ON THE NONUNIQUENESS OF EQUIVARIANT CONNECTED SUMS

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ABSTRACT. In both ordinary and equivariant 3-dimensional topology there are strong uniqueness theorems for connected sum decompositions of manifolds, but in ordinary higher dimensional topology such decompositions need not be unique. This paper constructs families of manifolds with smooth group actions that are equivariantly almost diffeomorphic but have infinitely many inequivalent equivariant connected sum representations for which one summand is fixed. The examples imply the need for restrictions in any attempt to define Atiyah-Singer type invariants for odd dimensional manifolds with nonfree smooth group actions. Applications to other questions are also considered.

The connected sum, which is essentially the simplest way of modifying the disjoint union of two manifolds to make it connected, is one of the most fundamental constructions in geometric topology. A particularly striking difference between low and high dimensional topology is that connected sum decompositions are unique in low dimensions but not in high dimensions (see [He, Ch. 7] and [JR] for low dimensional cases and [BrSt], [Wa, Cor. 1, p. 136] and the citations in [Br1, p. 23] for high dimensional cases).

One basic property of connected sums is that the connected sum of an arbitrary oriented n-manifold with an n-sphere is isomorphic to the original manifold, and a well studied special case of the uniqueness question in ordinary differential topology involves connected sums of the form $N\#\Sigma$ where Σ is a so-called exotic sphere that is homeomorphic but not diffeomorphic to S^n; special cases play an important role in the classification of smooth simply connected manifolds with a given homotopy type (*cf.* [Br3, Ch. V]). The set of all oriented diffeomorphism classes of exotic spheres forms an abelian group Θ_n with respect to connected sum, and the extent to which such decompositions are unique is determined by a subgroup $I(N) \subset \Theta_n$ called the *inertia group* of N^n (*cf.* [Ko], [L1], [Wi1–3]). Since the groups Θ_n are finite [KM], connected sums of the special type yield a finite set of oriented diffeomorphism classes.

In this paper we shall study equivariant analogs of these questions for smooth actions of a compact Lie group G. Specifically, if N denotes a suitably equivariantly oriented smooth G-manifold and $x \in N$ is a fixed point such that the G-representation on the tangent space is equivalent to a fixed representation V, then one can form equivariant connected sums at x with *G-twisted V-spheres* that are given by two copies of the unit disk in V with the boundaries identified by an equivariant diffeomorphism (*cf.* [MSc2, §2]). Once again the appropriately oriented G-twisted V-spheres form an abelian group Γ_V^G with respect to connected sum if $\dim V^G \geq 2$, and the extent to which the equivariant connected sum decompositions are unique is determined by an equivariant inertia group $I^G(N, A) \subset \Gamma_V^G$, where A denotes the component of N^G containing the point x.

Although this formally parallels the nonequivariant setting, one major difference is that the groups Γ_V^G are sometimes infinite, and therefore it is possible to obtain infinite families of smooth G-manifolds in some cases (*e.g.*, see [MSc2, Thm. 4.6]).

The main objective of this paper is to construct several families of G-manifolds N for which Γ_V^G is infinite but there are only finitely many distinct oriented equivariant diffeomorphism classes of manifolds of the form $N\#\Sigma$ where Σ runs through all G-twisted V-spheres. Here are more detailed statements of some main results.

Theorem 0.1. *Let $G = S^1$, suppose that N^k is a closed oriented k-manifold (with no group action furnished) where $k \geq 5$ is odd, let $m \geq 2$, and let $V = \mathbb{R}^k \times \mathbb{C}^m$ where G acts trivially on the real coordinates and by complex multiplication on the complex coordinates. Define a smooth action of G on $N^k \times \mathbb{C}\mathbf{P}^m$ by taking the trivial action on the first factor and the projectivization of the previously mentioned linear action over $\mathbb{C}^m$ on the second.* THEN *the group Γ_V^G is infinite, but there are only finitely many distinct oriented G-equivariant diffeomorphism classes of manifolds of the form $N^k \times \mathbb{C}\mathbf{P}^m \# \Sigma$ where Σ represents an element of Γ_V^G. In fact, for each k and m it is possible to choose N^k such that there is only one oriented G-diffeomorphism class.*

Theorem 0.2. *If $H = \mathbb{Z}_p$ where p is an odd prime and $m \geq \frac{1}{2}(p+3)$, then the conclusions of Theorem* 0.1 *remain true with H replacing G provided $k \equiv 1 \mod 4$.*

Numerous other examples of these sorts exist for actions of $\mathbb{Z}_p$, but the descriptions require numerous digressions; for more details see Examples 2.6.1–2.6.3 and Theorems 2.7–2.9.

