Math 472 - Non simply connected surgery

I - Poincaré Complexes

Def Let (X, Y) be a CW pair dominated by a finite complex, $\Lambda = Z[\pi_1(X)]$ group ring. Consider the infinite chains in $(\tilde{X}, \tilde{Y})$, $\tilde{X}$ = univ covering, $\tilde{Y} = \tilde{X}|Y$ induced covering. Let $w: \pi_1(X) \to Z_2$ be a homomorphism; then w determines an involution of Λ; $x = \sum a_g g \Rightarrow \bar{x} = \sum w(g) a_g g^{-1}$.

A Poincaré ~~pair~~ is a CW pair (X, Y) as above with an "orientation" hom w and a "fundamental class" $w \in H_n^t(\tilde{X},\tilde{Y}; Z)$ (∞ coeffs) s.t. the covering transformation $g \in \pi_1$ changes the sign of w by $w(g)$ and such that

$$\cap w : H^q(X;\Lambda) \longrightarrow H_{n-q}(X, Y; \Lambda)$$

is an isomorphism for twisted coefficients in Λ.

Note If B is a rt. Λ module, involution yields a left B module B_t. $H_p^t(X, Y; B) \cong H_p(C(\tilde{X}) \otimes_\Lambda B_t)$. It follows that the fundamental class here induces an isomorphism

$$H^q(X; B) \longrightarrow H^t_{n-q}(X, Y; B).$$

Note $\partial(PCPX) = PCPX$ w/ont ∂ also assumed

with twisted coefficients as usual

Proposition 1 If M is a PL mfld (with bdy perhaps) then M is a Poincaré complex and w is the first Stiefel-Whitney class.

Proof Isomorphism is obtained basically by means of the dual cpx., and finite proof goes over since using infinite coefficients. If we take fixed bases, isomorphism is in fact simple by some subdivision lemma in Milnor.

Consider objects of the form $\{(W, \partial W), f, F\}$

$\nu \oplus 1$ $\nu\downarrow$
$Y \subseteq X$

$f: (\tilde{W}, \partial\tilde{W}) \longrightarrow (\tilde{X}, \tilde{Y})$, deg $+1$ with possibly infinite chains, F trivialization of $\tau_W \oplus F^*\nu$ s.t. restr. to bdy trivializes $\tau_{\partial W} \oplus F^*\nu_0$, and s.t. $w_{(X,Y)} \circ f_* = w_{(W,\partial W)}$. Can define cobordism of such objects as for simply connected.

<u>Problem</u> Suppose $f|\partial W$ is a homotopy equivalence. Is there a cobordism of the original object to an object $(W', \partial W')$ s.t. cob$|$ bdies is an h-cob, and new object is homotopy equivalent to (X, Y) as pairs? Also same question for simple homotopy, s-cob, etc..

<u>Answer</u> Obstruction lies in a group $L^h_n(\pi_1(X), w)$ which is functorial, period 4, provided $n \geq 5$, $Y = \emptyset$ or $n \geq 6$. If $n \geq 6$, every obstruction is realizable.

L_n known for $\begin{cases} 0 & \text{(Kervaire-Milnor)} \\ Z^m \ w=0 & \text{(Shaneson)} \\ Z_2 \text{ either } w & \text{(Wall)} \end{cases}$

<u>Thm. 2</u> Suppose $f: (W, \partial W) \longrightarrow (X, Y)$ deg $+1$, w pres. Then f^* is a split mono and f_* is a split epi.

<u>Proof</u> Take mapping cylinders to assume $M = N \cap X$, ϕ incl, and $C_*(\phi) = C_*(\tilde{Y}, \tilde{N} \cup \tilde{X})$ — a free fin Λ-gen ch cpx. If $i \le r$ $\exists$ s.e.s.

$$0 \to Z_i \to C_i \to Z_{i-1} \to 0 \quad (*)$$

so Z_{i-1} proj $\Rightarrow Z_i$ proj $\Rightarrow Z_r$ proj.

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
C_{r+2} \arrow[r] & C_{r+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_r \arrow[r] \arrow[d] & 0 \\
 & C_r \arrow[ur, "\eta"] & Z_r \text{ proj} \arrow[l] \arrow[d] & \\
 & & H_r \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

Notice $H^{r+1}(C; B_r) = 0$
$\Rightarrow$ cochain
$C_{r+1} \xrightarrow{\partial} B_r$ is a
coboundary; i.e. $\exists$
$\eta : C_r \to B_r$, $\eta \partial = \partial$

But check that this gives a splitting of $B_r \to Z_r \Rightarrow$
$Z_r = B_r \oplus H_r \Rightarrow H_r$ projective.

