Y$ be continuous and equivariant. Then f extends to a continuous equivariant map from A to Y if a sequence of obstructions valued in the i-dimensional Bredon cohomology groups of (X, A), with coefficients in the $(i-1)$-dimensional homotopy groups of fixed set components of Y, is trivial.* □

PROPERTY 2.2. (Classification theorem) *Let G, X, Y be as above, and let $h_j : X \to Y$ be continuous and equivariant for $j = 0, 1$. Then h_0 and h_1 are equivariantly homotopic if a sequence of obstructions valued in the i-dimensional Bredon cohomology groups of X, with coefficients in the i-dimensional homotopy groups of fixed set components of Y, is trivial.* □

Note the difference in coefficient groups between 2.1 and 2.2.

PROPERTY 2.3. (Barratt-Federer spectral sequence) *Let G, X, Y be as above, and also assume X is finite-dimensional. Modulo some mildly exceptional behavior in dimensions 1 and 0 there is a spectral sequence such that*

$$E^2_{i,j} = {}_{BR}H_G^{-i}\,(X;\, {}_G\pi_j(Y))$$

where ${}_G\pi_j(Y)$ is a coefficient system derived from the homotopy groups of components of fixed point sets Y^H (where H runs through subgroups of G) and $E^\infty_{i,j}$ gives a series for $\pi_{i+j}(F_G(X, Y))$, where $F_G(X, Y)$ is the space of G-equivariant continuous maps from X to Y with the compact open topology. □

Precise statements of 2.1–2.3 appear in [DuS, Sec. 1]. Unfortunately, the preceding results are computationally less useful than their nonequivariant counterparts because Bredon cohomology groups are far more difficult to compute than ordinary singular cohomology. One the other hand, results from [Sc2] yield alternatives to 2.1–2.3 that involve ordinary cohomology groups; not surprisingly, there is a price to pay for this – roughly speaking, each Bredon cohomology group is replaced by a finite list of ordinary cohomology groups. In particular, the analog of Property 2.3 given by [Sc2] is the Barratt-Federer/Fáry spectral sequence in [DuS, Thm. 1.5].

One can in fact extend nearly all the basic concepts and results in algebraic topology to the category of $G-CW$ complexes, including an equivariant version of the Whitehead theorem for recognizing equivariant homotopy equivalences (see [Bre2]), Postnikov decompositions, and localization at a subring of the rationals [MMT]. This can be interpreted as a special case of a more general observation (see [DF, p. 131]; in this connection also see [May] and [DuS, statement (1.8)]). There is also a corresponding analog of equivariant stable homotopy theory; references for the latter include tom Dieck's book on the Burnside ring [tD1], an exhaustive account by G. Lewis, J. P. May, and M. Steinberger [LMS], and a more recent survey article by G. Carlsson [Car].

Application to smooth G-manifolds

Of course, if we wish to apply the preceding machinery to compact smooth G-manifolds it is necessary to find appropriate interpretations of the latter as $G-CW$ complexes. There are two ways of doing this. Results of A. Wassermann [Wa] yield a version of Morse Theory for G-invariant smooth functions on smooth G-manifolds, and from this it is elementary to show that smooth G-manifolds have the G-homotopy type of finite-dimensional $G-CW$ complexes; in fact, for compact smooth G-manifolds one can choose the $G-CW$ complexes to be finite (also see [Ko] for a self-contained account of these results). On the other hand, for many purposes it is more useful to invoke a stronger result due to S. Illman [IL3]: *Every smooth G-manifold (G finite) has a G-equivariant triangulation that is smooth in the sense of J. H. C. Whitehead* (see [Mun] for the corresponding result in ordinary differential topology).

3. Isovariant homotopy theory

Recall that a finite group action on a reasonable (say paracompact Hausdorff) space X is *free* if the isotropy subgroups at all points are trivial. Basic results in topology state that a free action of a finite group G on such a space is determined by its orbit space X/G and a homotopy class of maps from X/G to a universal base space $BG \simeq K(G,1)$. In [Pa] R. S. Palais extended this to a classification theory for G-spaces that are not necessarily free; this involves a special class of maps that Palais called ISOVARIANT. Specifically, a G-equivariant map $f : X \to Y$ is said to be isovariant if for all $x \in X$ the isotropy subgroups satisfy $G_x = G_{f(x)}$; an equivariant map f automatically satisfies $G_x \subset G_{f(x)}$, but it is easy to see that equality often does not hold. During the nineteen sixties and seventies isovariant maps were also discussed in connection with various topological problems, and questions about isovariant homotopy arose naturally; much of this is discussed at various points of [DuS]. The usefulness of isovariant homotopy for classifying G-manifolds became explicit in the work of Browder and Quinn [BQ] on stratified surgery theory. Various *ad hoc* techniques for dealing with isovariant homotopy theory gradually emerged, and by the mid nineteen eighties it was clear that one could analyze isovariant homotopy effectively by the basic techniques of algebraic topology. The key idea is expressed in [DuS] as follows:

> *Isovariant homotopy for smooth G-manifolds is essentially equivalent to equivariant homotopy theory for suitable diagrams of smooth G-manifolds.*

We shall explain this statement in several steps, beginning with a discussion of diagram categories. If $\mathbb{D}$ is a small category and S is some category of topological spaces, then a *$\mathbb{D}$-diagram with values in S* is merely a covariant functor $\mathbb{D} \to S$ and a *morphism of $\mathbb{D}$-diagrams* is a natural transformation of functors. Results of W. Dwyer and D. M. Kan [DK] show that one can extend much of classical homotopy theory and obstruction theory to categories of $\mathbb{D}$-diagrams with values in the category of CW complexes, and subsequent generalizations of E. Dror Farjoun [DF] yield corresponding results for suitable categories of $\mathbb{D}$-diagrams of G-CW complexes. More precisely, we need to choose a small category associated to some finite partially ordered set $\mathbb{P}$ that is closed under taking greatest lower bounds, and we also must restrict attention to $\mathbb{P}$-diagrams satisfying a few simple admissibility conditions (see [DuS, conditions $(i)-(iii)$ preceding (1.6)]. The basic results of equivariant algebraic topology extend directly to such equivariant diagrams and are summarized in [DuS, (1.6)–(1.10)].

Before discussing the types of diagrams we need, a technical remark is in order. Strictly speaking, the results of [DuS] only deal with a restricted class of group actions, but this class includes all actions of the cyclic groups $\mathbb{Z}_{p^r}$, where p is a prime, and all fixed point free actions of $\mathbb{Z}_{pq}$, where p and q are distinct primes. Extensions to more general actions will appear in a sequel to [DuS]. We shall first discuss the types of diagrams needed in an important special case and then outline an inductive extension to the general case.

The semifree case

The conclusions of [DuS] apply directly to actions that are *semifree* (= free off the fixed point set), so we shall describe the appropriate diagrams of G-manifolds in this important special case. If M is a compact smooth G-manifold, let M^G denote its fixed point set as usual, let α_M be the equivariant normal bundle of M^G in M, let $D(\alpha_M)$ and $S(\alpha_M)$ be the associated unit disk and sphere bundles respectively, and let $M \dot{-} M^G$ denote the closure of $M - D(\alpha_M)$. Then the relevant diagram of spaces, which is denoted by $B(\mathbf{QF}_M)$ in [DuS], is the following partially ordered set:

$$\begin{array}{ccccc} S(\alpha_M) & \longrightarrow & D(\alpha_M) & \longleftarrow & M^G \\ \downarrow & & \downarrow & & \\ M \dot{-} M^G & \longrightarrow & M & & \end{array}$$

In the preceding special case the main result of [DuS, Sec. 4] can be stated as follows:

THEOREM 3.1. (compare [DuS, Thm. 4.5]) *Let G be a finite group, let X and Y be compact smooth semifree G-manifolds, and let $B(\mathbf{QF}_X)$ and*

$B(\mathbf{QF}_Y)$ be the diagrams described above. Then there is a canonical isomorphism:

$$\left[\begin{array}{c} G-\textit{equivariant} \\ \textit{homotopy classes of} \\ \textit{continuous equivariant} \\ \textit{diagram morphisms} \\ B(\mathbf{QF}_X)\to B(\mathbf{QF}_Y) \end{array}\right] \cong \left[\begin{array}{c} G-\textit{isovariant} \\ \textit{homotopy classes of} \\ \textit{continuous isovariant} \\ \textit{maps of spaces} \\ X\longrightarrow Y \end{array}\right]$$

□

The proof of this result is a fairly straightforward application of standard compactness considerations and the covering homotopy property for fiber bundles; details appear in [DuS, Sects. 2–4].

Results of Dwyer and Kan [DK] and E. Dror Farjoun [DF] show that one can "do algebraic topology" in the diagram category corresponding to the left hand side of the correspondence displayed in the theorem. In particular, the associated obstruction theories in the diagram categories (*cf.* [DuS, Sec. 1]) are essentially obstruction theories for isovariant homotopy theory. Furthermore, as noted in [DuS, (1.8)–(1.10)] one also has Postnikov decompositions, localization at subrings of the rationals, and spectral sequences for the homotopy groups of isovariant function spaces analogous to those of J. Møller and the author [Mø, Sc2]. At the end of [DuS, Sec. 4] these ideas are applied to prove isovariant analogs of the Whitehead Theorems for recognizing (ordinary or equivariant) homotopy equivalences. Here is one special case:

THEOREM 3.2. *Let G be a finite group, let X and Y be compact,* UNBOUNDED, *semifree smooth G-manifolds, let $f : X \to Y$ be an isovariant map, and also assume that all fixed point sets X^H and Y^H are orientable. Then f is an isovariant homotopy equivalence if and only if for each isotropy subgroup H the map f induces homotopy equivalences from X^H to Y^H and from $X^H - \mathrm{Sing}(X^H)$ to $Y^H - \mathrm{Sing}(Y^H)$.*

Notation and remark. The *singular set* of M^H, denoted by $\mathrm{Sing}(M^H)$, is the set of all points in M^H whose isotropy subgroups PROPERLY contain H.

By definition an isovariant map $X \to Y$ automatically takes $X^H - \mathrm{Sing}(X^H)$ to $Y^H - \mathrm{Sing}(Y^H)$. $\square$

Related isovariant analogs of the Whitehead Theorem also appear in [DuS, Thm. 4.10 and Cor. 4.11]).

Actions with more general isotropy structures

The preceding results extend to arbitrary compact differentiable G-manifolds (where G is finite as usual). In principle the argument combines the techniques used in the semifree case with an inductive framework that we shall describe below; details will appear in a sequel to [DuS].

DEFINITION. If G is a compact Lie group acting with finitely many orbit types (*e.g.*, if G is finite), then the *isotropy depth* of the action is the largest nonnegative integer d such that one has a sequence of isotropy subgroups

$$H_0 \subsetneqq H_1 \subsetneqq \cdots \subsetneqq H_d.$$

In particular, an action with only one isotropy type (*e.g.*, a free action) has isotropy depth zero, and a semifree action has isotropy depth one; the results of [DuS] include extensions of 3.1 and 3.2 to actions with isotropy depth one. Aside from semifree actions, the class with isotropy depth one also includes fixed point free, effective actions of $\mathbb{Z}_{pq}$, where p and q are distinct primes (since the group in question contains exactly two nontrivial proper subgroups, and neither contains the other).

One of the main steps in extending Theorem 3.1 is the description of an analog to $B(\mathbf{QF}_M)$ for actions of isotropy depth greater than one. Suppose we are given a compact smooth G-manifold M; we begin by choosing a G-invariant metric on M and using it to construct a system of invariant tubular neighborhoods of the fixed point sets M^H where H runs through the isotropy subgroups of the action. If we are given an action of isotropy depth zero, then the finiteness of G implies that M splits into a disjoint union of codimension zero submanifolds M^K where K runs through the conjugates of

H, and the *isovariance diagram* $\mathbf{IsD}(M)$ is given by the partially ordered set whose members are the sets M^K. One then assumes that $\mathbf{IsD}(Y)$ has been defined for smooth actions of isotropy depth less than d such that $\mathbf{IsD}(Y)$ satisfies some appropriate conditions (*e.g.*, it contains all fixed point sets Y^H where H runs through the isotropy subgroups of the action).