Although the existence of examples with infinitely many distinct representations as equivariant connected sums has some interest in its own right, there is an independent motivation from another direction; namely, these examples have negative implications for any attempt to generalize the invariants of Atiyah and Singer [AS, p. 590] to odd dimensional closed smooth G-manifolds with nonfree actions. If p is an odd prime, then the results of [MSc2] yield a viable theory of such invariants for smooth $\mathbb{Z}_p$ manifolds whose equivariant tangent bundles have suitable stable triviality properties. The examples of this paper show that one cannot form a nontrivial theory of generalized Atiyah-Singer invariants unless some restrictions are placed on the class of manifolds under consideration (see Theorem 3.1 and Examples 3.4.1–2); in fact, this is true even when the value group for the invariants of [MSc2, §1] is highly nontrivial.

In the first two sections we construct examples with disconnected fixed point sets for which nontrivial invariants of Atiyah-Singer type cannot be defined. The first section describes some basic examples of equivariantly almost diffeomorphic semifree S^1-manifolds for which there are only finitely many equivariant diffeomorphism classes, and the second section uses these examples and T. Yoshida's work on surgery obstructions of twisted products [Yo] to construct the various examples of smooth $\mathbb{Z}_p$-manifolds for which Γ_V^G is infinite but there are only finitely many distinct oriented equivariant diffeomorphism classes of manifolds of the form $N \# \Sigma$ where Σ runs through all G-twisted V-spheres. In Section 3 the negative implications for defining generalized Atiyah-Singer invariants are discussed. Finally, Section 4 treats a related question; specifically, we shall correct one case of the main theorem in [Sc6] using some examples related to Section 1.

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1. Large S^1-equivariant inertia groups

Given a compact Lie group G and a finite dimensional G-representation V, we define Γ_V^G as in [MSc2, §2] to be the quotient of the group of all equivariant orientation preserving self diffeomorphisms of the unit sphere $S(V)$ modulo those that extend to the unit disk $D(V)$; as in [MSc2, Prop. 2.2], this set has a natural group structure that is abelian if $\dim V^G \geq 2$.

For the sake of clarity we shall review some material from [MSc2]. Given a class $\alpha \in \Gamma_V^G$ one can construct an associated G-equivariantly twisted V-sphere $\Sigma(\alpha)$ by choosing a representative h for α and gluing together two copies of the disk $D(V)$ using h. Elementary considerations show that the oriented equivariant diffeomorphism type of $\Sigma(\alpha)$ does not depend upon the choice of representative. If M is a closed oriented smooth G-manifold containing a fixed point x for which the tangential representation of G on the tangent space ot x is equivalent to V, then one can form an equivariant connected sum $M \#_x \Sigma(\alpha)$ of M with an arbitrary twisted G-sphere $\Sigma(\alpha_G)$ for an arbitrary $\alpha \in \Gamma_V^G$, at the point x. Letting A be the component of the fixed point set of M that contains x, we define $I^G(M; A)$ to be the set of all α such that M and $M \#_x \Sigma(\alpha)$ are orientation preservingly equivariantly diffeomorphic such that A corresponds to $A \# \Sigma(\alpha)^G$ (this is automatic if there is only one fixed point component of the appropriate dimension). As in [MSc2] this set is a group that is called the *equivariant inertia group*. By construction, $I^G(M, A)$ is trivial if and only if $M \#_x \Sigma(\alpha)$ and $M \#_x \Sigma(\beta)$ are inequivalent for $\alpha \neq \beta$, and in general the equivariant inertia group measures the nonuniqueness of equivariant connected sums.

If M is a closed smooth $\mathbb{Z}_p$-manifold (where p is an odd prime) and $G = \mathbb{Z}_p$, then [MSc2, Thm. 4.9] shows that the rank of an equivariant inertia group $I^G(M; A)$ is small if the equivariant tangent bundle of M is equivariantly stably trivial. The results of the next two sections will show that equivariant inertia groups can be relatively large if the nonequivariant rational characteristic classes of M are nontrivial; in fact, we shall construct examples for which $I^G(M; A)$ has finite index in Γ_V^G. Since the construction of such $\mathbb{Z}_p$-manifolds is based on a construction for semifree S^1-manifolds with large equivariant inertia groups, we shall construct the latter in this section and use them to produce the examples of $\mathbb{Z}_p$-manifolds in Section 2. The results of this section are closely related to those of [Ms4].

Preliminaries. We shall first summarize some basic properties of equivariant Kervaire-Milnor groups for semifree S^1 and S^3 representations that will be needed.

If G is a compact Lie group, one can define equivariant Kervaire-Milnor groups Θ_V^G as in [Sc5, §5] or [Sc7, §II.3] for most representations V, and the G-twisted V-sphere construction

$$[h] \to \Sigma([h]) := D(V) \cup_h D(V)$$

defines a group homomorphism ϕ from Γ_V^G to Θ_V^G. We shall prove that ϕ is an isomorphism if $G = S^1$ or S^3, the action of G on V is semifree, and the dimension restrictions

$$\dim V^G \geq 5 \qquad \dim V - \dim V^G \geq 2 \cdot \dim G + 2$$

hold (see [Hs] for the case $\dim V - \dim V^G = \dim G + 1$). This depends upon the following geometric result:

Proposition 1.1. *Let $G = S^1$ or S^3, and let W be a compact smooth semifree G-manifold such that*

(1) *∂W splits into $\partial_0 W \amalg \partial_1 W$, and likewise for ∂W^G,*
(2) *both W and W^G are simply connected h-cobordisms,*
(3) *$\dim W^G \geq 6$ and $\dim W - \dim W^G \geq 4$ if $G = S^1$ and 8 if $G = S^3$.*

Then W is G-diffeomorphic to $\partial_0 W \times [0,1]$.