2) Since $(*)_r$ splits for $i = r$ (Z_{r-1} proj),
$\exists$ map $C_r \twoheadrightarrow Z_r \twoheadrightarrow H_r = Z_r / B_r \Rightarrow$
H_r f.g. if C_r is.

3) Let ${}^2C = C_{r+2} \to C_{r+1} \to B_r \to 0$;
contractible by hyp. $C = {}^2C \oplus H_r \oplus {}^3C$,
${}^3C = Z_{r-1} \to C_{r-1} \to C_{r-2} \to C_{r-3}$ also contr.
As usual ${}^2C^{odd} \cong {}^2C^{even}$, ${}^3C^{odd} \cong {}^3C^{even} \Rightarrow$

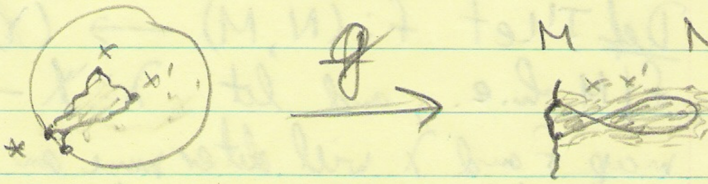
$$B_r \oplus Z_{r-1} \oplus \bigoplus_{i \ne 0} C_{i+2r} \cong \bigoplus_i C_{r+2i+1}$$

Add H_r to each side to get

$$\text{free} = C_r \oplus C_{i+2r} = H_r \oplus \text{free}.$$

Proof of theorem Case 1 $\dim X = 2k$. Can assume f is k-connected on N, $(k-1)$-conn. on M by prelim many surgeries. $K_r(N,M) = H_{r+1}(f;N)$ etc. want to kill this. If A is any Λ-mod., $H_i(f|N;A) = 0$, $i < k+1 \Rightarrow K_k(N,M)$ is stably free; adding trivial handles $S^k \times D^k$ along the bdy may assume $K_k(N,M)$ free. Basis elts. are rep. by immersions $(D^k, S^{k-1}) \to (N^{2k}, M^{2k-1})$. Since $\pi_1(N,M) = 0$, can use Whitney trick to show free generators represent embedded disks

FIGURE
can fill in
R.H.S.



Proceed as in s. conn. case.

Case 2 $\dim X = 2k+1$, f, $f|N$ k-conn. Have exact seq

$$0 \to K_{k+1}(N) \to K_{k+1}(N,M) \to K_k(M) \to K_k(N) \to K_k(N,M) \to 0$$

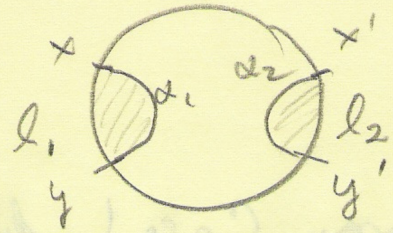
Claim can make $K_k(N,M) = 0$ by same arg as before $\Rightarrow K_{k+1}(N) = 0$ also. Get new sequence

$$0 \to K_{k+1}(N,M) \to K_k(M) \to K_k(N) \to 0$$

proj stably free as before $\Rightarrow$ all K's free can be assumed. Let $g: (D^{k+1}, S^k) \to (N,M)$ rep. a basis elt.

Claim can deform g through a regular homotopy so that $f_i | S^k$ is an embedding

⑥



$g(x) = g(x')$

$g(y) = g(y')$

l_1 + l_2 join double pts. But there exist whole curves full of double points which go to same thing if we consider g on D^{n+1}; i.e., have curves α_1, α_2 joining double points which go to same curve in g. Since $l_1\alpha_1$, $l_2\alpha_2$ extend to disks in D^{n+1}, $g \circ l_1 + g \circ l_2 \simeq$ const curve in N. Since incl is an iso of π_1, is also true in M. Can therefore use Whitney trick. Again proceed as in simply connected case.