Consider next a smooth action on M with isotropy depth equal to d, and let $F_d \subset M$ be the set of points whose isotropy subgroups are maximal. It follows that F_d is an invariant union of smooth submanifolds (neatly embedded if $\partial M \neq \varnothing$). Let $S\langle d\rangle$ and $D\langle d\rangle$ be the unit sphere and disk bundles for an invariant tubular neighborhood of F_d (with the abuse of language convention at the end of Section 1), and let $C\langle d\rangle := M \dot{-} F_d$ be the closure of the complement of $D\langle d\rangle$. The diagram $\mathbf{IsD}(M)$ of G-invariant subspaces of M is then constructed in pieces as follows: Since $S\langle d\rangle$ and $C\langle d\rangle$ have isotropy depth $\leq d-1$ we can take $\mathbf{IsD}(M; M \dot{-} F_d)$ to be the union of $\mathbf{IsD}(S\langle d\rangle)$ and $\mathbf{IsD}(C\langle d\rangle)$. The set F_d is a pairwise disjoint union of the subsets M^H where H runs through the maximal isotropy subgroups of the action, so define $\mathbf{IsD}(M; F_d)$ to be the family of all such sets $M_{(K)}$. Next, define $\mathbf{IsD}(M; D\langle d\rangle)$ to be the family of all subsets $D\langle d\rangle^H$; the partially ordered set $\mathbf{IsD}(M)$ is then defined to be a family of subsets generated from $\mathbf{IsD}(M; M \dot{-} F_d)$ and $\mathbf{IsD}(M; D\langle d\rangle)$ by adjoining suitable unions $P \cup Q$ where $P \in \mathbf{IsD}(M; M \dot{-} F_d)$ and $Q \in \mathbf{IsD}(C\langle d\rangle)$.

With these definitions of isovariance diagrams Theorem 3.1 generalizes to a similar canonical isomorphism:

$$\begin{bmatrix} G-\text{equivariant} \\ \text{homotopy classes of} \\ \text{continuous equivariant} \\ \text{diagram morphisms} \\ \mathbf{IsD}(M) \to \mathbf{IsD}(N) \end{bmatrix} \cong \begin{bmatrix} G-\text{isovariant} \\ \text{homotopy classes of} \\ \text{continuous isovariant} \\ \text{maps of spaces} \\ M \longrightarrow N \end{bmatrix}$$

where M and N are compact smooth G-manifolds. Roughly speaking, this can be done inductively by first adjusting the map on $D\langle d\rangle$ and then using an inductive hypothesis to adjust the map on $M \dot{-} F_d$ leaving $S\langle d\rangle$ fixed.

4. Isovariance versus equivariance

One basic question in equivariant homotopy theory is to determine whether a continuous map between two G-spaces is homotopic to an equivariant map (*cf.* [LW], [MP], and [La]). In this section we are interested in a corresponding question for isovariant homotopy theory; namely, whether a continuous equivariant map is equivariantly homotopic to an isovariant map. Several special cases have been studied in by geometric methods. One example will be discussed in Section II.2; other results include Illman's work on isovariance and equivariant general position [IL4] and results with applications to embedding manifolds in the metastable range (see Haefliger [Hae, Prop. 2, p. 245/06] and Harris [Har, Prop. 13, p. 24]). Each approach seems to yield insights not apparent from the others.

Default hypothesis: To keep the notation relatively uncomplicated we shall only consider semifree actions on connected manifolds whose fixed point sets are connected.

Similar results also hold for more general actions, but the terminology quickly becomes far more complicated.

We begin with an elementary observation.

PROPOSITION 4.1. *Let M and N be as above, let α_M and α_N be the equivariant normal bundles of the fixed point sets, let $D(-)$ denote an associated unit disk bundle, and let $f : M \to N$ be an equivariant map. Then f is equivariantly homotopic to a map h such that $h(D(\alpha_M)) \subset D(\alpha_N)$. Furthermore, if h_0 and h_1 are two equivariant maps with this property and Φ_t is an equivariant homotopy from h_0 to h_1, then there is an equivariant homotopy Ψ_t from h_0 to h_1 such that $\Psi_t(D(\alpha_M)) \subset D(\alpha_N)$ for all $t \in [0,1]$.*

This is essentially a special case of [DuS, Prop. 5.1]. □

Theorem 3.1 and Proposition 4.1 combine to yield the following lifting condition for finding an isovariant map in an equivariant homotopy class:

THEOREM 4.2. *Let M and N be as above, and let $f : M \to N$ be a continuous equivariant map satisfying the conditions of Proposition 4.1. Then f is equivariantly homotopic to an isovariant map if the associated maps $S(\alpha_M) \to D(\alpha_N)$ and $M \dot{-} M^G \to N$ lift – equivariantly and compatibly – to $S(\alpha_N)$ and $N \dot{-} N^G$ respectively.*

This is a special case of [DuS, Thm. 5.3]. □

One can obtain cohomological isovariance obstructions by combining Theorem 4.2 with obstruction theory in several different ways. Typical results along this line are given in [DuS, Thms. 5.4–5.5]. The remarks at the end of [DuS, Sec. 5] discuss variants and special cases of these theorems.

Comparative computations

Another way of studying the difference between equivariant and isovariant homotopy is to compare the homotopy groups of an equivariant function space $F_G(X, Y)$ with those of the corresponding isovariant function space $IF_G(X, Y)$, say if X and Y satisfy the basic assumptions of [DuS]. For example, one could use the appropriate Barratt-Federer spectral sequences from [Sc2] and [DuS] in each case, and as a first step it might be worthwhile to take tensor products with the rationals and obtain information about rational homotopy groups.

The final section of [DuS] provides a comparison along these lines when $G = \mathbb{Z}_{pq}$ and $X = Y = S(V)$, where p and q are distinct odd primes and V is a linear G-representation such that V contains at least two free irreducible summands. If $F_G(V)$ is the space of equivariant self maps of $S(V)$ then the Barratt-Federer spectral sequence of [Sc2] shows that $\pi_k(F_G(V)) \otimes \mathbb{Q} = 0$ for all but finitely many k. On the other hand, if $IF_G(V)$ is the space of isovariant self maps of $S(V)$ then the results of [DuS, Sec. 6] show that each rational homotopy group $\pi_k(IF_G(V)) \otimes \mathbb{Q}$ is finite dimensional, but the sequence of dimensions d_k satisfies $\limsup_{k \to \infty} d_k / k^n = \infty$ for every positive integer n. There is an explanation for this difference in terms of the Barratt-Federer spectral sequences: In the equivariant setting one has cohomology groups with coefficients in the homotopy groups of spheres, and rationally these vanish in all but at most two dimensions. On the other hand, in the isovariant setting one has cohomology groups with coefficients in the homotopy groups of wedges of spheres, and the ranks of these groups tend to grow exponentially as the dimension increases.

PART II

ISOVARIANT HOMOTOPY, CLASSIFICATION PROBLEMS, AND GENERAL POSITION

Background material

As in Part I we assume basic concepts in algebraic topology and transformation groups in Bredon, tom Dieck, Dovermann-Schultz, Milnor-Stasheff, and Spanier. Beyond this, we shall frequently mention the concept of *simple homotopy equivalence* as presented in Milnor's article [MLN2] or M. Cohen's book [Co], and we shall also use data from the *Sullivan-Wall surgery exact sequence.* The standard reference for the latter is Section 10 of Wall's book, *Surgery on Compact Manifolds* (:= [WL]). Most of the material of immediate interest in this article is summarized in [Brw2], and in particular the Sullivan-Wall sequence is presented in [Brw2, p. 29] with a minimum of technical diversions. However, we shall use notation that differs slightly from that of [WL] and [Brw2], mainly because we shall also need to consider variants of the structure sets described in those references.

Both [WL] and [Brw2] deal with structure sets $\mathcal{S}^{Diff}(X)$ of simple homotopy structures on a simple Poincaré complex X and with certain algebraically defined surgery obstruction groups $L_k(\pi)$. As noted in Section 17 of [WL] one can define analogous homotopy structure sets, surgery obstruction groups, and exact sequences for homotopy structures on a Poincaré complex; for this theory, equivalent structures are h-cobordant rather than diffeomorphic. To distinguish between the two structure set theories in the smooth category we shall denote the simple homotopy objects by $\mathcal{S}^{s,Diff}$ and $L_k^s(\pi, w)$, and we shall denote the ordinary homotopy objects by $\mathcal{S}^{h,Diff}$and $L_k^h(\pi, w)$; here w refers to the homomorphism $\pi \rightarrow \mathbb{Z}_2$ defined by the first Stiefel-Whitney class. We shall also use somewhat different notation for the bordism classes of degree one normal maps that are called $N^{Diff}(X)$ in [Brw2]. Standard results in algebraic and geometric topology imply that

$N^{Diff}(X)$ is isomorphic to the set of homotopy classes $[X, F/O]$, where F/O is the space classifying stable fiber homotopy trivializations of stable vector bundles of X; this space is considered in [Brw3, Sec. II.4] where it is called G/O (we use F/O rather than G/O because the two names are essentially used interchangeably in the literature and G will frequently denote a finite group in the discussion that follows). A full discussion of the isomorphism $N^{Diff}(X) \approx [X, F/O]$ appears in [Brw3, Thm. II.4.4, pp. 46–49]. This description of $N^{Diff}(X)$ is useful because there is an exact sequence of abelian groups

$$\cdots \to \widetilde{KO}(\Sigma X) \to \{\Sigma X, S^0\} \to [X, F/O] \hookrightarrow \widetilde{KO}(X) \to \widetilde{KSph}(X)$$

where $\widetilde{KO}$ denotes reduced real K-theory, $\{-, S^0\}$ denotes stable cohomotopy, and $\widetilde{KSph}$ denotes the analog of reduced K-theory for stable spherical fiber spaces.

Finally, we shall also note the existence of *relative structure sets* $\mathcal{S}^{c,Diff}(X, \partial X)$ for Poincaré complexes with formal boundaries; here $c = s$ or h. The basic idea is to take simple homotopy (resp. homotopy) equivalences $(M, \partial M) \to (X, \partial X)$ such that the map of boundaries is a diffeomorphism. There is an extension of the Sullivan-Wall exact sequence to such objects modulo some adjustments; the Wall groups are simply those for the fundamental group and first Stiefel-Whitney class of X, but the normal bordism set $N^{Diff}(X, \partial X)$ in this case is equivalent to $[X \cup \mathrm{Cone}(\partial X), F/O]$.

Recent work of M. Dawson [Daw] includes an independent proof of the main results in Section 1 and applications to smooth variants of the Cappell-Weinberger *replacement theorems* (*e.g.*, see Theorem 5.1 below for a statement of one such result in the locally linear topological or PL categories). Additional remarks on this work appear in Section 5.

1. Isovariant homotopy structures

The main ideas of surgery theory began to emerge in the nineteen fifties, and they became well established with the work of M. Kervaire and J. Milnor [KM] on classifying smooth manifolds that are homotopy equivalent to spheres (*i.e.*, homotopy spheres). Subsequent work of W. Browder and S. P. Novikov yielded far reaching extensions of [KM] to existence and classification questions for simply connected manifolds of a fixed homotopy type, and still further work of Wall extended the theory to manifolds with arbitrary fundamental groups [WL].

It soon became clear that surgery theory also yielded valuable information on existence and classification questions for group actions on manifolds (*cf.* [Brw1]). In particular, many striking applications to free differentiable group actions on spheres were made during the nineteen sixties (*e.g.*, see [HH], [Hs], [LdM]). Systematic efforts to study nonfree actions also began in the nineteen sixties with work of Browder and Petrie [BP] and Rothenberg and Sondow [RSo] on classifying smooth G-actions that are semifree and homotopically linear – in other words, both M and M^G are closed manifolds that are homotopy equivalent to spheres (see also [Brw1] and [Sc3]). Actions of this type can be viewed as smooth G-manifolds that are equivariantly homotopy equivalent to a linear G sphere $S(V)$ given by the unit sphere in some orthogonal, semifree representation of G on a finite dimensional real inner product space V. In analogy with the Browder-Novikov-Wall extension of [KM] to arbitrary closed manifolds, it is natural to search for an extension of the Browder-Petrie and Rothenberg-Sondow work to more general G-manifolds.

A major step in this direction was due to W. Browder, who presented his ideas in a lecture at a conference in 1971 (see p. *vii* in the book containing [MnY1]). This work was later extended by F. Quinn and published jointly in [BQ]. The basic idea was to consider manifolds that are *isovariantly* homotopy equivalent to a given model such that the isovariant equivalence satisfies a *transverse linearity* condition. For the sake of simplicity we shall only describe this for semifree actions. In such cases the isovariant homotopy equivalence $f : M \to N$ is supposed to be a map of triads from $(M; M \overset{*}{-} M^G, D(\alpha_M))$ to $(N; N \overset{*}{-} N^G, D(\alpha_N))$, where α_Y refers to the (equivariant) normal bundle of Y^G in Y as in Part I, and the induced map from $D(\alpha_M)$ to $D(\alpha_N)$ is assumed to be (orthogonally) linear and fiber preserving. If one specializes this theory to G-manifolds modeled by linear G-spheres, one obtains a theory that maps naturally into the Browder-Petrie and Rothenberg-Sondow theories and includes many infinite families of examples from (both of) the latter.