The proof is analogous to Rothenberg's proof of an equivariant s–cobordism theorem [Rbg, Thm. 3.4, p. 291] (see also [Ha]): First one observes that $W^G \cong \partial_0 W^G \times [0,1]$ by the simply connected h-cobordism theorem, then one thickens this to a tubular neighborhood N of W^G, and finally one extends to all of W by applying the simply connected h-cobordism theorem to $(W - \operatorname{Int} N)/G$.■

Corollary 1.2. *If $G = S^1$ or S^3 and V is a semifree G-representation satisfying $\dim V^G \geq 5$ and $\dim V - \dim V^G \geq 2\dim G + 2$, then the map ϕ determines an isomorphism from Γ_V^G to Θ_V^G.*■

If G and V are as above and G acts semifreely on a closed oriented smooth manifold M such that the tangential representation at a component A of M^G is V, then one can define $I^G(M;A)$ as in [MSc2, §2] (see also the preceding discussion), and the proof that $I^G(M;A)$ is a subgroup of Θ_V^G goes through unchanged.

The main results. Since we are interested in examples of G-manifolds with infinite equivariant inertia groups and the results of [BP] imply that Θ_V^G is finite if V is even-dimensional, for the rest of this section we shall assume that $\dim V$ is odd.

Suppose as before that $G = S^1$ or S^3, and let V be a semifree G-representation such that $\dim V^G \geq 5$ and $\dim V - \dim V^G \geq 2 \cdot \dim G + 2$ (*i.e.*, 4 if $G = S^1$ and 8 if $G = S^3$). For each V as above we shall construct relatively simple examples of smooth semifree S^d-manifolds (where $d = 1$ or 3) such that the tangent space representation at points in the component $A \subset M^G$ is given by V and $I^G(M;A)$ has finite index in Θ_V^G. Since the groups $\Theta_V^G \otimes \mathbb{Q}$ are often nonzero by the results of [BP], it follows in particular that $I^G(M;A) \otimes \mathbb{Q} \neq 0$ in these cases.

The statement of the result requires some notation. Let Λ denote the complex numbers or quaternions and let $d = \dim_{\mathbb{R}} \Lambda - 1$, so that S^d is the unit sphere in Λ. Consider the linear action of S^d on $\Lambda\mathbf{P}^m$ defined by

$$g \cdot [w_0 : \cdots : w_m] = [w_0 : \cdots : w_{m-1} : g w_m]$$

where $g \in S^d$ and $[- : \cdots : -]$ denotes the standard homogeneous coordinates on

$$\Lambda\mathbf{P}^m := \left(\Lambda^{m+1} - \{0\}\right) / \left(\text{right mult. by } \Lambda - \{0\}\right).$$

The fixed point set of this action is the union of the standardly embedded $\Lambda\mathbf{P}^{m-1}$ (last homogeneous coordinate $= 0$) and the point $[0 : \cdots : 0 : 1]$. Let V be the semifree S^d-representation $\mathbb{R}^k \oplus \Lambda^m$ where S^d acts trivially on the first coordinate and by scalar multiplication on the second.

The statement of the main result requires some additional information about the groups Θ_V^G. Specifically, the exact sequence for Θ_V^G (see [Sc4, (6.2)]) contains a map

$$\gamma_G : hS_{k+1}(\Lambda\mathbf{P}^{m-1}) \longrightarrow \Theta_V^G$$

given by taking a homotopy equivalence

$$h : (W, \partial W) \longrightarrow (D^{k+1} \times \Lambda\mathbf{P}^{m-1}, S^k \times \Lambda\mathbf{P}^{m-1})$$

such that $h|\partial W$ is a diffeomorphism, lifting h to a G–equivariant homotopy equivalence of principal G–bundles

$$E(h) : (W', \partial W') \longrightarrow (D^{k+1} \times S^{(d+1)m-1}, S^k \times S^{(d+1)m-1})$$

such that $E(h)|\partial W'$ is an equivariant diffeomorphism and forming

$$\Sigma(h) = S^k \times D^{(d+1)m} \cup_{\partial E(h)} W'.$$

The results of [BP] show that $\gamma_G \otimes \mathbb{Q}$ is an isomorphism.