<u>III - Obstructions to Performing Surgery</u>

<u>Def</u> J^1 Let $f: (N, M) \longrightarrow (Y, X)$ be a map s.t. $f|M$ h.e., and let $\lambda: X \to K$. $K \xrightarrow{w} K(Z_2, 1)$. A map f and λ will determine an elt. of $J^1(K, w)$. Such an object $\sim 0 \Leftrightarrow \exists$ cobordism $\rho: (W, \partial W) \to (Z, \partial Z)$ s.t. $\partial W = N \cup N'$, $\rho|N = f$, $\rho|N'$ h.e., $\partial Z = X \cup X'$, $X \cap X' = Y$. Add via disjoint union. Always have $A + -A \sim 0$; take $\rho = I \times N \xrightarrow{f \times 1} I \times Y$. Say $A \sim B \Leftrightarrow A - B \sim 0$, and denote eq. classes by $L^1(K, w)$. Is a gp under disjt union. Consider those elements of J^1 (J^2) (K, w) s.t. $\lambda: \pi_1(Y) \xrightarrow{\cong} \pi_1(K)$, Y conn., and consider restricted equivalence where Z conn, $\pi_1(Z) \xrightarrow{\lambda_2, \cong} \pi_1(K)$. $J^2(K, w)$ under restricted equivalence is $L^2_n(K, w)$.

N.B. Add concepts involving bdles/X, trivialization. For $f^*(\nu) \oplus \tau_N$ etc.

Suppose $f: (N, M) \longrightarrow (Y, X)$ deg $+1$; ~~this yields an elt. of~~ $\cancel{L^2_n(K(\pi, 1), w)}$ ~~& f cob. h.e.~~ $\Leftrightarrow f \sim 0$ in restricted sense.

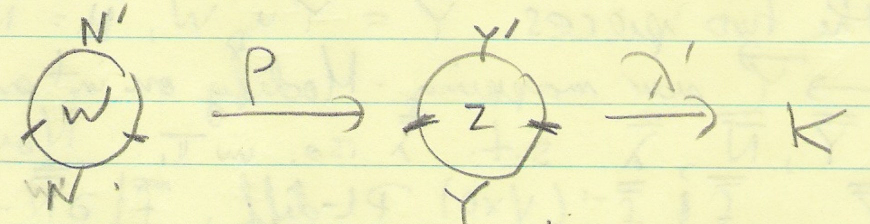
Have canonical maps $L^2(K, w) \to L^1(K, w)$

$$\begin{tikzcd} K' \arrow[r, "f"] \arrow[dr, "w'"'] & K \arrow[d, "w"] \\ & RP^\infty \end{tikzcd}$$

and if given above, have induced map $L'(K', w') \to L'(K, w)$ functorial map of abel gps.

<u>Thm. 1</u> A map $f: (N, M) \to (Y, X)$ from a restricted object is cobordant to a homotopy equivalence if and only if $f \sim 0$ in restricted sense. (dim $N \geq 5$)

<u>Proof</u> ($\Rightarrow$) Clear ($\Leftarrow$) By assumption inclusion



induces an isomorphism from $\pi_1(Y)$ to $\pi_1(Z)$ and N' and Y' are homotopy equivalent. By an extension of the results of II to triads, can cobord p to a homotopy equivalence of triads. Result follows from this.

<u>Thm. 2</u> Suppose $n \geq 4$ and K has a finite 2-skel. Then the natural map $L^2_n(K, w) \to L^1_n(K, w)$ is 1-1 onto.

$\| \Rightarrow$ Assume (Y, X) is a smooth or PL pair $\Leftarrow \|$

Also assume $\pi_1(K)$ is finitely presented

Using Wall's work on Poincaré complexes can drop assumption on (Y, X).

<u>Proof</u> Modify Y on the interior s.t. $\lambda: Y \to K$ is an iso in π_1, & let $Z = Y \times I \cup_s I_s \times D^n \cup_t D^2_t \times D^{n-1}$ and modify N by inverse images of S^0's and S^1's, which are S^0's and S^1's in greater profusion perhaps. This yields $N' \to Y'$, $W \to Z$, $N' \to Y'$ restricted object $\Rightarrow$ map onto. Proof that map 1-1 similar.

