Math 472 – Non simply connected surgery

I – Poincaré Complexes

Def Let (X, Y) be a CW pair dominated by a finite complex, $\Lambda = Z[\pi_1(X)]$ group ring. Consider the infinite chains in $(\tilde{X}, \tilde{Y})$, $\tilde{X}$ = univ covering, $\tilde{Y} = \tilde{X}|Y$ induced covering. Let $w : \pi_1(X) \to Z_2$ be a homomorphism; then w determines an involution of Λ; $x = \sum a_g g \Rightarrow \bar{x} = \sum w(g) a_g g^{-1}$.

A Poincaré pair is a CW pair (X, Y) as above with an "orientation" hom w and a "fundamental class" $w \in H_n^t(\tilde{X}, \tilde{Y}; Z)$ (∞ coeffs) s.t. the covering transformation $g \in \pi_1$ changes the sign of w by $w(g)$ and such that

$$\cap w : H^q(X; \Lambda) \longrightarrow H_{n-q}(X, Y; \Lambda)$$

is an isomorphism for twisted coefficients in Λ.

Note If B is a rt. Λ module, involution yields a left B module B_t. $H_p^t(X; B) \cong H_p(C(X) \otimes_\Lambda B_t)$. It follows that the fundamental class here induces an isomorphism

$$H^q(X; B) \longrightarrow H^t_{n-q}(X, Y; B).$$

Note $\partial(PCPX) =$ PCPX w/out ∂ also assumed

with twisted coefficients as usual

Proposition 1 If M is a PL mfld (with bdy perhaps) then M is a Poincaré complex and w is the first Stiefel-Whitney class.

Proof Isomorphism is obtained basically by means of the dual cpx., and finite proof goes over since using infinite coefficients. If we take fixed bases, isomorphism is in fact simple by some subdivision lemma in Milnor.

Consider objects of the form $\{(W, \partial W), f, F\}$

$\nu_0 \downarrow \nu$

$Y \subseteq X$

$f : (\tilde{W}, \partial \tilde{W}) \longrightarrow (\tilde{X}, \tilde{Y})$, deg $+1$ with possibly infinite chains, F trivialization of $\tau_W \oplus F^* \nu$ s.t. restr. to bdy trivializes $\tau_{\partial W} \oplus F^* \nu_0$, and s.t. $w_{(X,Y)} \circ f_* = w_{(W,\partial W)}$. Can define cobordism of such objects as for simply connected.

<u>Problem</u> Suppose $f | \partial W$ is a homotopy equivalence. Is there a cobordism of the original object to an object $(W', \partial W')$ s.t. cob$|$ bdies is an h-cob, and new object is homotopy equivalent to (X, Y) as pairs? Also same question for simple homotopy, s-cob, etc.

<u>Answer</u> Obstruction lies in a group $L^h_n(\pi_1(X), w)$ which is functorial, period 4, provided $n \geq 5$, $Y = \phi$ or $n \geq 6$. If $n \geq 6$, every obstruction is realizable.

L_n known for $\begin{cases} 0 & \text{(Kervaire-Milnor)} \\ Z^m \;\; w=0 & \text{(Shaneson)} \\ Z_2 \;\; \text{either } w & \text{(Wall)} \end{cases}$

<u>Thm. 2</u> Suppose $f : (W, \partial W) \longrightarrow (X, Y)$ deg $+1$, w pres. Then f^* is a split mono and f_* is a split epi.

<u>Proof</u> (w/out bdy) $\Lambda_1 =$ gp ring W, $\Lambda_2 =$ " — X. This yields a — pres map $C_*(\tilde{W}) \to C_*(\tilde{X})$ and hence also maps of coeffs. Have usual diagram

$$\begin{array}{ccc} H^r(W;B) & \xleftarrow{f^*} & H^r(X;B) \\ {\scriptstyle [W]\cap}\downarrow & & \downarrow{\scriptstyle [X]\cap} \\ H^t_{n-r}(W,B) & \xrightarrow{f_*} & H^t_{n-r}(X;B) \end{array}$$

Hence $[W]\cap$ induces an iso of $\operatorname{Cker} f^* = K^r(W;B)$ with $\operatorname{Ker} f_* = K^t_{n-r}(W;B)$. Thus if f is k conn, f_*, f^* are isos for $r < k$, $r > n-k$. Summand conditions obvious. The K's give exact sequences of the pairs $(W, \partial W)$ and can form excisions, etc.

II – A Basic Surgery Theorem

<u>Thm. 1</u> Suppose (X,Y) satisfies $\dim X \geq 6$ and $\pi_1(Y) \xrightarrow{\cong} \pi_1(X)$. Then any map of pairs $(M, \partial M) \to (X,Y)$ sat. aux. conds. is cobordant to a homotopy equivalence.

<u>Lemma 2</u> Let $\phi: (N,M) \to (Y,X)$ be a map of CW pairs, Y conn., $\Lambda = Z(\pi_1(Y))$ and suppose $H_i(\phi) = 0$, $i < r$. 1) ~~Then~~ If for any Λ mod B, $H^{r+1}(\phi;B) = 0$, then $H_r(\phi)$ is projective.
2) If N and Y are finite, $H_r(\phi)$ is finitely Λ-generated.
3) If $H_i(\phi) = 0$, $i \neq r$, then $H_r(\phi)$ is stably free.