Theorem 1.3. *In the setting above, suppose that* $m \geq 2$ *and* $k \geq 5$*, and let* N^k *be a closed oriented* k*-manifold. Then* $I^G(\Lambda\mathbf{P}^m \times N; \{\text{pt.}\} \times N)$ *contains the subgroup* $\gamma_G(hS_{k+1}(\Lambda\mathbf{P}^{m-1}))$*, where* γ_G *is given as above.*

Corollary 1.4. *Under the given conditions* $I^G(\Lambda\mathbf{P}^m \times N; \{\text{pt.}\} \times N)$ *has finite index in* Θ^G_V*.*

This follows because $\gamma_G \otimes \mathbb{Q}$ is bijective by the results of [BP].■

Proof of Theorem 1.3. Let Σ represent an element of Θ^G_V such that the equivariant normal bundle of Σ^G in Σ is (equivariantly) trivial. By construction every class in the image of γ_G has such a representative; in fact, if $\dim V > 2 \dim V^G$ then all classes have such representatives (*cf.* [Br2] if $G = S^1$, and modify the argument using [BeS, §11] if $G = S^3$).

By Corollary 1.2 every Σ representing a class in Θ^G_V is obtained by pasting two copies of $D(V)$ together by an equivariant self-diffeomorphism of $S(V)$. If we assume that the equivariant normal bundle of Σ^G in Σ is equivariantly trivial, then one can choose the equivariant diffeomorphism $f : S(V) \longrightarrow S(V)$ within its equivariant isotopy class so that f maps a tubular neighborhood of $S(V^G) \times D(V_G)$, where $V_G = V/V^G$ as usual, to itself by $f^G \times$ identity.

Let $\overset{\circ}{N}$ be the closed complement of some smoothly embedded k-disk in N, and attach $D(V_G) \times \overset{\circ}{N}$ to $S(V) \times [0, 1]$ along $D(V_G) \times S^{k-1}$ via the natural equivariant embedding

$$D(V_G) \times S^{k-1} \hookrightarrow S(V) = S(V) \times \{1\} \subseteq S(V) \times [0, 1].$$

Suppose now that Σ^G belongs to the nonequivariant inertia group $I(N)$. If $h : N\#\Sigma^G \longrightarrow N$ is an orientation-preserving diffeomorphism, it is well known that one can find h' isotopic to h such that h' induces a diffeomorphism of $\overset{\circ}{N}$ to itself such that $\partial h' = f^G$ (*e.g.*, see [Kr, §11C, p. 108]). One can then define an equivariant self-diffeomorphism f_1 on the resulting S^d-manifold, say W_1, by taking $f \times$ identity on $S(V) \times [0, 1]$ and identity $\times h'$ on $D(V_G) \times \overset{\circ}{N}$; strictly speaking one must also round corners equivariantly, but this can be done by standard considerations. The boundary of W_1 has two components, one of which is $S(V) \times \{0\}$ and the other of which is a free S^d-manifold we shall call P. The orbit map $P \longrightarrow P/S^d$ is a principal bundle projection, and thus we can form $W_2 = W_1 \cup_\partial P \times_G D^{d+1}$. Clearly f_1 induces an equivariant self-diffeomorphism of P, and one can extend this to $P \times_G D^{d+1}$ by taking a balanced product with the identity; using this extension we can also extend f_1 to an equivariant self-diffeomorphism f_2 of W_2 whose restriction to $\partial W_2 \cong S(V) \times \{0\}$ is equal to f.

Elementary considerations show that the G-manifold W_2 is equivariantly diffeomorphic to $\Lambda\mathbf{P}^m \times N - \operatorname{Int} D(V)$, where $D(V)$ is a linear neighborhood of (p_0, y) with p_0 given by the isolated fixed point of the action on $\Lambda\mathbf{P}^m$ and y arbitrary. Define an equivariant orientation-preserving diffeomorphism $F : (\Lambda\mathbf{P}^m \times N)\#\Sigma \longrightarrow \Lambda\mathbf{P}^m \times N$ by the following composite:

$$\begin{tikzcd}
(\Lambda\mathbf{P}^m \times N)\,\#\Sigma \arrow[r, "F"] \arrow[d, "\approx"'] & \Lambda\mathbf{P}^m \times N \\
W_2 \cup_f D(V) \arrow[r, "f_2 \cup \mathrm{id}"] & W_2 \cup D(V) \arrow[u, "\approx"']
\end{tikzcd}$$

If we choose a suitable equivariant orientation on $\Lambda\mathbf{P}^m \times N$, then F will be equivariant orientation preserving because h and h' preserve equivariant orientations. ∎

A result related to Corollary 1.4 is obtained in [Ms4] by a different method.

The proof of Theorem 1.3 in fact yields a strengthening of Corollary 1.4.