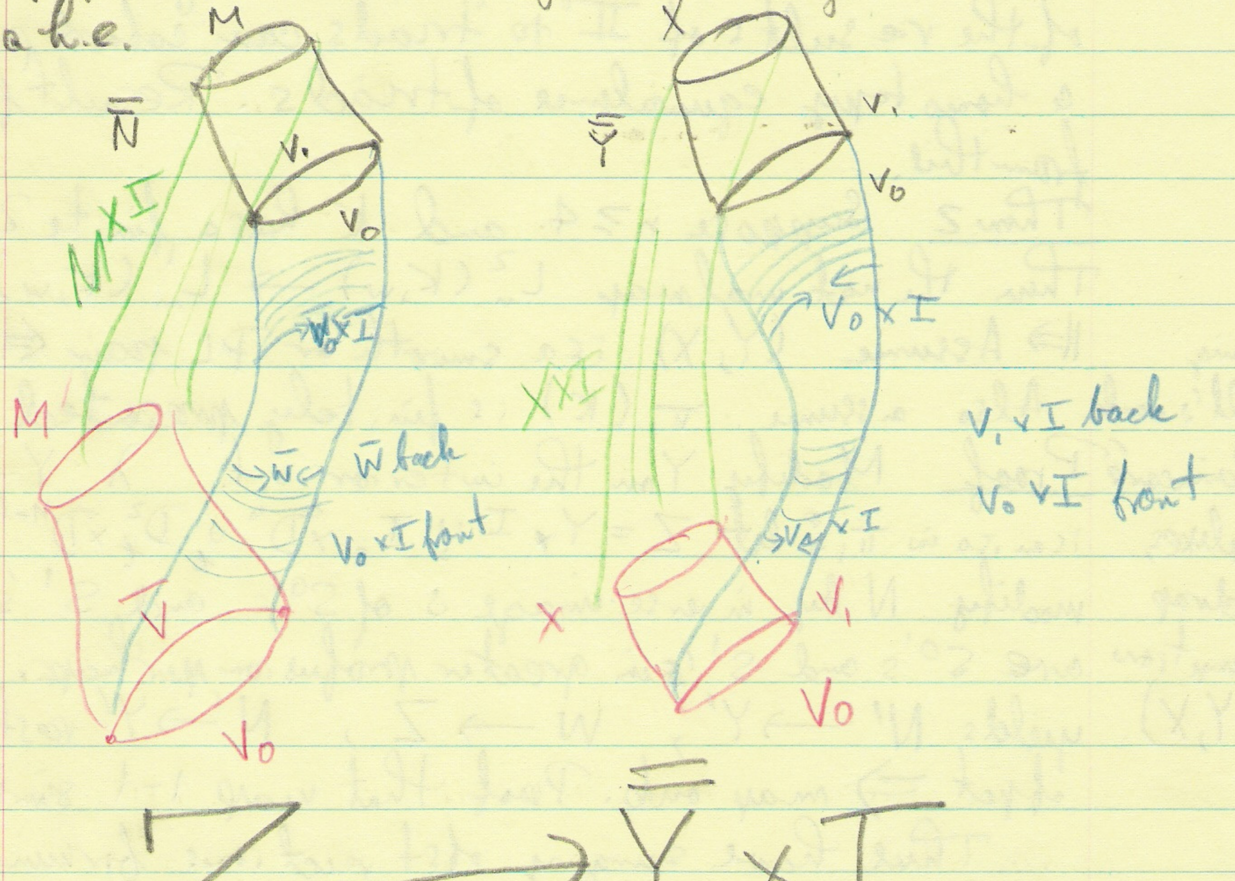
Thus have surgery obstructions forming an abelian group.

(8)

Prop. 3. Let $f:(N,M)\longrightarrow (Y,X)\xrightarrow{\lambda} K\xrightarrow{w}\mathbb{Z}_2$, $n\geq 6$, $V=$ PL mfld w/ bdy. Then f is equivalent in $L_m(K,w)$ to some $W=V\times I$, $g:(N',M')\to(W,\partial W)$ X), $\pi_1(V)\xrightarrow{\cong}\pi_1(K)$
$g|\,g^{-1}(V\times 0)$ PL homeo (diff)
$g|\,g^{-1}(V\times 1)$ h.e.

Proof. Take the disjoint union of f and 1_W and connect the two pieces. $\bar Y = Y\cup_D W$, $\bar N = N\cup_D W$, $\bar f:\bar N\longrightarrow \bar Y$ new mapping. Modify on interiors to get $\bar{\bar Y}$, $\bar{\bar N}$, $\bar{\bar\lambda}$ s.t. $\bar{\bar\lambda}$ iso. in π_1. Now have $V\subseteq\partial\bar{\bar Y}$, $\bar{\bar f}|\,\bar{\bar f}^{-1}(V\times 0)$ PL-diff, $\bar{\bar f}|\,\partial\bar{\bar N}-\bar{\bar f}^{-1}V\times 0\longrightarrow\partial\bar{\bar Y}-V$ homotopy equivalence. By modification of fundamental thm. for triads, get a cobordism to a h.e.