One of the most important properties of the Browder-Quinn setting is the existence of a surgery exact sequence that is formally parallel to the Sullivan-Wall sequence (*cf.* [BQ, Thm. 2.2, p. 29]):

$$\cdots \to L_{n+1}^{c,BQ}(X) \to \mathcal{S}_G^{c,BQ}(X) \to [X/G, F/O] \to L_n^{c,BQ}(X)$$

In this sequence X is a closed smooth G-manifolds, the symbol c denotes either s (for equivariant simple homotopy; *cf.* Illman [IL2] or Rothenberg [Ro]) or h (for ordinary equivariant homotopy), and the groups $L_*^{c,BQ}(X)$ are the *Browder-Quinn surgery obstruction groups* as defined and studied in [BQ] and [DoS2, Sec. 2]. Although these groups are written in terms of X, they are in fact determined by weaker data that is summarized in the *geometric reference* R_X of W. Lück and I. Madsen [LüMa, Definitions (2.3) and (3.1), pp. 512 and 516]. As noted in [BQ] and [DoS2], certain natural exact couples determine spectral sequences converging to the groups $L_*^{c,BQ}(X)$ such that the initial terms are ordinary Wall groups L_*^c, and therefore one can view the terms in the Browder-Quinn surgery sequence as computable up to determination of the homotopy groups of F/O and the appropriate Wall groups. In analogy with the Sullivan-Wall sequence, there are also relative versions of the Browder-Quinn sequence involving structure sets $\mathcal{S}^{c,BQ}(X, \partial X)$ represented by transverse linear G-homotopy equivalences $(M, \partial M) \to (X, \partial X)$, with $c = h$ or s as usual, such that ∂M maps to ∂X by a diffeomorphism.

The basic aim of isovariant surgery theory is to provide a setting that is broad enough to include both the Browder-Quinn theory and the work of Browder-Petrie and Rothenberg-Sondow, but is also more or less computable, at least in some cases beyond those of [BP], [RSo], and [BQ].

We shall begin by relating [BQ] to [BP] and [RSo] in the case of homotopically linear semifree group actions on spheres. The tangential representation at a fixed point will be assumed to have the form $V \approx \mathbb{R}^k \oplus \alpha$, where the representation α has no trivial summands (hence G acts freely on the unit sphere $S(\alpha)$). Deviating slightly from the the notation of [Sc3], let $CS_k(G,\alpha)$ be the set of equivariantly oriented h-cobordism classes of homotopically linear semifree group actions as described above, where the tangent space at a fixed point is G-isomorphic to the representation V; as noted in [RSo] these sets have natural abelian group structures if the dimension of the fixed point set is at least 2. There are also canonical abelian group structures on the relative Browder-Quinn structure sets

$$\mathcal{S}_G^{h,BQ}(\,D(V), S(V)\,),$$

and there is a natural forgetful map $\mathcal{S}_G^{h,BQ}(\,D(V), S(V)\,) \to CS_k(G,\alpha)$, given by gluing a copy of $D(V)$ to the boundary, that is additive. As in Part I let $F_G(\alpha)$ be the space of equivariant self maps of the unit sphere $S(\alpha)$. The orthogonal centralizer of α is a compact subgroup of the topological monoid $F_G(\alpha)$ and will be denoted by $C_G(\alpha)$; it follows that the quotient space construction defines a principal bundle

$$C_G(\alpha) \subset F_G(\alpha) \to F_G(\alpha)/C_G(\alpha).$$

One can then define a *knot invariant* homomorphism

$$\omega : CS_k(G,\alpha) \to \pi_k(F_G(\alpha)/C_G(\alpha)\,)$$

as in [Sc3, top of p. 311] or [Sc4, Sec. 2].

PROPOSITION 1.1. *In the notation of the preceding paragraph, there is a long exact sequence of the following form:*

$$\cdots \pi_{k+1}(F_G(\alpha)/C_G(\alpha)) \to \mathcal{S}_G^{h,BQ}(D(V), S(V)) \\ \to CS_k(G,\alpha) \to \pi_k(F_G(\alpha)/C_G(\alpha))$$

In other words, the forgetful map from the Browder-Quinn groups to the $CS_k(G,\alpha)$ groups is the "homotopy fiber of the knot invariant." □

The next step in the comparison is to note that each element of $CS_k(G,\alpha)$ is canonically isovariantly homotopy equivalent to $S(V \oplus \mathbb{R}) \approx D_+(V) \cup_\partial D_-(V)$. This suggests that the groups $CS(G,V)$ should be viewed as structure sets for relative G-isovariant homotopy structures on $(D(V), D(V))$. In fact, it is possible to describe structure set theories for arbitrary smooth semifree G-manifolds in the spirit of [BP] and [RSo]. As before there are two versions $\mathbf{I}\mathcal{S}^s_G$ and $\mathbf{I}\mathcal{S}^h_G$ for isovariant simple homotopy and ordinary isovariant homotopy equivalences respectively. There are also relative versions of these structure sets for isovariant homotopy structures that are diffeomorphisms on the boundary.

The exact sequence of [Sc3, (1.1), p. 311] plays an important role in many studies of the groups $CS_k(G,\alpha)$, and therefore one would like to have analogs of this for the structure sets $\mathbf{I}\mathcal{S}^c_G(M)$. In order to do this it is necessary to generalize the knot invariant, and this in turn requires a suitable analog of the homotopy group $\pi_k(F_G(\alpha)/C_G(\alpha))$. The approach below is an adaptation of ideas from [Sc9, Secs. 2–3].

DEFINITION. Let X be a G-space, let $A \subset X$ be G-invariant, and let ξ be a G-vector bundle over X. A *G-isovariant fiber homotopy linearization* of ξ is a pair (ξ, h) consisting of a G-vector bundle $\omega \downarrow X$ and a G-isovariant fiber homotopy equivalence $h : S(\omega) \to S(\xi)$ that is an orthogonal isomorphism over A. The set $F/O_{G,iso}(\xi \text{ rel } A)$ is the set of Such objects modulo the equivalence relation generated by fiber preserving orthogonal vector bundle isomorphisms $S(\omega') \to S(\omega)$ and isovariant homotopies $H_t : S(\omega) \to S(\xi)$ that are orthogonal over A.

Note. If G acts semifreely on M with $X = M^G$ and $\xi = \alpha_M$, then G acts freely on $S(\xi)$ and the isovariance condition merely requires that G act freely on $S(\omega)$.

If $f : M \to N$ is an isovariant homotopy equivalence of semifree smooth G-manifolds, then one can define a generalized knot invariant of f in $F/O_{G,iso}(\alpha_N)$ as follows: By the results of Section I.3 we can deform f isovariantly so that f maps $S(\alpha_M)$ to $S(\alpha_N)$, and the construction yields a unique isovariant homotopy class of maps $S(\alpha_M) \to S(\alpha_N)$; this map can be further deformed, again uniquely up to isovariant homotopy, to a map f' such that the following diagram commutes:

$$\begin{array}{ccc} S(\alpha_M) & \xrightarrow{f'} & S(\alpha_N) \\ \downarrow & & \downarrow \\ M^G & \xrightarrow{f^G} & N^G \end{array}$$

Since f^G is a homotopy equivalence there is a unique G-vector bundle β (up to isomorphism) such that $\alpha_M \cong \{f^G\}^*\beta$, and it follows that f' factors through an isovariant fiber homotopy equivalence $\eta : S(\beta) \to S(\alpha_M)$; by construction the class of (β, η) in the set $F/O_{G,iso}(\alpha_N)$ is well defined, and this is the generalized knot invariant of f.

A similar construction is valid for relative homotopy structures on a compact smooth semifree G-manifold with boundary, and in this case the knot invariant lies in the relative set $F/O_{G,iso}(\alpha_N \text{ rel } \partial N)$. The following result is a natural extension of Proposition 1.1 to arbitrary smooth semifree G-manifolds:

THEOREM 1.2. *If X is a closed smooth semifree G-manifold such that each component of X^G is at least 5-dimensional, then there is an exact sequence of structure sets*

$$\cdots F/O_{G,iso}(\alpha_X \times I \text{ rel } X \times \{0,1\}) \to \mathcal{S}^{c,BQ}(X) \to \mathbf{I}\mathcal{S}^c_G(X) \to F/O_{G,iso}(\alpha_X)$$

that extends infinitely to the left. All objects to the left of $\mathcal{S}^{c,BQ}(X)$ are groups, and all maps are compatible with group structures as in the Sullivan-Wall exact sequence. □

Reminder. In the Sullivan-Wall exact sequence the source and target for the surgery obstruction map $[X, F/O] \to L^c_n(\pi, w)$ are abelian groups but the map itself is not additive in general.

The preceding sequence relates $\mathcal{S}^{c,BQ}(X)$ to $\mathbf{I}\mathcal{S}^c_G(X)$. There is also an exact sequence for $\mathbf{I}\mathcal{S}^c_G(X)$ that generalizes the exact sequence for homotopy linear actions in [Sc3, (1.1), p. 311]. Before stating this result we need a notational convention:

> If Y is a compact bounded manifold then $\partial^* : \mathcal{S}^c(Y) \to \mathcal{S}^c(\partial Y)$ is given by restriction to the boundary.

THEOREM 1.3. *Let X satisfy the conditions of Theorem 1.2. Then there is an exact sequence of sets*

$$\cdots \mathcal{S}^{c,Diff}(X - \mathit{Int}(D(\alpha_X))/G, S(\alpha_X)/G) \to \quad I\mathcal{S}^c_G(X)$$

$$\downarrow$$

$$\mathcal{S}^{c,Diff}(X^G) \times F/O_{G,iso}(\alpha_X)$$

$$\downarrow$$

$$\mathcal{S}^{c,Diff}(S(\alpha_X)/G)/\mathit{Image}\ \partial^*$$

that extends infinitely to the left. All objects to the left of the raised dots are groups and the corresponding maps are compatible with group structures as in Theorem 1.2. □

As usual, there is a variant of this exact sequence for relative structure sets.

Extensions to more general actions

Theorems 1.2 and 1.3 provide a means for analyzing isovariant structure sets in terms of ordinary structure sets and equivariant/isovariant homotopy theory, provided the group action is semifree. Each of these extends to actions with more complicated orbit structure. In particular, a generalization along the lines of 1.2 was considered in earlier work by the author [Sc13]; the necessary modifications include

(*i*) an extension of $F/O_{G,iso}(-)$ from vector bundles to the vector bundle systems (known as Π-*bundles* in the papers of Dovermann-Petrie-Rothenberg [DP1–2, DR]) over Sing(X),

(*ii*) the definition of a generalized knot invariant for an isovariant homotopy equivalence, taking its value in the set described above.

With this machinery in place, it is a formal exercise to prove that the forgetful map $\mathcal{S}^{BQ,c}(X) \to \mathbf{I}\mathcal{S}_G^c(X)$ is essentially the homotopy fiber of the knot invariant constructed by (*ii*).

Theorem 1.3 is essentially a means for analyzing isovariant homotopy structures on X by splitting them into two pieces; namely, pieces over a tubular neighborhood of X^G and pieces over the free G-manifold $X \dot{-} X^G$. There are several ways of extending this to more general actions; we shall only discuss two extreme cases here. The first approach is to split an arbitrary smooth G-manifold into a smooth equivariant regular neighborhood $\mathcal{R}_X$ of the singular set $\mathrm{Sing}(X)$ and the free G-manifold $X \dot{-} \mathrm{Sing}(X)$. This approach was discussed in [Sc6]; we shall not attempt to provide a precise description because it requires a notion of isovariant structure set for the singular set $\mathrm{Sing}(X)$, which is generally not a smooth G-manifold (with a possibly noneffective group action). To describe a complementary approach, we shall assume for the sake of simplicity that all isotropy subgroups are normal (*e.g.*, this happens if G is abelian). Suppose that H is a maximal isotropy subgroup, and let α_H be the equivariant normal bundle of X^H in X. Then one has the following analog of Theorem 1.3:

THEOREM 1.4. *Let X satisfy the conditions of the preceding paragraph. Then there is an exact sequence of sets*

$$\begin{array}{ccc} \cdots \mathbf{I}\mathcal{S}_G^c(X - \mathit{Int}(D(\alpha_X)), S(\alpha_X)) & \longrightarrow & \mathbf{I}\mathcal{S}_G^c(X) \\ & & \downarrow \\ & & \mathbf{I}\mathcal{S}_{G/H}^c(X^H) \times F/O_{G,iso}(\alpha_X) \\ & & \downarrow \\ & & \mathbf{I}\mathcal{S}_G^c(S(\alpha_X))/\mathit{Image}\ \partial^* \end{array}$$

that extends infinitely to the left. All objects to the left of the raised dots are groups and the corresponding maps are compatible with group structures as in Theorems 1.2 – 1.3. □

This result provides a means for analyzing isovariant structure sets inductively with respect to the number of orbit types, for the two structure sets in the sequence aside from $\mathbf{I}\mathcal{S}_G^{BQ,c}(X)$ have fewer orbit types than the original action, and the same is true for the data in $F/O_{G,iso}(-)$. In subsequent work we shall study special cases of this sequence in connection with questions from Section 3 below.

