Bridge, pinochle, and conditional probability

Bridge and pinochle are card games which use different decks of cards. In bridge, the deck contains 52 cards in four suits (clubs, diamonds, hearts, spades) and face values running from 2 through jack, queen, king, ace. In pinochle, the deck contains 48 cards in pairs, with four suits with face values from from 9 through jack, queen, king, ace (so there are two of each type of card).

Problem. A fair coin is tossed. If it comes up heads, then a card is drawn from a bridge deck. If it comes up tails, then a card is drawn from a pinochle deck. Given that the card drawn is a jack, what is the probability that the coin came up tails?

Solution. Formally the sample space Σ consists of the ordered pairs in

$$\{h\} \times \{2,3,4,5,6,7,8,9,10,j,q,k,a\} \quad \bigcup \quad \{t\} \times \{x,y\} \times \{9,10,j,q,k,a\} \ .$$

Since tossing the coin and drawing the card are independent events, the probability of each element in the first subset is $\frac{1}{2} \times \frac{1}{52} = \frac{1}{104}$, while the probability of each element in the second subset is $\frac{1}{2} \times \frac{1}{48} = \frac{1}{96}$.

Let $J \subset \Sigma$ be the subset of all events whose last coordinate is j (*i.e.*, drawing a jack), and let H and T denote the subsets of events whose first coordinates are h and t respectively (*i.e.*, the coin came up heads or tails respectively). Then the probabilities $\mathbf{p}(E)$ for the various subsets of events are given by

$$\mathbf{p}(H \cap J) \ = \ \frac{1}{2} \cdot \frac{4}{52} \ = \ \frac{1}{26} \ , \qquad \mathbf{p}(T \cap J) \ = \ \frac{1}{2} \cdot \frac{8}{48} \ = \ \frac{1}{12} \ , \qquad \mathbf{p}(J) \ = \ \frac{1}{26} + \frac{1}{12} \ = \ \frac{19}{156} \ .$$

The problem asks for the *conditional probability* of tails given that a jack was drawn. Formally, this is

$$\mathbf{p}(T|J) \quad = \quad \frac{\mathbf{p}(T \cap J)}{\mathbf{p}(J)}$$

and by the preceding computations the conditional probability is given by

$$\frac{1/12}{19/156} \quad = \quad \frac{156}{228} \quad = \quad \frac{13}{19}$$

whose decimal expansion equals

$$0.684210526315789473\,684210526315789473\ldots$$

and thus we see that the odds that the coin came up tails are slightly higher than 2 to 1 (which is what one might have guessed).