H:

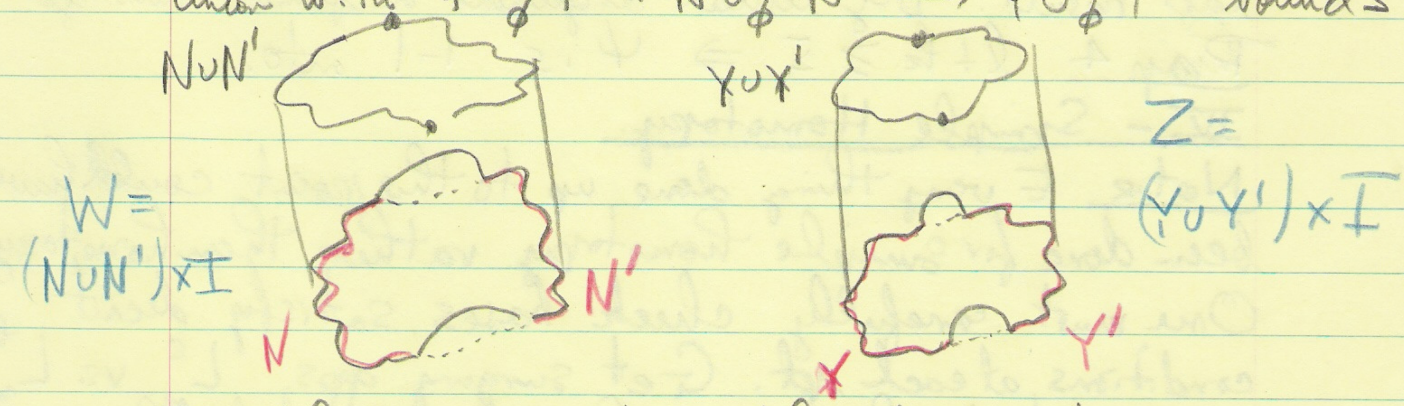


$$\bar{\bar Z}\longrightarrow Y\times I$$

So $\partial Z = \overline{\overline{N}} \cup M \times I \cup \overline{\overline{\overline{N}}} \cup V_0 \times I \cup \overline{W}$. and the

<u>Lemma 4</u> Suppose one has surgery problems $f:(N, M) \longrightarrow (Y, X)$ and $f':(N', M) \to (Y', X)$. Then $f \cup f' : (N \cup_M N' \longrightarrow Y \cup_X Y'$ represents the sum of f and f' in $L'_m(K, w)$.

<u>Proof</u> Need only show given surgery problem disjoint union with $f \cup_\phi f' : N \cup_\phi N' \longrightarrow Y \cup_\phi Y'$ bounds.



Do this as suggested in the above picture.

<u>Proof of 3 concluded</u> So H restricted to each of these yields a surgery problem with obstruction in $L'_m(K, w)$, and by the lemma, their sum is zero. Given obstruction represented by $\overline{N}$. But $\theta(\overline{N}) = \theta(V_0 \times I) = 0$ since homotopy equivalences. Furthermore, also $\theta(M \times I) = 0$. Hence $\theta(\overline{\overline{N}}) + \theta(\overline{W}) = 0$, or $\theta(\overline{W}) = -$ given obstruction. Reverse orientation to get result. $H' | \overline{W} : \overline{W} \longrightarrow W \times I$ deg $+1$, $\theta =$ given obstr., $H' | V_0$ diff (PL), $H' | \overline{V}$ h. e.

Let T be a closed manifold, $\pi_1(T) = G$. Then every surgery problem is equivalent to one $h: (N, \partial N) \longrightarrow (D^l \times T, S^{l-1} \times T)$ $l \geq 2$ where $h | h^{-1}(D^{l-1}_- \times T)$ is a diffeo (PL). Can easily

define connected sum on $L'(K, w)$ $h_1 \# h_2$:
$(N_1 \# N_2, \partial N_1 \# \partial N_2) \longrightarrow (D^l \times T, S^{l-1} \times T)$
Define $L^{l+k}(T)$ with the equivalent objects
$h_1: N_1 \rightarrow$, $h_2: N_2 \rightarrow$ if $\exists$ $H: W \longrightarrow D^l \times T \times I$
s.t. $H|\partial_i W = h_i$. Have map $\psi: L^{l+k}_{l+k}(T) \rightarrow$
$L_{l+k}(K(\pi_1(T), 1), w_1(T))$ which is onto by
last result. By usual argument on fund. thm. get

Prop 4 $l+k \geq 5 \Rightarrow \psi$ is 1-1 onto.

IV - Simple Homotopy

Note Every thing done up to this point could have been done for simple homotopy rather than homotopy. One must carefully check bases satisfy decent conditions at each pt. Get surgery gps. L^s_n vs. L^h_n. Fundamental theorem goes through and in fact have

Prop. 1 Suppose $\pi_1(X) \xrightarrow{\cong} \pi_1(Y)$. Then if two
$f: (N, \partial N) \longrightarrow (Y, X)$ and $f': (N', \partial N') \longrightarrow (Y, X)$
are cobordant, there is a diff (PL) d:
$(N, \partial N) \longrightarrow (N', \partial N')$ s.t. $f'd \simeq f$.