2. Isovariance and the Gap Hypothesis

During the nineteen seventies and early eighties, Petrie and several other topologists (beginning with S. Straus [Str]) found many striking applications of surgery to smooth G-manifolds satisfying the following basic condition:

GAP HYPOTHESIS. A smooth G–manifold M is said to satisfy the (standard version of the) *Gap Hypothesis* if for each pair of isotropy subgroups $H \supsetneqq K$ and each pair of components $B \subset M^H$, $C \subset M^K$ such that $B \subsetneqq C$ we have

$$(\ddagger) \qquad \dim B < \tfrac{1}{2}(\dim C).$$

This is basically a general position condition. Its usefulness arises because surgery theory involves the existence of smoothly embedded spheres whose dimensions are no more than half the dimensions of the ambient manifolds. If the Gap Hypothesis holds, then one can choose the appropriate embedded spheres in each fixed set component $C \subset M^K$ to miss all the components $D \subset M^H$ such that $D \subsetneqq C$. This means that all the constructions involving embedded spheres can be done equivariantly on the set $M^K \cap M_{(K)}$, which has only one isotropy type (namely, K). In effect, this reduces an equivariant surgery problem to a sequence of nonequivariant problems over the orbit spaces $M_{(K)}/G$. A similar reduction arises in the Browder-Quinn theory even if the Gap Hypothesis does not hold; this follows directly from the isovariance and transverse linearity conditions of [BQ].

Most of Petrie's work dealt with the existence of smooth G-actions on disks and spheres with properties quite unlike those of orthogonal actions (compare [Pet1–2] and [DPS]). In a somewhat different direction, K. H. Dovermann and M. Rothenberg modified Petrie's approach to construct classification theories for G-manifolds in a given equivariant homotopy type *provided the Gap Hypothesis holds* (see [DR] and [LüMa]).

One of the central problems in equivariant surgery is to understand the role of the Gap Hypothesis more clearly (*cf.* [Sc12, Sec. 4]), and thus it is natural to seek relationships between the isovariant homotopy structure theory of Section 1, which does not require the Gap Hypothesis, and equivariant surgery theories that somehow rely on the Gap Hypothesis as in [DR] or [LüMa] (related examples are also discussed in [DoS2, Sec. II.3]). As noted in [Daw], the isovariant structure sets of Section 1 lie somewhere between such equivariant surgery theories and the Browder-Quinn theories. The following result of S. Straus [Str] and W. Browder [Brw4] establishes a stronger and more precise relationship; in particular, the theories of [DR] and [LüMa] are equivalent to the theories of Section 1 when the Gap Hypothesis holds.

THEOREM 2.1. *Let $f : M \to N$ be an equivariant homotopy equivalence of closed smooth G-manifolds that satisfy the Gap Hypothesis. Then f is equivariantly homotopic to an isovariant homotopy equivalence. Furthermore, if $M \times [0,1]$ satisfies the Gap Hypothesis then this isovariant homotopy equivalence is unique up to isovariant homotopy.*

This result and the machinery of Sections I.4 and II.1 suggest a two step approach to analyzing smooth G-manifolds within a given equivariant homotopy type if the Gap Hypothesis does not necessarily hold; namely, the first step is to study the obstructions to isovariance for an equivariant homotopy equivalence and the second step is to study the isovariant structure sets of the preceding section.

Sketch of the proof of Theorem 2.1. We shall only deal with semifree G-manifolds in order to illustrate the ideas without addressing the bookkeeping problems that arise for more general actions; furthermore, for the sake of simplicity we shall use a slightly stronger version of the Gap Hypothesis with $\dim B + \varepsilon < \frac{1}{2}(\dim C)$ for some small positive integer ε. Finally, we shall only consider the existence question; the uniqueness result follows by applying similar methods to $M \times [0,1]$.

The original proofs of Straus and Browder rely heavily on methods and results from nonsimply connected surgery. The argument presented here does not completely eliminate geometric topology, but it only requires simple considerations involving embeddings in the general position range and transversality. Of course it would be interesting to know if the proof can be done entirely with homotopy theoretic machinery.

The first step in the proof is to deform f equivariantly so that it maps $D(\alpha_M)$ isovariantly to $D(\alpha_N)$ such that $S(\alpha_M)$ is sent to $S(\alpha_N)$. This will follow quickly if $S(\alpha_M)$ and $S(\{f^G\}^*\alpha_N)$ are equivariantly fiber homotopy equivalent. To prove the latter, one first uses a result of K. Kawakubo [Ka] to show that the equivariant tangent bundle τ_M is stably equivariantly fiber homotopy equivalent to $f^*\tau_N$. Restricting to fixed point sets, we conclude next that the restrictions of these bundles to N^G are also equivariantly stably fiber homotopy equivalent; in other words, $\{f^G\}^*\tau_{N^G} \oplus \{f^G\}^*\alpha_N$ is equivariantly stably fiber homotopy equivalent to $\tau_{M^G} \oplus \alpha_M$. The classifying space versions of the standard splittings for equivariant stable homotopy theory (*e.g.*, the discussion at the end of [Se]) then imply that $\{f^G\}^*\alpha_N$ is equivariantly stably fiber homotopy equivalent to α_M. Since the dimensions of the latter bundles are at least somewhat larger than the dimensions of M^G and N^G, the stable range theorems of [Sc1] and [Sc5] imply that the unit sphere bundles of $\{f^G\}^*\alpha_N$ and α_M are already equivariantly fiber homotopy equivalent before stabilization.

The second step is to analyze the set of points where the modified map fails to be isovariant. We can apply transversality on the complement of $D(\alpha_M)$, without changing the map on $S(\alpha_M)$, so that a further equivariant deformation of f is transverse to N^G on the complement of M^G. It follows immediately that the set Y of nonisovariant points is a smooth invariant submanifold such that $\dim Y = \dim M^G$. Using general position and the fact that f is an equivariant homotopy equivalence, one can then show that Y lies in some tubular neighborhood of the fixed point set (the inclusion of Y can be deformed into $D(\alpha)$ because f is an equivariant homotopy equivalence, and by general position one can modify this into an isotopy of Y into $D(\alpha)$).

The third step is to show that the map obtained in the previous step is equivariantly homotopic to an isovariant map if and only if the class of the nonisovariant set in an appropriate bordism theory vanishes. By the results of Section I.4, the obstruction to deforming the map f_2 obtained thus far is the obstruction to finding an equivariant lifting of $f_2|M \dot{-} M^G$ from N to $N \dot{-} N^G$. Because the Gap Hypothesis holds, one can use the Blakers-Massey Theorem to view this lifting obstruction as the obstruction to finding an equivariant nullhomotopy for the composite of $f_2|M \dot{-} M^G$ with the collapse map $N \to N/N \dot{-} N^G \approx D(\alpha_N)/S(\alpha_N)$.

The next to last step is to notice that the obstruction from the preceding step need not vanish, but it has a canonical indeterminacy given by the possible choices of the equivariant fiber homotopy equivalence from α_M to $\{f^G\}^*\alpha_N$. In fact, since we are in the stable range the homotopy classes of such equivalences are given by $[N^G, F_G]$. Finally, an analysis of the obstructions in the third step shows that one can kill the isovariance obstruction by choosing a (possibly) different equivariant fiber homotopy equivalence on the equivariant sphere bundles. □

3. Homotopy linear actions on spheres

As indicated in Section 1, the original interest in classifying smooth manifolds in a given isovariant homotopy type involved certain smooth group actions on homotopy spheres. In this section we shall discuss some basic questions in this area that can be analyzed, at least to some extent, by the methods of the preceding sections.

From a purely formal viewpoint we are interested in smooth G-manifolds that are isovariantly homotopy equivalent to linear actions on spheres. However, for historical and practical reasons it is more useful to deal with actions satisfying apparently weaker assumptions and to prove that all such actions are isovariantly homotopy equivalent to the appropriate linear example.

The basic concepts and constructions for homotopy linear actions are summarized in [Sc10, Secs. 5–6]. We shall begin with a modified version of the definition in [Sc10, Sec. 5, p. 274].

DEFINITION. Let φ_0 be a linear representation of G on $\mathbb{R}^n + 1$ that splits as $\varphi \oplus \mathbb{R}$ (with trivial action on the second summand). If H is a subgroup of G let $n(H) + 1$ denote the dimension of the real vector space φ_0^H (hence $n(H) \geq 0$). A smooth G-action γ on a smooth manifold Σ^n is said to be *strongly φ-homotopy linear* ($\equiv$ *strongly φ-homotopically linear* or *strongly φ-semilinear*) if the following hold:

(1) For each $H \subset G$ the fixed point set of H is homeomorphic (but not necessarily diffeomorphic) to $S^{n(H)}$.
(2) If $H \subset K \subset G$ and $n(H) - n(K) = 2$ then $\Sigma^H - \Sigma^K$ is homotopy equivalent to S^1.
(3) The induced G representations at the tangent spaces of points in Σ^G are all equivalent to φ.

It is fairly elementary to show that each such action is G-homotopy equivalent to the unit sphere $S(\varphi_0)$, or equivalently to the one point compactification of φ (*cf.* [Sc10, Prop. 5.1]); in fact, Σ^n is usually G-homeomorphic to this linear sphere (see the remarks on [Sc10,p. 274] following Proposition 5.1), and in the remaining cases the results of [DuS, Sec. 4] show that Σ is isovariantly homotopy equivalent to the linear action.

Connected sums. If $G = \{1\}$ then a strongly homotopy linear G-manifold is a manifold homeomorphic to S^n (*i.e.*, an *exotic sphere*) by the Generalized Poincaré Conjecture [MLN1, p. 109]; a diffeomorphism classification of such objects was developed in the previously mentioned work of Kervaire and Milnor during the late nineteen fifties and early nineteen sixties [KM]. An elementary but highly useful step in their program was the use of objects with orientations and the introduction of an abelian group structure on the oriented diffeomorphism classes of exotic spheres by means of connected sums (see [Sc10, p. 275] and the references cited there). One can proceed similarly with strongly homotopy linear φ-spheres: Given two such G-manifolds Σ_1 and Σ_2, let $D_i \subset \Sigma_i$ be G-diffeomorphic to the disk

$D(\varphi)$ and glue $\Sigma_1 - \mathrm{Int}(D_1)$ to $\Sigma_2 - \mathrm{Int}(D_2)$ equivariantly along the common boundary; once again one needs a suitable concept of orientation to ensure this construction is well defined, and this can be done as in [Sc9, Sec. 1]. As noted in [Sc10, Prop. 5.2, p. 276], this yields a monoid structure on the set of all (suitably equivariantly oriented) diffeomorphism classes of strongly φ-semilinear spheres, the resulting monoid is abelian if the fixed point set dimension is at least 2, and if we factor out the submonoid of actions that bound equivariantly contractible G-manifolds, then the resulting quotient is a group (abelian if the fixed point sets are at least 2-dimensional). Following [Sc10] we shall denote this group by $\Theta^G(\varphi)$.

Digression—some motivation

Although one can certainly study the groups $\Theta^G(\varphi)$ for their own sake, these groups also arise naturally in connection with certain questions of independent interest. Before proceeding with further results on such actions we shall describe some of these contexts.

EXAMPLES. **1.** *Smooth actions of arbitrary p-groups on exotic spheres.* In fact, as noted in [Sc6] this was one of the original motivations for studying the classification of smooth G-manifolds in a given isovariant homotopy type. If one is given an action of a finite abelian p-group on an exotic sphere, then Smith theory shows that all the fixed point sets are mod p homology spheres. This implies that an arbitrary such action admits an isovariant map to a linear model with degree prime to p. If the dimension of the fixed point set of G is at least 2, then one can use these maps and the methods of Section 1 to define p-localized versions of the knot invariant with values in the abelian groups

$$F/O_{G,iso}(S(\varphi^H \oplus \mathbb{R}) \times [\varphi/\varphi^H] \text{ rel } \{\text{basept.}\})_{(p)}$$

where H is an arbitrary isotropy subgroup of the action. For the special case of cyclic p-groups where H is the minimal nontrivial isotropy subgroup, this was done previously in [Sc4] and [Sc9, Sec. 4], where the invariants were used to obtain restrictions on the fixed point structure of smooth $\mathbb{Z}_{p^r}$-actions on exotic spheres. In subsequent work the more general invariants will be used to study actions of other abelian p-groups on exotic spheres in relatively low dimensions.