Theorem 1.5. *In the setting of Theorem* 1.3, *the equivariant inertia group*

$$I^G(\Lambda\mathbf{P}^m \times N; \{\text{pt.}\} \times N)$$

contains the subgroup $\Theta_V^G(N)$ *of all classes* $\langle\Sigma\rangle$ *such that* $\langle\Sigma^G\rangle \in I(N)$ *and the equivariant normal bundle of* Σ^G *in* Σ *is equivariantly trivial.* ∎

Corollary 1.6. *If* $k \geq 5$, $I(N) = \Theta_k$ *and either of the conditions*

(1) $G = S^1$ *and* $m = 2$

(2) $(d+1)m > k$,

holds, then $I(\Lambda\mathbf{P}^m \times N; \{\text{pt.}\} \times N) = \Theta_V^G$.

Corollary 1.7. *If* $(d+1)m > k \geq 5$ *then there is a* 1*-connected* G*-manifold* N^k *such that* $I^G(\Lambda\mathbf{P}^m \times N^k; \{\text{pt.}\} \times N^k) = \Theta_V^G$.

Proof that 1.6 implies 1.7. The results of Winkelnkemper [Wi1–3] prove the existence of a closed 1-connected manifold N^k such that $I(N^k) = \Theta_k$ (*Note:* The subsequent proof in [Kr, §11C, pp. 108–109] is closely related to the argument proving Theorem 1.3). Since we are assuming condition (2) in Corollary 1.6, we can apply the latter to show that $I^G(\Lambda\mathbf{P}^m \times N^k; \{\text{pt.}\} \times N^k) = \Theta_V^G$.∎

Proof of Corollary 1.6. By Theorem 1.5 it suffices to show that $\Theta_V^G = \Theta_V^G(N)$. Since $I(N^k) = \Theta_k$ it suffices to verify that the equivariant normal bundle of Σ^G in Σ is equivariantly trivial. If $G = S^1$ and $m = 2$ this follows from [L2]. On the other hand, if $G = S^1$ and $2m > k$ the result appears in [Br2], while if $G = S^3$ and $4m > k$ the result can be found in [MSc1].∎

DERIVATION OF THEOREM 0.1. The results of [BP] show that Γ_V^G is infinite if $G = S^1$ and $V = \mathbb{R}^k \oplus \mathbb{C}m$ where $k \geq 5$ and $m \geq 2$. The statement about finitely many oriented equivariant diffeomorphism types follows from Corollary 1.4, and the existence of examples for which there is only one class follows from Corollary 1.7.∎

2. Large $\mathbb{Z}_p$-equivariant inertia groups

For general choices of G and V one has canonical restriction maps $\mathrm{res}_H : \Gamma_V^G \to \Gamma_{\mathrm{res}(V,H)}^H$ where H is a closed subgroup and $\mathrm{res}(V,H)$ is the restriction of the G-representation to H. By Corollary 1.2 we can view res_H as a homomorphism from Θ_V^G to $\Theta_{\mathrm{res}(V,H)}^{H,s}$ if G and V satisfy the conditions of the corollary and H is a finite subgroup. If $G = S^3$ and $H = S^1$ then results from [BP] imply that $\mathrm{res}(S^1 \to S^3; V) \otimes \mathbb{Q} : \Theta_V^G \otimes \mathbb{Q} \to \Theta_{\mathrm{res}(V,H)}^H \otimes \mathbb{Q}$ is injective.

Suppose now that M is an equivariantly oriented G-manifold such that the component A of the fixed point set M^G has tangential representation V, where $\dim V^G \geq 1$. If H is a closed subgroup of G such that $V^H = V^G$, then A is also a component of M^H and there is an elementary but important relationship between the G- and H-equivariant inertia groups of $(M; A)$.

Proposition 2.1. *In the notation of the preceding paragraph the restriction map* res_H : $\Gamma_V^G \to \Gamma_{\mathrm{res}(V,H)}^H$ *sends* $I^G(M; A)$ *into (but not necessarily onto)* $I^H(M; A)$.■

If G acts semifreely and H is a nontrivial subgroup then the hypothesis on M^H automatically applies. Since the ranks of the S^1-equivariant inertia groups of the semifree S^1-manifolds $N^k \times \mathbb{C}\mathbf{P}^m$ are large by Theorem 1.3, it is natural to consider the implications of Proposition 2.1 in this case if, say, $H = \mathbb{Z}_p$ where p is prime. Results of T. Petrie [P] on the restriction map yield the following conclusion:

Theorem 2.2. *Let $k \geq 5$ be odd, let $m \geq 2$, let p be an odd prime, and assume that $m \geq \frac{1}{2}(p+3)$. If N^k is an arbitrary closed oriented manifold, $H = \mathbb{Z}_p$, and V is the restriction of the linear semifree S^1-action on $\mathbb{R}^k \oplus \mathbb{C}^m$, then the codimension of $I^H(N^k \times \mathbb{C}\mathbf{P}^m; N^k \times \{\mathrm{pt.}\}) \otimes \mathbb{Q}$ in $\Theta_V^{H,s} \otimes \mathbb{Q}$ is*

$$\begin{cases} 0 & \text{if } k \equiv 1 \mod 4 \\ \leq 1 & \text{if } k \equiv 3 \mod 4. \end{cases}$$