Proof $h: W \longrightarrow Y \times I$ extension. Can modify leaving N, N' fixed s.t. h' is a simple homotopy equivalence. Hence get $V \longrightarrow Y \times I$ s.h.eq $\Rightarrow$ V is an s-cobordism; apply the s-cobordism thm.

Thm. 2 (Rothenberg) There is an exact sequence
$\rightarrow L^s_{n+1}(\pi, w) \rightarrow L^h_{n+1}(\pi, w) \rightarrow T_{n+1}(\pi) \rightarrow$
$L^s_n(\pi, w) \longrightarrow L^h_n(\pi, w) \rightarrow \ldots$
Where $T_n(\pi)$ is the subquotient of $Wh(\pi)$ given by elts. $x = (-1)^n \bar{x}$ mod things of the form $y + (-1)^n \bar{y}$.
($-$ obtained from usual conj$_2$).

M any $(n-2)$ diml closed mfld

$\pi_1(M) = \pi$, $w = w_1(M)$.

<u>Proof</u> Define A as follows: the elts. of A are (f, F), $f : (W, \partial W) \longrightarrow (D^2 \times M, S^1 \times M)$ s.t. f h.eq., $f|\partial W$ s.h.e., $f|f^{-1}(D^1_- \times M)$ diff (PL). Can add W_1, W_2 along $f^{-1}(D^1 \times M)$. Two objects are equivalent if $\exists$ h cob W_1, W_2 and W_1 cob. 0 $\Leftrightarrow$ equivalent to s.h.eq. This makes A into an abelian group; there is an exact sequence with A replacing $T_n(\pi)$. So suff. to construct an isomorphism. Take the Whitehead torsion of the map f.

$$\begin{tikzcd} & f : (W, \partial W) \arrow[r, "f"] & (D^2 \times M, S^1 \times M) \\ \partial_* W \arrow[ur, "\iota_*"] \arrow[rr, "f_*"] & & D^1_* \times M \arrow[u, "\text{s.h.eq}"'] \end{tikzcd} \qquad * = +, -$$

Notice $Wh(f_*) = Wh(\iota_*) + Wh(f)$ $\quad * = +, -$
or $Wh(\iota_+) + Wh(f) = Wh(f_+)$, $Wh(\iota_-) + Wh(f) = Wh(f_-)$. Since f_+ s.h.eq, f_- diff (PL), invts. vanish and $Wh(\iota_+) + Wh(f) = Wh(\iota_-) + Wh(f) = 0$
But if $Wh(\iota_+) = \tau$, $Wh(\iota_-) = (-1)^n \bar{\tau} \Rightarrow$
$-\tau = Wh\, f = (-1)^{n-1} \bar{\tau} \Rightarrow Wh(f)$ satisfies condition. This gives map $A \longrightarrow T_n(\pi)$ (Well-def (why?))

<u>Map onto</u> Can construct a map f with any Whitehead invt. s.t. f_- diff (PL); f_+ will be a s.h.e. by same computation as before.

<u>Map 1-1</u> Suppose $Wh(f) = \gamma + (-1)^n \bar{\gamma}$. Take ~~any h cobordism with this torsion invt. Then~~

$$\begin{tikzcd} D^2 \times M = W_1 \arrow[r, "g"] \arrow[d, "2"] & D^2 \times M \arrow[d] \\ V \arrow[r, "\ell"] & D^3 \times M \\ D^2 \times M \arrow[u] \arrow[r, "1_M"] & D^2 \times M \arrow[u] \end{tikzcd} \qquad Wh\,\ell = -(\gamma + (-1)^n \bar{\gamma})$$

Attach an h-cobordism to $\partial_+ W$ with torsion $\gamma+(-1)^n\bar\gamma$; the new outside boundary is still $\partial_+ W$ and the map of the expanded manifold is a simple homotopy equivalence; fact that cobordant obvious.