2. *Fixed point sets of differentiable actions on spheres.* Results of L. Jones [Jo], A. Assadi [As] and others show that certain variants of the groups $\Theta^G(\varphi)$ carry the obstructions to realizing a mod p homology k-sphere as the fixed point set of a smooth semifree $\mathbb{Z}_{p^r}$-action on some homotopy $(k+2m)$-sphere, where $m \geq 2$ (*cf.* [As]). The basic idea is simple: If A is the homology sphere, let A_0 denote A with the interior of a closed disk removed. Then one can realize A_0 as the fixed point set of a smooth semifree $\mathbb{Z}_{p^r}$-action on D^{k+2m}. The induced action on the boundary then determines an element of the appropriate variant group $\mathrm{Var}\Theta^G(\varphi)$, and one can extend the action to an action on a homotopy sphere if and only if this element is zero. Related ideas are used in [Sc14] to construct examples of smooth $\mathbb{Z}_{pq}$-actions on spheres, where p and q are distinct odd primes, such that the Pontrjagin numbers of the fixed point set are nontrivial.

3. *Equivariant smoothings of topological G-manifolds.* Results of Lashof and Rothenberg [LaR] show that the smoothability of a G-manifolds and the classification of equivariant smoothings reduce to equivariant bundle-theoretic questions, at least if there are no 4-dimensional components in the fixed point sets of the isotropy subgroups. This is formally parallel to ordinary smoothing theory for topological manifolds (*cf.* [KiSb]). However, in ordinary smoothing theory the results of [KM] and [KiSb] translate the bundle-theoretic problems into well known questions of homotopy theory, but comparable insights into equivariant smoothing theory only exist in special cases. This is already evident in known results on the topological classification of linear representations (*e.g.*, see [CS1–3, CSSW, CSSWW]), which is the first step in analyzing the bundle-theoretic problems in [LaR]. Partial results on the higher order steps appear explicitly in [LaR] and [MR], and implicitly in [Sc7], [KL], and [KwS6]. Standard techniques of engulfing theory ([Hud, Ch. VII] or [RSa, Ch. 7]) imply that a strongly φ-homotopically linear G-manifold is equivariantly homeomorphic to $S(\varphi \oplus \mathbb{R})$ if the dimension of φ^G is sufficiently large [CMY, IL5, Ro, Sc7], and thus information about the groups $\Theta^G(\varphi)$ has immediate implications for equivariant smoothing theory (*cf.* the results on rational characteristic classes in [Sc7]). It is conceivable that information on the groups $\Theta^G(\varphi)$ can also shed light on equivariant smoothing theory for more general G-manifolds; in particular, the results of [LaR, pp. 215 and 264–265] suggest this.

4. *Rational invariants for classifying smooth G-manifolds up to finite ambiguity.* In a sequence of papers culminating with [RT], Rothenberg and Triantafillou described an equivariant analog of D. Sullivan's rational invariants for diffeomorphism classification of certain smooth simply connected manifolds up to finite ambiguity [Su]. However, their invariants only provide an equivariant *almost diffeomorphsim* classification up to finite ambiguity in many cases; in other words, to complete the picture one needs a smooth equivariant classification up to finite ambiguity for all G-manifolds that are equivariant connected sums $M_0 \# \Sigma$, where M_0 is fixed and Σ is a homotopy linear G-sphere. Questions of this type have been studied by M. Masuda [Ms] and will be considered further in joint work of Masuda and the author [MSc].

Exact sequences

Of course, the usefulness of the groups $\Theta^G(\varphi)$ depends on the extent to which they can be computed. The original work of [KM] can be summarized in a long exact sequence

$$\cdots \to P_{n+1} \to \Theta_n \to \pi_n(F/O) \to \cdots$$

where the groups P_k are 0 if k is odd, infinite cyclic if k is divisible by 4, and cyclic of order two if $k \equiv 2 \mod 4$ (compare [Lev] or [Sc10, Thm. 6.1, p. 277]). One particular consequence of this sequence is the finiteness of the groups Θ_n if $n \geq 4$. The subsequent work of [BP] and [RSo] yielded somewhat different exact sequences for $\Theta^G(\varphi)$ when G acts semifreely on φ (see [Sc3, (1.1)]; also compare [Sc10, Thm. 6.3, p. 277]). As indicated in

[Sc10, Sec. 6] one can use these exact sequences to obtain fairly complete information on the rationalized group $\Theta^G(\varphi)\otimes\mathbb{Q}$. In particular, the following conclusion is an elementary consequence of the exact sequences in [Sc10, Thm. 6.3]:

PROPOSITION 3.1. *If G acts semifreely on φ such that $\dim\varphi \geq 5$ and $\dim\varphi - \dim\varphi^G \geq 3$, then the dimension of the rational vector space $\Theta^G(\varphi)\otimes\mathbb{Q}$ is at most $|G|+3$.* $\square$

This estimate is not really the best possible, but it shows that the ranks of the rationalized groups have uniform bounds depending only on the order of the group.

In [Sc9] the approach for semifree actions is extended to a more general class of actions that are called *ultrasemifree*; precise descriptions of the basic exact sequences appear in [Sc9, (6.2), p. 275], and rational computations with these exact sequences are discussed in [Sc9, Sec. 7]. For these cases one again obtains bounds for the ranks of the groups $\Theta^G(\varphi)$ that only depend upon the order of G.

The machinery of Section 1 allows one to extend everything to more general actions in a straightforward manner:

THEOREM 3.2. *Let φ be a G-representation such that all isotropy subgroups are normal (e.g., suppose G is abelian) and $\dim\varphi^G \geq 2$. Let H be an isotropy subgroup for the action on φ, and let $\varphi_H := \varphi/\varphi^H$. Then there is the following long exact sequence of abelian groups:*

$$\begin{array}{c} \mathbf{I}S^h_G(D(\varphi^H\oplus\mathbb{R})\times S(\varphi_H), \partial(-)) \\ \downarrow \\ \Theta^G(\varphi) \\ \downarrow \\ \Theta^{G/H}(\varphi^H)\oplus F/O_{G,iso}(S(\varphi^H\oplus\mathbb{R})\times\varphi_H, \text{basept.}) \\ \downarrow \\ \mathbf{I}S^h_G(D(\varphi^H)\times S(\varphi_H), \partial(-)) \\ \downarrow \end{array}$$

In particular, it seems likely that the preceding exact sequence should imply that the abelian groups $\Theta^G(\varphi)$ are finitely generated, but we have not verified this.

Although the exact sequence in Theorem 3.2 has not yet been used to do many computations beyond those for semifree and ultrasemifree actions, the following result from [Sc16] indicates the potential usefulness of Theorem 3.2.

PROPOSITION 3.3. *Let p and q be distinct odd primes, and let ω be an orthogonal representation of $\mathbb{Z}_{pq}$ such that the following hold:*

(i) If ω_p and ω_q are the fixed sets of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ respectively, then each has dimension at least 4 and their intersection is the zero subspace.

(ii) The dimension of $\omega_1 := \omega/(\omega_p + \omega_q)$ is at least 4.

Then there are finitely generated subgroups $V_k \subset \Theta^G(\mathbb{R}^k + \omega)$ such that if $v_k = \dim V_k \otimes \mathbb{Q}$ then for all positive integers n the sequence $\{v_k/k^n\}$ is unbounded. □

This contrasts sharply with the results on $\dim \Theta^G(\varphi)$ in the semifree and ultrasemifree cases (refer back to Proposition 3.1).

The groups $\Theta^G(\varphi)$ and the Gap Hypothesis

If the G-manifold φ satisfies the Gap Hypothesis, then the "rather long" equivariant surgery sequence of Dovermann and Rothenberg [DR] provides another means for computing $\Theta^G(\varphi)$. In particular, the methods and results of [DR] yield a canonical bound on the dimensions of the rational vector spaces $\Theta^G(\varphi) \otimes \mathbb{Q}$ in terms of $|G|$; consequently, Proposition 3.3 shows the existence of many new rational classification invariants for G-homotopically equivalent smooth G-manifolds beyond the usual invariants that appear when the standard Gap Hypothesis holds.

4. Borderline cases of the Gap Hypothesis

To simplify the discussion, in this section we shall only consider degree one equivariant normal maps ($f : M \to X$, bundle data) such that f maps the singular set of M to the singular set of X by an equivariant homotopy equivalence. Since the key inductive step in equivariant surgery involves situations of this type, our hypothesis is basically a way of concentrating on a single inductive step. In any case, for such maps the appropriate Gap Hypothesis assumption is that

$$\dim X \geq 2 \cdot \dim(\mathrm{Sing}(X)) + 2.$$

In this section we are interested in examples that lie just outside this Gap Hypothesis range but have been studied effectively by the standard techniques of equivariant surgery. There are two reasons for our interest in such cases. First of all, some are needed in Part III. Second, special cases of these results have implications for equivariant and isovariant homotopy theory that are not presently obtainable by purely homotopy-theoretic methods; needless to say, it would be enlightening to have intrinsically homotopy theoretic proofs for such results.

Following [DoS2, Section III.2] we define the *Gap Hypothesis balance* to be $\Delta(X) := \dim X - 2 \cdot \dim(\mathrm{Sing}(X))$; with this terminology the appropriate version of the Gap Hypothesis is $\Delta(X) \geq 2$. The cases of interest here are $\Delta(X) = 0$ or 1; as one might expect, the similarities with the Gap Hypothesis range decrease as $\Delta(X)$ gets smaller. In particular, examples of M. Rothenberg and S. Weinberger (described in [DoS2, Sec. I.6]) indicate that the situation becomes even more complicated if $\Delta(X)$ is negative. We begin by discussing the situation when the Gap Hypothesis holds.

The cases $\Delta(X) \geq 2$

In these cases the methods of equivariant surgery yield the following conclusion (compare [DR] or [DoS2, Sec. I.5]):

THEOREM 4.1. *Suppose that $f : M \to X$ and appropriate bundle data determine an equivariant degree one normal map of closed smooth 1-connected n-manifolds ($n \geq 5$) such that the associated map of singular sets is an equivariant homotopy equivalence and $\Delta(X) \geq 2$. Then f is normally cobordant* *to an equivariant simple homotopy equivalence if an ordinary Wall surgery obstruction* $\sigma(f) \in L_n^s(\mathbb{Z}[G], w)$ *is trivial.*

As indicated in [DoS2, Sec. I.4], the crucial idea in the proof is that the pair $(M, M - \mathrm{Sing}(M))$ is highly connected, and this allows one to deform all surgical constructions into the complement of the singular set. It follows that the equivariant surgery problem essentially reduces to an ordinary surgery problem on the orbit space of the free part of the action. □

The case $\Delta(X) = 1$

The crucial new insights in this case are due to M. Morimoto. In [Ba] A. Bak defines a quotient of the Wall group $L_{2k+1}^c(\mathbb{Z}[G], w)$ that is denoted by $\mathbf{W}_{2k+1}^c(\mathbb{Z}[G], \Gamma G(X); w)$; the set $\Gamma G(X)$ is a set of order two elements $g \in G$ such that $\dim X^g = k$ (*cf.* [Mto1, p. 467]) and w denotes the first Stiefel-Whitney class. Following Morimoto, we shall call the group $\mathbf{W}_*^c(-)$ the *Bak group* associated to the given data. Frequently the Bak group is isomorphic to the corresponding Wall group. In particular, this is true if G has odd order or k is even and the orientation homomorphism is trivial. However, the example in [Mto1, Corollary C, page 468] shows that the projection from the Wall group to the Bak group has a nontrivial kernel in some cases, and the results of [BaMo] yield additional examples.

THEOREM 4.2. *Suppose that* $f : M \to X$ *and appropriate bundle data determine an equivariant degree one normal map of closed smooth* 1*-connected* $(2k+1)$*-manifolds* $(k \geq 2)$ *such that the associated map of singular sets is an equivariant homotopy equivalence and* $\Delta(X) = 1$ *(hence the singular set is* k*-dimensional). Then* f *is normally cobordant to an equivariant simple homotopy equivalence if the image of an ordinary Wall surgery obstruction* $\sigma(f) \in L_n^s(\mathbb{Z}[G], w)$ *in the Bak group* $\mathbf{W}_{2k+1}^s(\mathbb{Z}[G], \Gamma G(X); w)$ *is trivial.*

An analogous result holds for equivariant homotopy equivalences if one replaces L^s *by* L^h *and* $\mathbf{W}^s$ *by* $\mathbf{W}^h$.

