Legendrian curves in 3–dimensional contact manifolds

Definition. Let M^n be a smooth n-manifold without boundary, and let ω be a nowhere zero smooth 1-form on M. A piecewise smooth curve $\gamma : [a,b] \to M$ is said to be *tangent* to ω if $\omega(\,\gamma'(t)\,)$ is identically zero. Here we assume γ' is the usual tangential lifting of γ to a smooth curve in the tangent space $\mathbf{T}(M)$, and at the finite set of nonsmooth points we assume that the one-sided tangent vectors $\gamma'(t+)$ and $\gamma'(t-)$ satisfy $\omega(\,\gamma'(t+)\,) = \omega(\,\gamma'(t-)\,) = 0$.

Given M and ω, there is a natural equivalence relation $\mathcal{R} = \mathcal{R}^\omega$ on M such that two points p and q are $\mathcal{R}$-related if and only if they can be joined by a piecewise smooth curve in M which is tangent to ω; it is straightforward to verify this is an equivalence relation.

If the form ω comes from a codimension 1 foliation of M, then a piecewise smooth curve γ is tangent to ω if and only if γ is contained in a single leaf of the foliation. It follows that the $\mathcal{R}$-equivalence classes in this case are simply the leaves of the foliation (check this carefully). On the other hand, if ω comes from a contact structure on a connected smooth 3-manifold, then we have the following result:

THEOREM. *Let ω be a contact 1-form on the connected smooth 3-manifold M. Then every pair of points in M can be joined by a regular piecewise smooth curve which is everywhere tangent to ω.*

Regularity means that the tangent vectors $\gamma'(t)$ at smooth points and one-sided tangent vectors $\gamma'(t\,\pm)$ of nonsmooth points are all nonzero. A curve which satisfies the conditions of the theorem is said to be a (regular piecewise smooth) *Legendrian curve*.

Reduction to a special case

We begin by showing that the theorem will follow from a purely local result.

LOCAL THEOREM. *Suppose that $\omega_0 = dx - z\,dy$ is a standard contact form on $\mathbb{R}^3$, and let U_0 be an open neighborhood of $\mathbf{0}$ in $\mathbb{R}^3$. Then there is an open subneighborhood $V_0 \subset U_0$ of $\mathbf{0}$ such that every point in V_0 can be joined to $\mathbf{0}$ by a regular piecewise smooth Legendrian curve in U_0.*

Proof that the Local Theorem implies the main result. Suppose that M^3 is a connected smooth manifold with a smooth contact form ω, and let $\mathbf{p} \in M$. By the Darboux Lemma for contact structures, there is an open neighborhood U of $\mathbf{p}$ in M and a diffeomorphism h from U to an open subset $U_0 \subset \mathbb{R}^3$ such that $h(\mathbf{p}) = \mathbf{0}$ and $h^*(\omega_0|U_0) = \omega|U$. By the Local Theorem there is an open subneighborhood $V_0 \subset U_0$ such that all points in V_0 can be joined to $\mathbf{0}$ by regular piecewise smooth Legendrian curves in U_0. If $V = h^{-1}[V_0]$, then it follows that every point in V can be connected to $\mathbf{p}$ by regular piecewise smooth Legendrian curves in U.

If we now define a binary relation $\mathcal{R}$ on M as before, then the preceding paragraph shows that the resulting equivalence classes are open. Since M is connected, it follows that there is a single equivalence class for $\mathcal{R}$.

Proof of the Local Theorem

A typical vector field on an open subset $W \subset \mathbb{R}^3$ has the form

$$\mathbf{X}(x,y,z) \;=\; P(x,y,z)\,\frac{\partial}{\partial x} \;+\; Q(x,y,z)\,\frac{\partial}{\partial y} \;+\; R(x,y,z)\,\frac{\partial}{\partial z}$$

where P, Q and R are smooth real valued functions on W. This vector field is tangent to ω_0 if and only if $P = z \cdot Q$. There are no restrictions on the smooth functions Q and R, and at every point $\mathbf{v}_0 = (x_0, y_0, z_0)$ a basis for the kernel of $\omega_0(\mathbf{v}_0)$ is given by the vector fields

$$\frac{\partial}{\partial z} \qquad \text{and} \qquad z\,\frac{\partial}{\partial x} \;+\; \frac{\partial}{\partial y}\ .$$

We shall analyze the integral curves for certain specific vector fields on $\mathbb{R}^3$, and pieces of these curves will define the piecewise smooth curves we need. By construction, such curves are Legendrian. In our examples the functions Q and R will depend only on the y and z variables; this will make it fairly easy to solve the systems of differential equations we shall consider.