V - Algebraic View of Surgery Groups

Even dimensional case $L_{2k}(G)$. Represent by $f: M \to Y$ deg $+1$, $K_r(f) = H_{r+1}(\tilde{Y}, \tilde{M})$ proj stably free $\Rightarrow$ can assume free. Recall standard diagram from simply connected surgery:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & & & \\
 & & & H_{k+1}(M_0) \arrow[d] & & & \\
 & & & H_{k+1}(M') \arrow[d, "\cdot v"'] & & & \\
0 \arrow[r] & H_k(M_0) \arrow[r, hook] & H_k(M) \arrow[r, "\rho"] & \mathbb{Z} \arrow[r] \arrow[d] \arrow[l, bend left, "v"] & H_{k-1}(M_0) \arrow[r] & H_{k-1}(M) \arrow[r] & 0 \\
 & & & H_k(M_0) \arrow[d] \arrow[ull] & & & \\
 & & & H_k(M') \arrow[d] & & & \\
 & & & 0 & & &
\end{tikzcd}
$$

(Replace H by K and $\mathbb{Z}$ by $\Lambda = \mathbb{Z}[\pi]$.)

$\rho(w) = w \cdot v$, intersection with the surgery class v. Choose so that $w \cdot v = 1$. If

can write $H_k(M) = \overset{w}{\mathbb{Z}} \oplus \overset{v}{\mathbb{Z}} \oplus A$ $w \cdot v = 1$ and v can be represented by a properly embedded sphere, then can kill off w, v to get $H_k = A$. For $H_k(M_0) \cong \mathbb{Z}v \oplus A$ and $H_k(M')$ is the quotient by the image of $\mathbb{Z}$, which corresponds to the preimage of the class $v \in H_k(M)$

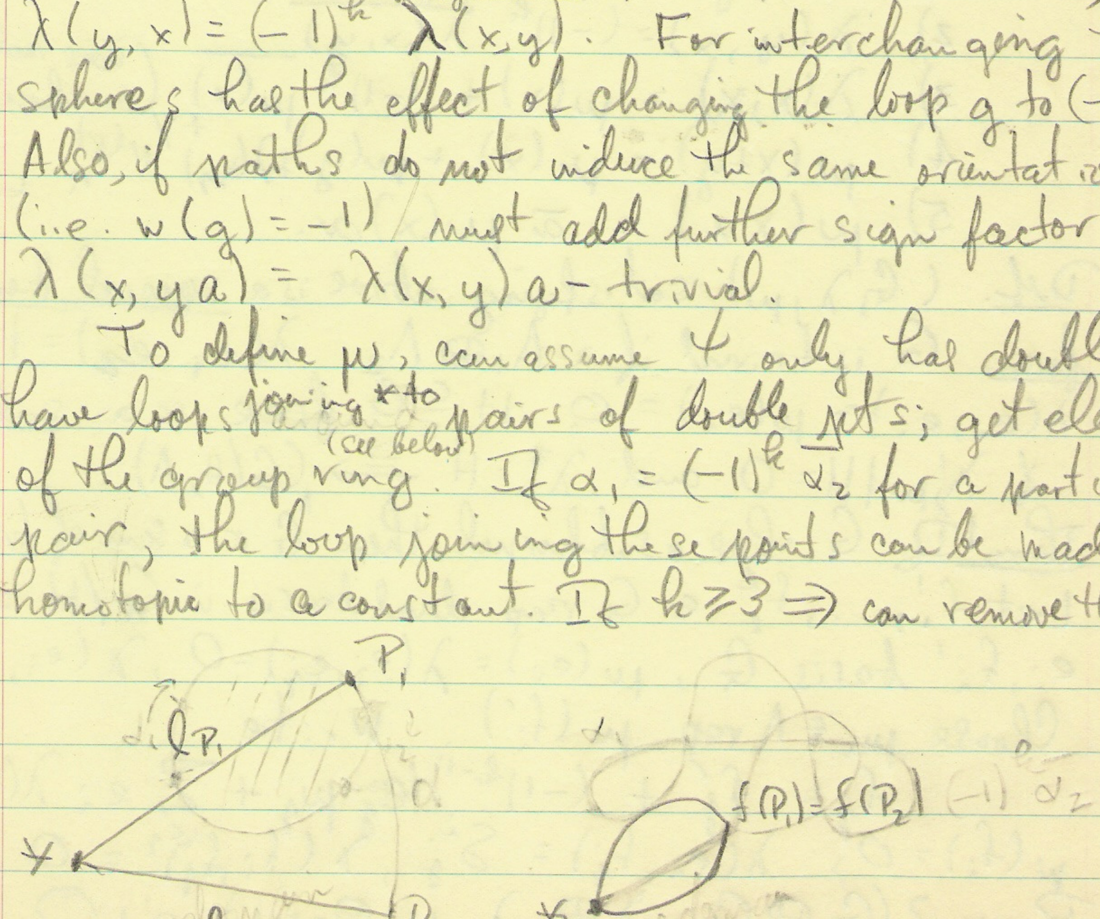
In general setting, class in $K_k(M)$ gives rise to based regular homotopy class of immersions $S^k \to M^{2k}$; i.e. a map $S^k \longrightarrow V_{2k,k}(\tilde{M}) = E$. Will define maps

$$\lambda : \pi_k(E) \otimes \pi_k(E) \longrightarrow \Lambda \quad \text{(intersection)}$$

$$(k>1) \qquad \mu : \pi_k(E) \longrightarrow \Lambda \big/_{gp} (v - (-1)^k \bar{v}) = I \quad \text{(selfintersection)}$$