Under the hypotheses of Theorem 4.2 the pair $(M, M - \mathrm{Sing}(M))$ is not quite so highly connected, and one can deform some but not all surgical constructions back to the free part of the action on M. The methods employed in the proof of Theorem 4.1 still yield a surgery obstruction in the appropriate Wall group, but this obstruction is not necessarily well-defined. However, if one passes to the Bak group, then one does obtain a well-defined obstruction. □

Results of Dovermann [Do2] relate the preceding to questions involving isovariance. Namely, if f is isovariant on the singular set then the given conditions allow one to surger f into an isovariant map that is an equivariant homotopy equivalence on the singular set. In this setting the Wall group element represents the obstruction to surgering f into an isovariant homotopy equivalence. Thus the kernels of the maps from Wall groups to Bak groups carry obstructions for transforming certain equivariant equivalences into isovariant equivalences.

The case $\Delta(X) = 0$

In this case results are only known for the case $G \approx \mathbb{Z}_2$, and the main results are due to Dovermann [Do1] (see also [DoS1] and [Sc15, Thm. 2.5]).

THEOREM 4.3. *Let $f : X \to Y$ come from a suitably defined degree one $\mathbb{Z}_2$-normal map of smooth 1-connected $2k$-dimensional $\mathbb{Z}_2$-manifolds, where f induces a homotopy equivalence of fixed point sets and the latter are k-dimensional. Then f is normally cobordant to a $\mathbb{Z}_2$-homotopy equivalence, relative to the fixed point sets, if and only if the following hold:*

(i) If k is even, the $\mathbb{Z}_2$-signatures of X and Y are equal.

(ii) If k is odd, the ordinary Kervaire invariant of f is trivial and a mod 2 rank invariant of the surgery kernel of f is also trivial.

Unlike the preceding cases, one must now consider homotopy classes in $\pi_q(M)$ that cannot be deformed into the complement of singular set; the obstruction to doing this is measured by a homological intersection number. One needs a modified concept of Hermitian form (called *quasi-quadratic* in [Do1]); the invariants of such forms turn out to be the algebraic invariants described in the statement of the theorem. □

Theorem 4.3 has some curious homotopy theoretic implications that are not yet derivable from other techniques. As indicated earlier, it would be enlightening to have intrinsically homotopy theoretic arguments, both for the sake of making everything more self contained and also in the interests of proving further results along the same lines.

The first implication involves equivariant function spaces. Following [BeS], if the finite group G acts freely and orthogonally on the unit sphere $S(V)$ in the Euclidean space V, let $F_G(V)$ be the space of G-equivariant self maps of $S(V)$. Of course, if H is a subgroup of G there is a natural forgetful map ρ from $F_G(V)$ to $F_H(V)$. Also, there is a stabilization map from $F_G(V)$ to a space $F_G := \lim F_G(V \oplus W)$ where W runs through all isomorphism classes of free G-representations. The main results of [BeS] state that F_G is homotopy equivalent to the free infinite loop space $\Omega^\infty S^\infty(K(G,1)_+)$ and the forgetful map from F_G to F is induced by the transfer map in stable homotopy (*cf.* [BG]). In particular, by the Kahn-Priddy Theorem [KP] the map $F_G \to F$ induces a split surjection in positive dimensional homotopy groups if $G \approx \mathbb{Z}_2$. Dovermann's results yield the following unstable analog of the Kahn-Priddy theorem.

PROPOSITION 4.4. *In the terminology above, suppose that $G \approx \mathbb{Z}_2$ and $V = \mathbb{R}^n$ with the antipodal involution. If $j : F_G(V) \to F_G \to F$ is the composite of stabilization with the forgetful map, then the image of $j_* : \pi_n(F_G(V)) \to \pi_n(F) \cong \pi_n$ contains all positive-dimensional elements whose Hopf invariants are even and whose Kervaire invariants are zero.*

In particular, it follows that the image of j_* has index at most two in every dimension and the index is one except for a very sparse set of dimensions (recall that an element can have an odd Hopf invariant only in dimensions 1, 3, and 7, and an element can have a nonzero Kervaire invariant only in dimensions of the form $2^m - 2$).

Proposition 4.4 follows by combining the results of [Sc4] on knot invariants with the results of [Sc8] on realizing exotic spheres as fixed point sets of involutions on homotopy spheres (in this connection also see [Lö]); the role of Dovermann's work is that the crucial results from [Sc8] and [Lö] depend upon [Do1].

Questions: Can one eliminate the condition on Kervaire invariants in Proposition 4.4? Basic results in homotopy theory show that the condition on Hopf invariants CANNOT be eliminated. Also, can one prove a result similar to 4.4 for the image of $\pi_n(F_G(V)) \to \pi_n(F)$ when $V = \mathbb{R}^{n-1}$? An answer to this question would have implications for realizing exotic n-spheres as fixed point sets of smooth involutions on homotopy $(2n-1)$-spheres.

Here is another consequence of Theorem 4.3 to a question of interest in nonequivariant homotopy theory (compare [Str], [Sc15]):

PROPOSITION 4.5. *Let M and N be closed, homotopically equivalent 1-connected n-manifolds where $n \geq 3$. Then the deleted symmetric squares $\{M \times M - \Delta_M\}/\mathbb{Z}_2$ and $\{N \times N - \Delta_N\}/\mathbb{Z}_2$ are homotopically equivalent.* □

Question: Can one eliminate the simple connectivity hypothesis in this result? Straus proves an analogous result for deleted reduced cyclic p-th powers where p is an odd prime, and the latter result has no hypothesis on the fundamental group [Str]. Results of P. Löffler and R. J. Milgram [LöMi] suggest that the answer to the question is yes. An obvious suggestion for approaching this is to prove a version of Dovermann's result for involutions on nonsimply connected manifolds with $\Delta(X) = 0$.

5. Isovariance and nonsmoothable group actions

Many questions arising in Parts I and II are also meaningful and interesting for group actions that are not smooth. In this section we shall describe some results along these lines.

Beginning in the mid nineteen eighties Cappell and Weinberger developed a variety of surgery-theoretic techniques for constructing exotic topological or PL group actions with a given isovariant homotopy type. Some of their results for circle actions are summarized in [CW1]. The following *replacement theorem* is a simple but basic example of their results for finite group actions:

THEOREM 5.1. *Let $G = \mathbb{Z}_p$ where p is an odd prime, and let W^n be a closed simply connected manifold with a locally linear topological or PL G-action Φ such that $M + W^G$ is also simply connected and* $\dim M \geq 5$. *If $h : M' \to M$ is a homotopy equivalence, then there is another locally linear topological or PL (resp.) G-action Φ' on W such that*

(*i*) *the two actions are equivalent on the complements of their fixed point sets,*

(*ii*) *the fixed point set of Φ' is M',*

(*iii*) *the G-manifolds (W, Φ) and (W', Φ') are G-isovariantly homotopy equivalent (but the equivalence is not necessarily isovariantly homotopic to a transverse linear map in the sense of Browder and Quinn).* □

Cappell and Weinberger also obtain several extensions and refinements of Theorem 5.1; details of this work appear in [CW2], and further results on obstructions to replacement appear in [DW]. In [Daw] Dawson uses his version of the results in Section 1 to study similar replacement questions for smooth actions when the codimension of the fixed point set is small. Dawson has also obtained results on *replacement of tangential representations* up to homotopy, where one replaces the G-representation Ω at the tangent space of a fixed point by an isovariantly homotopy equivalent representation Ω' and attempts to find an action G-homotopy equivalent to the orignal one with the same fixed point set and the modified tangential data.

In a forthcoming book [Wb1] Weinberger develops powerful and fairly general machinery for classifying certain topological actions up to isovariant homotopy equivalence. Specifically, his results apply to group actions for which the fixed point sets of subgroups define a weak analog of a Thom-Mather stratification (*e.g.*, a CS stratification in the sense of Siebenmann [Si2] or a homotopy stratification in the sense of Quinn [Q2]). Since this work involves several deep concepts that are not needed in the smooth category (for example, results on ends of maps [Q1]), we shall not attempt to explain the main ideas here. This work has already produced some further developments and applications due to Weinberger and M. Yan [Wb2, WY, Y], including counterexamples to equivariant analogs of the Borel rigidity conjecture for aspherical manifolds [Wb2].

In view of [Wb1] it would be useful to have an extension of the results of Part I to isovariant maps of G-manifolds with weak stratifications of an appropriate type. Perhaps the most obvious complication is that fixed point sets need not have closed tubular neighborhoods (*e.g.*, see the examples near the beginning of [Q1]). An extension of the results in [DuS] to nonsmoothable actions will probably require diagrams involving Quinn's notion of *homotopy collar* [FQ, pp. 214–215] for the sets $M_{(H)}$ associated to a G-manifold M (homotopy collars are called *homotopy completions* in [Q1, Sec. 7.8]) and *homotopy links of* ∞ *in the one point compactification* in [Q2]).

Finally, we note that the Browder-Straus proof of Theorem 2.1 goes through for certain classes of nonsmoothable actions; for example, the proof applies to semifree PL actions on manifolds such that the fixed point sets are also manifolds. Of course, it would be enlightening to have an alternate proof as in the smooth category, with little or no input from surgery theory.

PART III

FINITE GROUP ACTIONS ON HOMOLOGY 3-SPHERES

Background references

In contrast to Parts I and II, the emphasis in this final part will be on specific geometric problems involving transformation groups on 3-manifolds. The following references contain most of the background material we shall need:

(1) Article by M. Davis and J. Morgan in Bass–Morgan (eds.), *The Smith Conjecture* (:= [DaMo]).

(2) Article by Edmonds in Contemp. Math. Vol. 36 (1985) (:= [Ed]).

(3) Article by Raymond in Transactions Amer. Math. Soc. Vol. 131 (1968) (:= [Ra]).

(4) Notes from lectures of Thurston on the geometry and topology of 3-manifolds (:= [Th1]; the main points are summarized in [Th2]).

During the past fifteen years the concepts of *orbifold* and *orbifold fundamental group* have become fairly standard in 3-dimensional topology. Since both arose from considerations involving transformation groups, we shall use these concepts as needed. The basic definitions and examples can be found in Thurston's notes [Th1, Ch. 13, especially p. 13.5] or the first few sections of [DaMo].

1. A survey of known results

For several decades topologists have known that dimensions 3 and 4 form a transitional range from the geometric rigidity of line and surface topology to the freedom of movement one has in dimensions ≥ 5. References on this topic are numerous and include an old article of Siebenmann [Si1], work of A. Casson and C. Gordon [CG], and books of M. Freedman and F. Quinn [FQ], S. Donaldson and P. Kronheimer [DK], and S. Akbulut and J. McCarthy [AM]. Our interest in this article lies with transitional properties of group actions on spheres and manifolds closely resembling spheres (*e.g.*, manifolds homeomorphic but not necessarily diffeomorphic to the standard sphere). As indicated by the title above, we shall concentrate on the 3-dimensional case; some references for the 4-dimensional case include [Ed] for work done through the early nineteen eighties, [BKS] and [DM] for results about the fixed point sets of smooth actions on S^4, [KwS2–4] for results on topological actions on S^4 with isolated singular points, and [KwS1] and [KwS5] for results concerning topological circle actions on S^4 and other 4-manifolds.

In 1- and 2-dimensional topology all group actions on manifolds are equivalent to smooth actions that preserve nontrivial geometric structures (*cf.*, [Ed, p. 341]). For example, all compact Lie group actions on S^1 and S^2 are equivalent to orthogonal actions. On the other hand, it is well known that a smooth action of a finite cyclic group on S^4 need not be orthogonal because the fixed point set can be a nontrivially knotted 2-sphere (see Giffen [Gi] and Gordon [Go1]). The known results for dimension 3 lie somewhere between these two adjacent cases.