Proof. If $k + 1 \neq 2m$ then by [MSc2, Prop. 3.2] it will suffice to show that

$$I^H(N^k \times \mathbb{C}\mathbf{P}^m, N^k \times \{\mathrm{pt.}\}) \otimes \mathbb{Q}$$

contains the image of $\widetilde{L}_{k+2m+1}^s(H) \otimes \mathbb{Q}$. Let $G = S^1$ and consider the following commutative diagram

$$\begin{array}{ccccc} & & hS_{k+1}\left(\mathbb{C}\mathbf{P}^{m-1}\right) & \xrightarrow{\gamma_G} & \Theta_V^G \\ & & \Big\downarrow B^! & & \Big\downarrow \mathrm{res}_H \\ \widetilde{L}_{k+2m+1}^s(H) & \xrightarrow{\widetilde{\Delta}} & shS_{k+1}(L(V_H)) & \xrightarrow{\gamma_H} & \Theta_V^{H,s} \end{array}$$

in which $B^!$ is defined by taking the induced circle bundle whose first Chern class comes from p times the generator in $H^2(\mathbb{C}\mathbf{P}^{m-1}; \mathbb{Z})$. As in [MSc2], V_H denotes the restriction of the free S^1 representation on $\mathbb{C}^m$ and $L(V_H)$ is the lens space $S(V_H)/H$. The results of Petrie [P] show that

$$\mathrm{Image}\, B^! \otimes \mathbb{Q} \supseteq \mathrm{Image}\, \widetilde{\Delta} \otimes \mathbb{Q}$$

if $2m \geq p + 3$. By Theorem 1.3 and Proposition 2.1 we know that the image of $\mathrm{res}_H \otimes \mathbb{Q}$ lies in $I^H(N \times \mathbb{C}\mathbf{P}^m; N \times \{\mathrm{pt.}\}) \otimes \mathbb{Q}$, and therefore a diagram chase shows that the image of $\gamma_H \widetilde{\Delta} \otimes \mathbb{Q}$ is also contained in $I^H(N \times \mathbb{C}\mathbf{P}^m; N \times \{\text{pt.}\}) \otimes \mathbb{Q}$. This completes the proof if $k+1 \neq 2m$.

Suppose now that $k+1 = 2m$. As noted in [MSc2, §3] the exact sequence in the bottom row of the commutative diagram extends to the right with a term of the form $\pi_k(F_H(V_H), C_H(V_H)) \oplus \Theta_k$ (notation as in [MSc2, §3]), and the proof of [MSc2, Prop. 3.2] is based on the finiteness of $\pi_k(F_H(V_H), C_H(V_H))$ when $k+1 \neq 2m$. In particular, if $\pi_k(F_H(V_H), C_H(V_H))$ is finite whenever $V_H = V/V^H$ is $\mathbb{C}^m$ and $k = 2m-1$, then one can use the method of [MSc2, Prop. 3.2] to obtain the same conclusion as before; note that $C_H(V_H)$ is the unitary group U_m in this case. Consider the following commutative diagram, in which the horizontal arrows are given by evaluation at the basepoint:

$$\begin{array}{ccc}
\pi_{2m-1}(U_m) & \longrightarrow & \pi_{2m-1}(F_H(\mathbb{C}^m)) \\
\Big\downarrow {\scriptstyle e_*} & & \Big\downarrow {\scriptstyle e'_*} \\
\pi_{2m-1}(S^{2m-1}) & \longrightarrow & \pi_{2m-1}(S^{2m-1})
\end{array}$$

The results of [Sc2] imply that $e'_* \otimes \mathbb{Q}$ is bijective, and basic results on the homotopy theory of unitary groups also show that $e_* \otimes \mathbb{Q}$ is also bijective. Therefore $\pi_k(F_H(V_H), C_H(V_H))$ is finite, and by the remarks at the beginning of this paragraph this implies the theorem in the exceptional case $k+1 = 2m$.■

Corollary 2.3. *In the setting above, if* $k \equiv 3 \mod 4$ *then there is an equivariantly* 1*-connected* V*-framable manifold* M *such that*

$$I^H(N \times \mathbb{C}\mathbf{P}^m \# M; N \times \{\text{pt.}\} \# M^H)$$

has finite index in $\Theta_V^{H,s}$.

Recall that a semifree G-space X is equivariantly 1-connected if both X and X^G are 1-connected; see [MSc2, paragraph preceding Prop. 4.8] for the concept of V-framed manifold.