The first class of examples. Let α be a fixed real number, and consider the vector field

$$\mathbf{X}(x,y,z) \quad = \quad z\,\cos\alpha\,\frac{\partial}{\partial x} \;+\; \cos\alpha\,\frac{\partial}{\partial y} \;+\; \sin\alpha\,\frac{\partial}{\partial z}\ .$$

These vector fields are tangent to ω_0, and we can easily find their integral curves with initial conditions of the form $(c, 0, 0)$ as follows: The last two coordinates satisfy $y' = \cos\alpha$ and $z' = \sin\alpha$, and since $y(0) = z(0) = 0$ it follows that $y(t) = t\,\cos\alpha$ and $z(t) = t\,\sin\alpha$. These in turn imply the differential equations

$$x'(t) \quad = \quad z(t)\,y'(t) \quad = \quad t\,\cos\alpha\sin\alpha$$

with $x(0) = c$, from which we conclude that

$$x(t) \quad = \quad \frac{\cos\alpha\sin\alpha}{2}\,t^2 \;+\; c\ .$$

CLAIM. *All points of these integral curves lie on the surface*

$$x \quad = \quad \frac{yz}{2} \;+\; c$$

and conversely every point on such a surface lies on an integral curve of some vector field of the type under consideration.

Verification of this claim is an elementary exercise and is left to the reader. It follows that for each c, all points on the surface $L(c)$ with defining equation $x = \frac{1}{2}\,yz + c$ are $\mathcal{R}$-equivalent to each other, where $\mathcal{R}$ is defined with respect to the form ω_0.

Let Φ be the change of variables map on $\mathbb{R}^3$ sending (u, v, w) to $(u + \frac{1}{2}\,vw,\ v,\ w)$, so that Φ is a diffeomorphism which sends each plane $u = c$ to the level surface $L(c)$. Let $\varepsilon > 0$ and choose $\delta > 0$ so that Φ maps the open set defined by $|u| < \delta$ and $v^2 + w^2 < \delta^2$ into the disk defined by $x^2 + y^2 + z^2 < \varepsilon^2$. If U_0 is the latter and $U_1 \subset U_0$ is the image of the former, then it follows immediately that for every real number $c \in (-\delta, \delta)$ all points of $U_1 \cap L(c)$ can be joined to each other by regular piecewise smooth Legendrian curves.

The next step is to find vector fields which are tangent to ω_0 and have integral curves which are transverse to the surfaces $x = \frac{1}{2}\,yz + c$.

The second class of examples. Now consider the vector field

$$\mathbf{X}(x,y,z) \quad = \quad z^2\,\frac{\partial}{\partial x} \;+\; z\,\frac{\partial}{\partial y} \;-\; y\,\frac{\partial}{\partial z}\ .$$

We want to analyze the integral curves with initial conditions of the form $(a, 0, b)$ at $t = 0$; we assume that $b > 0$. It follows that the last two coordinates are given by $y(t) = b \sin t$ and $z(t) = b \cos t$, and the remaining coordinate is then given by $x'(t) = b^2 \cos^2 t$ with $x(0) = a$. Since $\cos 2t = 2\cos^2 t - 1$, the differential equation for x reduces to

$$x'(t) \quad = \quad \frac{b^2}{2}\,(\,1 \;+\; \cos 2t\,)$$

and clearly we can use this to find $x(t)$ explicitly.

CLAIM. *If $f(x, y, z) = x - \frac{1}{2}yz$, then*

$$g(t) \quad = \quad f\circ\gamma(t)$$

is strictly increasing near $t = 0$.

If this is true, then we have the following conclusion:

> *Suppose that the integral curve with initial condition $(a, 0, b)$ lies on the surface $x = \frac{1}{2}\,yz + c$ (this happens if and only if $c = a$). Then there is some $\delta > 0$ such that the integral curve contains a point of each surface $x = \frac{1}{2}yz + c'$ for all c' such that $|c' - c| < \delta$.*

This result implies that every point in $\mathbb{R}^3$ is $\mathcal{R}$-related to $\mathbf{0}$, for the first family of examples implies that each set $L(c)$ is contained in a single $\mathcal{R}$-equivalence class, say $\mathcal{R}(c)$, and the claim implies that $\mathcal{R}(c') = \mathcal{R}(c)$ for $|c' - c| < \delta$ for some $\delta > 0$ (which depends upon c); therefore the connectedness of $\mathbb{R}$ implies there is only one equivalence class.

A more careful inspection shows that if U_1 is the previously constructed neighborhood of $\mathbf{0}$ with defining inequalities $|u| < \delta$ and $v^2 + w^2 < \delta^2$, then if $|a|, b < \delta$ and γ is the integral curve with initial condition $(a, 0, b)$, then for $|t|$ sufficiently small the points $\gamma(t)$ lie inside U_1. This implies that every pair of points in U_1 can be joined by a regular piecewise smooth Legendrian curve which is completely contained in U_1.

Verification of the claim. Let $f(x, y, z) = x - \frac{1}{2}\,yz$. By the usual Chain Rule we have

$$g'(0) \quad = \quad \nabla f\big(\gamma(0)\big) \cdot \gamma'(0)$$

and our conditions on γ imply that the right hand side equals $\nabla f(a, 0, b) \cdot (b^2, b, 0)$. Since

$$\nabla f(x, y, z) \quad = \quad (1,\, -\tfrac{1}{2}\,z,\, -\tfrac{1}{2}\,y)$$

the expression for $g'(0)$ simplifies to $(1,\, -\frac{1}{2}\,b, 0) \cdot (b^2, b, 0) = \frac{1}{2}\,b^2$, which is positive. By continuity, we also know that $g'(t)$ will be positive provided t is close enough to zero, and therefore the restriction of f to the integral curve will be strictly increasing if t is sufficiently close to 0 (because its derivative is positive).