FACT 1.1. *There is a continuum of pairwise inequivalent topological $\mathbb{Z}_k$ actions on S^3 for every prime k.*

As noted in [Ed], this is due to Bing [Bi] and Alford [AL]. □

FACT 1.2. *All smooth actions of compact Lie groups on S^3 with nonempty fixed point sets are orthogonal. Modulo some possibly exceptional cases, all actions of compact Lie groups on S^3 with positive-dimensional singular sets are orthogonal.*

If $G = S^1$ this result is contained in [Ra], and for larger Lie groups the result is a straightforward exercise. When G is finite cyclic and the fixed point set is a circle this was conjectured by P. A. Smith and solved in the late nineteen seventies by the combined efforts of several mathematicians (see the book containing [DaMo]). Results for other finite groups appear in several different places, including papers of M. Davis and J. Morgan [DaMo], M. Feighn [Fn], and S. Kwasik and the author [KwS6]. About ten years ago W. Thurston announced that the second assertion in 1.2 holds without exception [Th3]; although workers in the area have few doubts about the correctness of this statement, it is not clear when a complete written proof will be available.□

FACT 1.3. *There are many smooth group actions on integral homology 3-spheres that are analogous to important examples of smooth actions on higher-dimensional spheres.*

A similar—and related—phenomenon occurs in the theory of isolated singularities of complex hypersurfaces (*cf.* Mumford [Mum]): Given a complex polynomial $f(\mathbf{z})$ in $n+1$ variables such that the origin is an isolated singularity, let Σ_f be the intersection of the zero set $\{\mathbf{z} \in \mathbb{C}^{n+1} | f(\mathbf{z}) = 0\}$ with a sphere of sufficiently small radius; it follows that Σ_f is a closed smooth $(2n-1)$-manifold. Mumford's result deals with the case $n = 2$ and states that a 3-dimensional manifold of the form Σ_f is diffeomorphic to S^3 if it is simply connected; on the other hand, there are many examples where Σ_f is a nonsimply connected integral homology sphere. The analogs of the latter in higher dimensions are homeomorphic but not necessarily diffeomorphic to the standard $(2n-1)$-sphere (see Milnor's book [MLN3] for further information on this topic).

EXAMPLES FOR FACT 1.3. (1) Perhaps the most basic of these are the *pseudofree* smooth circle actions on Seifert homology 3-spheres. These actions are free on the complement of a finite set of pairwise disjoint circles, and the isotropy subgroups for points on these circles are pairwise relatively prime integers $d_i > 1$. The family of such actions includes all fixed point free orthogonal circle actions on S^3; in fact, the orthogonal actions are precisely those for which there are at most two exceptional circles, and in all remaining cases the underlying homology spheres are not simply connected. In contrast to the orthogonal case, the number of exceptional orbits for an arbitrary action on a Seifert homology 3-sphere can be an arbitrary positive integer. There are several excellent descriptions of these manifolds in the literature, including Raymond's article [Ra], the book by P. Orlik [Or], and lecture notes by M. Jankins and W. Neumann [JN]. In higher dimensions one also has orthogonal pseudofree circle actions on S^{2n-1}: If $(d_1, \cdots, d_n)$ is a sequence of pairwise relatively prime positive integers, then the unit sphere in $\mathbb{C}^n$ for the linear action

$$t \cdot (z_1, \cdots, z_n) := (t^{d_1} z_1, \cdots, t^{d_n} z_n)$$

is free on the complement of m circles, where m is the number of integers d_i that are greater than one. Of course, it follows that the number of exceptional circles for an orthogonal pseudofree circle action on S^{2n-1} is at most n, and it is natural to ask if it is possible to construct pseudofree smooth circle actions on S^{2n-1} with larger numbers of exceptional circles for larger values of n. In [MnY1–2] D. Montgomery and C.-T. Yang succeeded in producing such examples when $n = 3$; a survey of this and analogous work in dimensions $(2n+1) \geq 9$ appears in [DPS, Sec. 2] (also see [Pet1]). To complete the picture, we note that the case $n = 2$ (*i.e.*, the 5-dimensional case) reflects the basic difficulties of 4-dimensional topology. One can use M. Freedman's work on 4-dimensional topological surgery [FQ] to construct topological pseudofree circle actions on S^5 that are locally linear, and applications of gauge theory to smooth pseudofree circle actions on S^5 have also been considered (*cf.* [FS]; see also *Math. Reviews* 88e:57032).

(2) Let p be an odd prime, and let D_{2p} be the dihedral group of order $2p$. If we are given a fixed point free, linear action of D_{2p} on $S(V) \approx S^{2n-1}$, then the fixed point sets of the order two subgroups are all $(n-1)$-spheres, and if H and K are two distinct subgroups of order two then $S(V)^H$ and $S(V)^K$ have linking number ± 1. If we are given an arbitrary fixed point free D_{2p} action on $M \approx S^{2n-1}$ such that the fixed point set of every (equivalently, of some) order two subgroup is an $(n-1)$-sphere, then Smith theory implies that the linking numbers of M^H and M^K are congruent to ± 1 mod p; results of J. Davis and T. tom Dieck [DtD, tD3] show that one can realize exotic

linking numbers by smooth D_{2p} actions on the $(2n-1)$-sphere if $n \geq 3$. In contrast, special cases of Fact 1.2 imply that no such examples can exist if $n = 2$ and $p > 5$ (*e.g.*, see [DaMo]). Despite this, C. Livingston has shown that fixed point free, smooth D_{2p}-actions with exotic linking numbers exist on integral homology 3-spheres [Liv].

(3) If A_5 denotes the alternating group on 5 letters, then A_5 can be viewed as the group of isometries of a regular dodecahedron or icosahedron, and consequently there are natural realizations of A_5 as a subgroup of SO_3. The homogeneous space SO_3/A_5 is the well-known *Poincaré homology 3-sphere* that we shall denote by $\Sigma(2,3,5)$; summaries of the properties of this manifold appear in [Bre3, Sec. I.8] and [KiSc]. As noted in [Bre3, pp. 55–56], the action of A_5 on $\Sigma(2,3,5)$ obtained by restricting the transitive action of SO_3 has exactly one fixed point. Although smooth actions with one fixed point cannot exist on a smooth homotopy 3-sphere [BKS], the methods of geometric topology have produced numerous examples of smooth actions on higher dimensional spheres with one fixed point during the past two decades; the first examples were due to E. V. Stein [Stn], with additional families of examples due to T. Petrie [Pet1–2] appearing shortly afterwards. We shall not attempt to summarize subsequent work here, but many further references appear in [BKS], applications to algebraic group actions are discussed in [DMP], and an article by E. Laitinen, M. Morimoto, and K. Pawałowski [LMP] provides the most recent information available at this time.

In the remaining sections of Part III we shall concentrate on the following question:

(‡) *What is the role of the standard one fixed point action on* $\Sigma(2,3,5)$ *in the family of all smooth one fixed point actions on homology 3-spheres?*

Standard considerations involving P. A. Smith cohomological fixed point theory, the local linearity of smooth actions near fixed points, and the subgroups of SO_3 show that A_5 is the only group that can act on a closed integral homology 3-sphere with exactly one fixed point.

One motivation for studying such actions is that symmetry considerations have led to interesting classes of 3-manifolds such as Seifert manifolds. On the other hand, smooth group actions with one fixed point have been studied extensively over the past two decades, both as test cases for the sorts of exotic smooth group actions that can exist on spheres and in connection with questions from algebraic transformation groups. In particular, recent work has shown that smooth one fixed point actions on nonsimply connected homology 3-spheres are rather exceptional low-dimensional examples. Standard geometrization results (*cf.* [Ed]) imply that finite group actions with a single fixed point do not exist on (homology) spheres of dimension 1 or 2, and more recent results of [BKS], [DM], and [Mto2] show the nonexistence of smooth actions with exactly one fixed point on homology 4- and 5-spheres. In contrast, such smooth actions exist on genuine spheres in all dimensions ≥ 6 (*e.g.*, see [Stn], [Pet2], [Mto2], [BaMo], and related examples of [BKS]).

A result of G. Bredon [Bre1] states that $\Sigma(2,3,5)$ is the only integral homology sphere that admits a transitive action of a compact Lie group that is not equivalent to an orthogonal action on a standard sphere. Since the one fixed point action of A_5 on $\Sigma(2,3,5)$ is the restriction of this exceptional transitive action, the evidence in this and the preceding paragraph may suggest that all one fixed point actions on homology 3-spheres are closely related to the standard examples in some fashion (*e.g.*, perhaps there is an equivariant degree one map into $\Sigma(2,3,5)$). However, our main results show the existence of many smooth one fixed point actions on irreducible homology 3-spheres. Some of these actions are clearly related to the standard actions on $\Sigma(2,3,5)$, but others are quite different.

2. Equivariant surgery in three dimensions

Our objective is to construct exotic examples of smooth one fixed point actions on homology 3-spheres by means of equivariant surgery and other techniques. Of course, it is well known that many basic results of surgery theory fail in dimension three; the purpose of this section is to summarize some aspects of surgery theory that are both valid and useful for 3-manifolds.

For many years geometric topologists have known that surgery theory applies to dimension three if one is willing to settle for homology equivalences rather than genuine homotopy equivalences; informal discussions of this appear in the writeup by F. Quinn on page 225 of the book containing [Brw2] and also in [FQ, p. 200, lines −3 to −1]. Here is a more formal statement that "homology surgery works for 3-manifolds."

FOLK THEOREM 2.1. *Suppose that* $f : (M, \partial M) \to (X, \partial X)$ *and appropriate bundle data determine a degree one normal map of compact 3-manifolds with boundary such that* $\partial f : \partial M \to \partial X$ *induces a* $\mathbb{Z}[\pi_1(X)]$*-homology equivalence (with possibly twisted coefficients). Then* f *is normally cobordant rel boundaries to a simple* $\mathbb{Z}[\pi_1(X)]$*-homology equivalence if the ordinary Wall surgery obstruction* $\sigma(f) \in L_3^s(\mathbb{Z}[\pi_1(X)], w)$ *is trivial.*

The proof of this follows directly from the methods of Wall's book for (2n+1)-manifolds with $n \geq 2$ subject to one complication: If one removes an embedded n-sphere from a $(2n+1)$-manifold, the fundamental group does not change if $n \geq 2$ but it usually changes drastically if $n = 1$. This means that one loses control of the fundamental group of the source manifold for the normal map, but the underlying homological arguments remain valid if we we work with twisted coefficients in the group ring $\mathbb{Z}[\pi_1(X)]$. □

In [BaMo] Bak and Morimoto formulate a version of this for equivariant surgery on 3-manifolds with orientation-preserving actions.

THEOREM 2.2. *Let* M *and* X *be closed smooth* G*-manifolds with orientation-preserving actions of a finite group* G*, let* $f : M \to X$ *and suitable bundle data define a degree one equivariant surgery problem, where* X *is simply connected and* f *maps* $\mathrm{Sing}(M)$ *to* $\mathrm{Sing}(X)$ *by an equivariant homotopy equivalence. Furthermore, assume that the projective class group* $\widetilde{K}_0(\mathbb{Z}[G])$ *is zero. Then* f *is equivariantly normally bordant to a* G*-homotopy equivalence if an obstruction in the Bak group*

$$\mathbf{W}_3^h(\mathbb{Z}[G], \Gamma G(X); 1)$$

is trivial. □

As noted in Section 2.4, the Bak group $\mathbf{W}_3^h(\mathbb{Z}[G], \Gamma G(X); 1)$ is defined in [Ba] as a quotient of the Wall group $L_3^h(\mathbb{Z}[G], 1)$.

We shall need a slight extension of the preceding result:

COMPLEMENT 2.3. *The preceding result remains valid if one merely assumes that X is obtained from a compact smooth 3-manifold by attaching G-free equivariant cells on the free part of X.* □

We shall also need some computational results for Bak groups and certain other algebraic K-theoretic groups. These are all established in [BaMo].

THEOREM 2.4. *Let G be the alternating group A_5. Then the projective class group of $\mathbb{Z}[G]$, the Whitehead group of G, and the Bak group $\mathbf{W}_3^h(\mathbb{Z}[G], \Gamma G(X); 1)$ are all trivial.* □

In contrast to the preceding result, the Wall group $L_3^h(\mathbb{Z}[A_5], 1)$ is nontrivial, for $L_3^h(\mathbb{Z}[\mathbb{Z}_2], 1) \approx \mathbb{Z}_2$ implies that $L_3^h(\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2], 1) \supset \mathbb{Z}_2 \times \mathbb{Z}_2$, and by transfer considerations it follows that $L_3^h(\mathbb{Z}[A_5], 1)$ also contains a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

3. Construction of exotic examples

Elementary considerations show that the one fixed point action on $\Sigma(2,3,5)$ is not the only smooth action on a homology 3-sphere with exactly one fixed point. For example, one can construct many examples by taking an equivariant connected sum of $60 = |A_5|$ copies of some homotopy sphere P^3 over the free part of the action on $\Sigma(2,3,5)$; this yields an infinite family of pairwise inequivalent one fixed point A_5-actions on homology 3-spheres. More generally, if H is an isotropy subgroup of the action on $\Sigma(2,3,5)$ and P^3 has a smooth action of H, then one can often form a *stratumwise equivariant connected sum* of $\Sigma(2,3,5)$ with $|G/H|$ copies of P along the fixed point sets of the conjugates of H; constructions of this type are used extensively by Meeks and Yau in [MeY1, Sec. 9]. In particular, one can take $H = A_4$ and obtain a one fixed point action on a connected sum of 6 copies of $\Sigma(2,3,5)$; we mention this because it appears to be the simplest example of an integral homology 3-sphere that supports a smooth one fixed point action and has a vanishing Rochlin invariant.