Sketch of proof. Let M be an example with a connected fixed point set given by [MSc2, Prop. 5.1], so that $I^H(M; M^H) \otimes \mathbb{Q}$ is not contained in the image of $T_V \otimes \mathbb{Q}$, where $T_V = \gamma_H \circ \widetilde{\Delta}$ in the first paragraph of the proof of Theorem 2.2. Since $I^H(N \times \mathbb{C}\mathbf{P}^m; N \times \{\text{pt.}\}) \otimes \mathbb{Q}$ contains the latter, one can use the elementary identity

$$I^H(M \# N; A \# B) \supseteq I^H(M; A) + I^H(N; B)$$

to show that $I^H(N \times \mathbb{C}\mathbf{P}^m \# M; N \times \{\text{pt.}\} \# M^H)$ has finite index in $\Theta_V^{H,s}$.■

Examples with other tangential representations. Several features of the examples in Theorem 2.2 lead naturally to further questions. In particular, since the tangential representations at fixed points are restrictions of semifree S^1-representations, one can ask whether similar examples exist for other $\mathbb{Z}_p$-representations. Furthermore, since one component of the fixed point set has codimension 2, one can ask whether similar examples exist for which the Standard Gap Hypothesis holds. In a related direction, since the fixed point sets consist of two components, one can ask whether similar examples exist with connected fixed point sets.

We shall construct examples in such cases by combining Theorem 2.2 with results of T. Yoshida [Yo] (and subsequent observations in [DS,§III.1] and [Ya]) together with the following idea that is implicit in [RT2, §2]:

Proposition 2.4. *Let $G = \mathbb{Z}_p$ where p is an odd prime, and let M be a closed 1-connected smooth G-manifold such that M is odd dimensional, each component of M^G has dimension ≥ 5, the action is effective, and $M - M^G$ is simply connected. Let V be the tangential representation at some component A of M^G for which $\dim V - \dim V^G \geq 4$. Then there is a commutative diagram of the form*

$$\begin{tikzcd}
\widetilde{L}^s_{n+1}(G) \arrow[r, "j"] \arrow[d, "="'] & L^{s,BQ}_{n+1}(D(V),S(V)) \arrow[r, "c^!_A"] \arrow[d, "\Delta_0(V)"'] & L^{s,BQ}_{n+1}(M) \arrow[d, "\Delta_0(M)"'] \\
\widetilde{L}^s_{n+1}(G) \arrow[r, "T'_V"] \arrow[d, "="'] & \mathcal{S}^{s,BQ}_G(D(V),S(V)) \arrow[r, "b^!_A"] \arrow[d, "F"'] & \mathcal{S}^{s,BQ}_G(M) \arrow[d, "\mathrm{CLS}"'] \\
\widetilde{L}^s_{n+1}(G) \arrow[r, "T_V"] & \Theta^{G,s}_V \arrow[r, "(M;A)\#"] & \mathfrak{D}_+(M)
\end{tikzcd}$$

in which

1. *j is the map in the Orbit Sequence for $L^{s,BQ}_{n+1}(D(V),S(V))$,*
2. *$c^!_A$ and $b^!_A$ are maps from the Browder-Quinn simple surgery sequence for the pair $(D(V),S(V))$ to the corresponding sequence for M that are given by modifying a structure on a closed linear disk neighborhood of a point in A,*
3. *$\Delta_0(V)$ and $\Delta_0(M)$ are maps in the appropriate Browder-Quinn surgery sequences,*
4. *T_V is the composite described in the proof of Corollary 2.3,*
5. *F is a forgetful ($\equiv$ restriction) homomorphism as in Section 1,*
6. *$\mathfrak{D}_+(M)$ denotes the set of G-oriented equivariant diffeomorphism classes of closed smooth G-manifolds that are G-simple homotopy equivalent to M,*
7. CLS *takes a Browder-Quinn structure to its oriented equivariant diffeomorphism class,*
8. *$(M;A)\#$ denotes the connected sum construction.*

Sketch of proof. For each subsequence the proof of commutativity is a fairly straightforward exercise. The composite $c^!_A \circ j$ is merely the canonical map in the orbit sequence for $L^{s,BQ}_{n+1}(M)$ because $M - M^G$ is 1-connected, and therefore this map does not depend on A; in other words, if X is another component of M^G then $c^!_A \circ j = c^!_X \circ j$.■

The necessary input from [Yo] can be stated in a purely formal manner.

Theorem 2.5. *Let $G = \mathbb{Z}_p$ where p is an odd prime, and let M be a closed 1-connected smooth G-manifold such that M is odd dimensional, each component of M^G has dimension ≥ 5, the action is effective, and $M - M^G$ is simply connected. Let V be the tangential representation at some component A of M^G for which $\dim V - \dim V^G \geq 4$. Let P be a closed 1-connected smooth G-manifold of dimension $4\ell > 0$ such that $\dim P - \dim P^G \geq 4$ and the equivariant Witt invariant of P is equal to that of the unit form ($\mathbb{Z}$, mult.) in the Witt ring of $\mathbb{Z}[G]$. If B is a component of P with tangential representation W and the image of $T_V \otimes \mathbb{Q}$ lies in $I^G(M;A) \otimes \mathbb{Q}$, then the image of $T_{V\oplus W} \otimes \mathbb{Q}$ lies in $I^G(M \times P; A \times B) \otimes \mathbb{Q}$.*