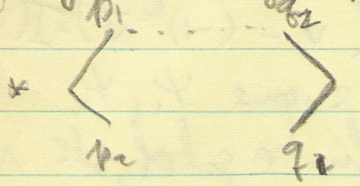
To define λ, take immersions f_1, f_2. Assume intersect transversely in a finite set of pts none of which are double pts. Joining intersecting points to the base point by α_1, α_2 gives a loop in M which gives the desired intersection. Take the element of π obtained sufficiently weighted; then λ is the element of the group ring $\sum_i w_g g$. λ is bilinear by general position. Furthermore, $\lambda(y, x) = (-1)^k \overline{\lambda(x, y)}$. For interchanging the spheres has the effect of changing the loop g to $(-1)^k g^{-1}$! Also, if paths do not induce the same orientation (i.e. $w(g) = -1$) must add further sign factor of (-1). $\lambda(x, ya) = \lambda(x, y)a$ – trivial.

To define μ, can assume f only has double pts. $\Rightarrow$ have loops joining $*$ to pairs of double pts; get element of the group ring (see below). If $\alpha_1 = (-1)^k \alpha_2$ for a particular pair, the loop joining these points can be made homotopic to a constant. If $k \geq 3 \Rightarrow$ can remove the pair.



For the curves $\alpha_1 + \alpha_2$

Notice first that if order of double points is changed, get a ~~new~~ intersection is $(-1)^k w(g)\,\varepsilon\, g^{-1}$ as before. Also if $\alpha_1 = \varepsilon_1 (-1)^k \bar{\alpha}_2$ for loops then by reorienting they are homotopic. In particular, the map



$\longrightarrow M$ extends to a 2-disk

$\longrightarrow M$. Now apply the Whitney trick.

<u>Important data:</u> $\lambda : K \otimes K \to \Lambda$ s.t. $\lambda^{\#} : K \to (K, \Lambda)$ iso and $\mu : K \to \Lambda / I_k$ s.t.

1) $\lambda(x, ya) = \lambda(x,y)\,a$
2) $\lambda(y, x) = (-1)^k\, \overline{\lambda(x,y)}$
3) $\lambda(x,x) = \mu(x) + (-1)^k\, \overline{\mu(x)}$ (Does not depend on choice of $\mu(x)$)
4) $\mu(x+y) = \mu(x) + \mu(y) + \lambda(x,y) + \chi_N(x)$ Euler class of ν bdle
5) $\mu(xa) = \bar{a}\,\mu(x)\,a$.

<u>Def.</u> (G, λ, μ) satisfying above is a <u>special Hermitian form</u>. G is trivial if $= \Lambda_1 \oplus \Lambda_2$, $\lambda(e_1, e_2) = 1$, $\lambda(e_i, e_i) = \mu(e_i) = 0$. $H \subseteq G$ free is a <u>subkernel</u> if $\lambda, \mu | H = 0$ and $\lambda^{\#} : H \xrightarrow{\cong} (G/H, \Lambda)$.

<u>Thm. 1</u> If G has a subkernel, then G is a sum of trivial forms.

Let $f_1', \dots, f_\ell' \in G$ rep dual basis in $G/H \Rightarrow$ e_i, f_i' basis G, $\mu(e_i) = \lambda(e_i, e_j) = 0$, $\lambda(e_i, f_j') = \delta^i_j$.

Choose $\mu_i \in \Lambda$ rep $\mu(f_i')$. Write

$$f_j = f_j' + (-1)^{k-1} e_j \mu_j + \sum_{i<j} e_i\, \lambda(f_i, f_j')$$

$\mu(f_i) = 0$, $\lambda(e_i, f_j) = \delta^i_j$, $\lambda(f_i, f_j) = 0$.

<u>Prop. 2</u> $(G \oplus G,\ \lambda \oplus -\lambda,\ \mu \oplus -\mu) \sim 0$.
(Consider $\{e_i + e_i'\}$; forms a subkernel).