The preceding examples are all obtainable from $\Sigma(2,3,5)$ by familiar sorts of constructions. In fact, there have been some informal regularity conjectures that all one smooth fixed point actions on homology 3-spheres are somehow modeled after $\Sigma(2,3,5)$. Perhaps the weakest of these conjectures is that the singular set of such an action is always equivalent to $\mathrm{Sing}(\Sigma(2,3,5))$. The main result of this section provides a negative answer to this particular question and describes all possible fixed point sets.

THEOREM 3.1. *There are exactly four equivariant homeomorphism classes of singular sets for smooth A_5-actions on $\mathbb{Z}$-homology 3-spheres with exactly one fixed point.*

The possibilities for the singular set may be described as follows: Suppose we are given a smooth action of A_5 on the homology 3-sphere Σ^3 with exactly one fixed point. For each subgroup C of order 2 the fixed set of C is the union of two semicircles with two endpoints in common; one of the endpoints is the fixed point of the A_5-action, and the other is fixed under the unique subgroup of A_5 that contains C and is isomorphic to the alternating group on four letters. Each semicircle also contains a point that is fixed under a subgroup of order 6 containing C and a point that is fixed under a subgroup of order 10 containing C. The union of all fixed sets of order 2 subgroups consists of 30 semicircles, and A_5 acts transitively on this set of semicircles. On the other hand, a direct analysis shows that there are exactly four ways of constructing an A_5-orbit of data $(\Gamma_C; x_6, H_6, x_{10}, H_{10})$, where C is a subgroup of order 2, Γ_C is homeomorphic to [0,1], the points x_6 and x_{10} belong to $\{\frac{1}{3}, \frac{2}{3}\}$, and H_6 and H_{10} are subgroups of order 6 and 10 respectively containing C. Furthermore, Smith theory implies that for each class of semicircles there is a unique 1-complex with cell-preserving A_5-action that is a potential singular set for a smooth action on a homology 3-sphere with one fixed point.

Sketch of proof of Theorem 3.1. Each of the 1-complexes K in the preceding paragraph can be realized as the singular set of a smooth action on *some* closed oriented 3-manifold by an elementary "rewiring" construction on $\Sigma(2,3,5)$ with its standard one fixed point action. Let A_K be the manifold so obtained; it follows from the construction that $H_1(A_K;\mathbb{Q}) \neq 0$. One can then use machinery developed by R. Oliver [OL] to add equivariant cells along the free part of A_K and obtain an equivariant CW complex B_K with the same structure on and near the singular set and such that B_K is homotopy equivalent to S^3; if we split A_K equivariantly as $D \cup_\partial E$ where D is a linear disk about a fixed point, the equivariant cells can all be added over E and one obtains a corresponding splitting $D \cup_\partial E'$ where E' is contractible. Therefore the inclusion of A_K in B_K can be viewed as a map of triads, and from this it follows that the inclusion is an isomorphism on $H_3(-;\mathbb{Z})$ and can be viewed as a map of degree one. In order to make this into an equivariant surgery problem, it is necessary to introduce some equivariant bundle data; the details of the rewiring construction imply that the equivariant tangent bundle is stably isomorphic to a product bundle on the complement of a finite invariant subset F, and it follows that equivariant bundle data can be given by crossing the map $A_K - F \to B_K$ with the identity on Ω for a suitable A_5-representation Ω. This suffices for surgery-theoretic purposes because the latter involve maps from positive-codimensional manifolds into A_K and such maps can always be deformed to avoid a finite subset (similar considerations arise in [DR], where bundle data with deficiencies are discussed in greater detail). The results of Bak and Morimoto [BaMo] (cited in Section 2) now show that

(*i*) one can do equivariant surgery away from the singular set of A_K to convert the map $A_K \to B_K$ into a $\mathbb{Z}$-homology equivalence if an obstruction in some quotient group of the Wall group $L_3^h(A_5;1)$ – specifically, the associated Bak group $W_3^h(A_5, \Gamma G(X);1)$ – is zero,

(*ii*) the Bak group in (*i*) is equal to zero.

Therefore one can modify A_K by equivariant surgery away from the singular set to obtain a homology sphere with a smooth one fixed point action. □

Fixed point free actions on homology 3-spheres

The methods of this section also yield results about fixed point free actions of finite groups on homology 3-spheres. Some of this is work in progress, but the results for $G = A_5$ can be stated fairly simply. There is a unique linear action of this type; namely, let V be the orthogonal complement of the diagonal in $\mathbb{R}^5$ where A_5 acts on the latter by permuting the coordinates, and take the induced action on the unit sphere $S(V)$. It is immediately clear that one can find examples with exotic orbit structures; in particular, this can be done by taking an equivariant connected sum of two one fixed point actions on homology 3-spheres at the fixed points. The methods of this section show that *the singular set of a fixed point free A_5-action on an integral homology 3-sphere can be an arbitrary 1-dimensional A_5-complex that satisfies the necessary conditions imposed by Smith theory*; the list of all such possibilities is fairly short, but it does contain more than the singular set of the linear action and the singular sets obtained from the equivariant connected sums described above. One reason for interest in such actions involves the linearity question for smooth actions of finite groups on S^3 (*cf.* [DaMo], [Fn], [KwS6]); fixed point free A_5-actions represent one basic type that is included in Thurston's announcement [Th3] but has not yet been verified elsewhere.

4. Actions on hyperbolic homology spheres

One obvious drawback of Theorem 3.1 is that the argument does not yield explicit examples of actions with exotic singular sets. In particular, it is natural to ask if such examples can be found on homology 3-spheres that are *irreducible* and geometric in the sense of Thurston [Th2]. More specifically, Thurston's hyperbolization theorem [Th4] implies that one can often find hyperbolic 3-manifolds with certain topological or geometric properties by applying suitable conversion procedures to general 3-manifolds with such properties, and in this connection one would like to know if smooth one fixed point actions can be found on hyperbolic homology 3-spheres. According to the main result of this section (Theorem 4.3), such examples can be found. The results of this section generate a variety of questions; some examples are presented after the proof of Theorem 4.3.

One can interpret Theorem 3.1 as a negative answer to questions about the existence of a single basic model for one fixed point actions on homology 3-spheres. However, the following result shows that the one fixed point actions on irreducible homology 3-spheres form a family of models for all such actions.

THEOREM 4.1. *If* Σ^3 *is a closed integral homology 3-sphere with a smooth one fixed point action of* A_5, *then* Σ^3 *is equivariantly diffeomorphic to an iterated equivariant connected sum along strata of the form* $\Sigma_0 \# |G/H_1|\Sigma_1 \cdots \# |G/H_r|\Sigma_r$, *where* Σ_i *is an irreducible homology 3-sphere with a smooth action of* H_i *if* $i > 1$ *and* Σ_0 *is an irreducible homology 3-sphere with a smooth one fixed point action of* A_5.

In particular, it follows that every one fixed point action on a homology 3-sphere has an *irreducible nucleus* given by a smooth one fixed point action on some connected summand of Σ.

The proof of Theorem 4.11 follows from the standard equivariant analogs of the Papakyriakopoulos Sphere Theorem [MeY2, JR] and an analysis of the ways in which the invariant separating spheres in Σ can meet the singular set of the action; this set turns out to be a 1-dimensional finite cell complex with a cell-preserving group action (that can be equivariantly subdivided to yield an equivariant regular simplicial action in the sense of, say, [Bre3, Ch. III]). □

In view of Theorem 4.1 and Thurston's Geometrization Conjecture [Th1–2], it is natural to ask next about one fixed point actions on irreducible geometric homology spheres. These manifolds fall into three distinct classes; namely, *Seifert fibered*, *non-simple Haken* (in other words, containing an incompressible torus), and *hyperbolic*. Since the Poincaré homology 3-sphere $\Sigma(2,3,5)$ is an example of a Seifert fibered 3-manifold (in fact, it is a Brieskorn variety), the most immediate question is whether other Seifert fibered homology 3-spheres support smooth one fixed point actions. This question has a simple negative answer:

PROPOSITION 4.2. *Let Σ^3 be a Seifert fibered homology 3-sphere. Then Σ^3 admits a smooth action of a finite group with one fixed point if and only if Σ^3 is diffeomorphic to the Poincaré homology 3-sphere $\Sigma(2,3,5)$.*

This follows from considerations involving the fundamental group of Σ and the orbifold fundamental group associated to the group action. □

In contrast to the preceding result, situation is completely different for hyperbolic and Haken homology 3-spheres:

THEOREM 4.3. *(i) There exist infinitely many irreducible non-simple Haken homology 3-spheres that support smooth actions of A_5 with exactly one fixed point.*

(ii) There exist infinitely many irreducible hyperbolic homology 3-spheres that support smooth actions of A_5 with exactly one fixed point.

Sketch of proof of Theorem 4.3. The constructions of examples depend heavily on Theorem 3.1, Theorem 4.1, and the nonexistence of smooth one fixed point actions on homotopy 3-spheres [BKS]. When combined, these imply that *every possible singular set is realized by a smooth one fixed point A_5-action on a nonsimply connected, irreducible homology 3-sphere.*

To prove statement (i), one first takes a simple closed curve in the free part of $\Sigma(2,3,5)$ that represents a nonzero element of $\pi_1(\Sigma(2,3,5))$, then forms a connected sum with a knot in some small coordinate 3-disk, and afterwards deforms it to be disjoint from all its translates under the action of A_5. Next, one takes a closed invariant tubular neighborhood U of these $|A_5|$ pairwise disjoint curves and replaces the interior of U with the $|A_5|$ copies of the interior of some nontrivial knot complement. One can do this such that the manifold in question becomes an irreducible homology sphere and the components of the boundary of U become incompressible tori.

The proof of statement (ii) is somewhat more delicate. As in the preceding discussion, one removes a suitably chosen union $\operatorname{Int} U$ of invariant open solid tori from the free part of the action to obtain a bounded Haken manifold V with a smooth one fixed point action and $\partial V = A_5 \times T^2$. Next one uses the splitting theorems of W. Jaco and J. H. Rubinstein [JR] to construct an equivariant splitting of V along incompressible tori into Seifert fibered and hyperbolic pieces; the hyperbolicity assertion uses Thurston's recognition principle for hyperbolic 3-manifolds ([Th4]; also see C. McMullen's article [McM] and the references cited there). The equivariant geometrization results of W. Meeks and G. P. Scott [MS] then imply that the fixed point of the induced action on V lies in a hyperbolic piece, say V_0, and an argument involving the dual graph of the splitting implies that one can attach solid tori to the boundary components of V_0 (nonequivariantly) to obtain a homology 3-sphere. Thurston's results on Dehn fillings [Th1–2] then imply that infinitely many such attachments yield a hyperbolic manifold; furthermore, elementary considerations imply that these Dehn fillings will yield integral homology spheres, and a more detailed analysis also shows that infinitely many of these constructions can be done equivariantly. □

The preceding results generate a variety of questions. Here are two examples:

(1) If Σ is an irreducible homology 3-sphere with a smooth A_5-action with one fixed point, is the Rochlin invariant always equal to 1? This is true for all examples checked thus far. Similarly, one can ask about the Casson invariant [AM] or other invariants from topological quantum field theory.

(2) In [Th3] Thurston announced results implying that the one fixed point actions on the hyperbolic homology 3-spheres of Theorem 4.3 are hyperbolic structure preserving. There are many examples of one fixed point actions on spheres in dimensions ≥ 6. Can one use equivariant surgery to convert such actions into one fixed point actions on hyperbolic homology spheres that preserve a hyperbolic structure? Results of M. Davis and T. Januszkiewicz [DJ] provide a means for converting manifolds and orbifolds to objects that are hyperbolic in the sense of M. Gromov [Gr] (also see [Bow], [GH]), and these suggest that one can find at least some hyperbolic one fixed point actions on higher dimensional integral homology spheres. More generally, it would be interesting to know which of the many exotic smooth finite group actions on spheres are simply connected analogs of hyperbolic actions on hyperbolic homology spheres.

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