As in [Yo], the equivariant Witt ring $W_+(\mathbb{Z}[G])$ is given by considering all $\pm$unimodular quadratic forms on torsion free finitely generated abelian groups, and then factoring out by metabolic forms (M, φ) that contain a submodule K with rank equal to half that of M such that the form is zero on K. Direct sum and tensor product make this into a commutative ring with unit, and the multiplicative unit is given by $(\mathbb{Z},\ \text{mult.})$

Proof. Note first that the hypotheses of Proposition 2.4 for $(M; A)$ imply the corresponding properties for $(M \times P; A \times B)$ provided we replace V by $V \oplus W$. This yields a commutative diagram as in 2.4 with $n + 4\ell$ replacing n, $V \oplus W$ replacing V, and $(M \times P; A \times B)$ replacing $(M; A)$. The methods of [BQ] yield the following commutative diagram in which μ is the Yoshida twisted product map for P as defined in [Yo]:

$$\begin{array}{ccc}
\widetilde{L}^s_{n+1}(G) & \xrightarrow{\ \mu\ } & \widetilde{L}^s_{n+4\ell+1}(G) \\
\downarrow & & \downarrow \\
L^{s,BQ}_{n+1}(M) & \xrightarrow{\ \times P\ } & L^{s,BQ}_{n+4\ell+1}(M \times P) \\
\downarrow & & \downarrow \\
\mathcal{S}^{s,BQ}_G(M) & \xrightarrow{\ \times P\ } & \mathcal{S}^{s,BQ}_G(M \times P)
\end{array}$$

Since the Witt invariant of P is the unit, the main result of [Yo] implies that μ is an isomorphism.

By hypothesis there is a subgroup $E_V \subset \widetilde{L}^s_{n+1}(G)$ of finite index such that $T_V(E_V)$ lies in $I^G(M; A)$. If $\alpha \in E_V$ let Σ_V represent $T'_V(\alpha) \in \mathcal{S}^{s,BQ}_G(D(V), S(V))$ and let $\Sigma_{V\oplus W}$ represent $T'_{V\oplus W}(\alpha) \in \mathcal{S}^{s,BQ}_G(D(V \oplus W), S(V \oplus W))$. A chase of the diagram above and the related diagrams from Proposition 2.4 (for both M and $M \times P$) shows that $(M\#\Sigma_V) \times P$ is G-orientation preservingly equivariantly diffeomorphic to $(M \times P)\#\Sigma_{V\oplus W}$ and that one has an equivariant diffeomorphism of this type that sends $(A\#\Sigma^G_V) \times B$ to $(A \times B)\#\Sigma^G_{V\oplus W}$. Let

$$f_1 : (M\#\Sigma_V) \times P \longrightarrow (M \times P)\#\Sigma_{V\oplus W}$$

be such a map. Since $[\Sigma_V] \in I^G(M; A)$ by our hypotheses, there is also an equivariant diffeomorphism $f_2 : (M; A)\#\Sigma_V \to M$ such that $f_2(A\#\Sigma^G_V) = A$. If we define h to be the composite $(f_2 \times \mathrm{id}_P) \circ f_1^{-1}$, then it follows immediately that h defines a diffeomorphism from $(M \times P; A \times B)\#\Sigma_{V\oplus W}$ to $M \times P$ such that $(A \times B)\#\Sigma^G_{V\oplus W}$ corresponds to $A \times B$. Since μ is bijective it follows that $\mu(E_V)$ also has finite index in $\widetilde{L}^s_{n+4\ell+1}(G) \approx \widetilde{L}^s_{n+1}(G)$; if we combine these observations we see that the subgroup $I^G(M \times P; A \times B) \otimes \mathbb{Q}$ contains the image of $T_{V\oplus W} \otimes \mathbb{Q}$.■

Here are a few examples for Theorem 2.5 that are important for our purposes. More precisely, we shall give examples of closed 1-connected smooth G-manifolds of dimension $4\ell > 0$ such that $\dim P - \dim P^G \geq 4$ and the equivariant Witt invariant of P is equal to that of the unit form $(\mathbb{Z},\ \text{mult.})$ in the Witt ring of $\mathbb{Z}[G]$.

Example 2.6.1. If X is any smooth G-manifold, then as in [DS] one can construct a smooth G-manifold $X\!\uparrow\!G$ by taking the product of $|G|$ = order of G copies of X with itself and letting G act by permuting the coordinates cyclically. If $P = \mathbb{C}\mathbf{P}^2\!\uparrow\!G$ then Theorem 2.5 applies to P by the results of [DS, §III.1] or [Ya]. Note that P^G is the diagonal copy of $\mathbb{C}\mathbf{P}^2$. This gives an example for $s = |G|$; one can obtain examples for $\ell = t \cdot |G| > |G|$ by using other examples of periodicity manifolds from [DS], or more simply one can just take a product of t copies of P.

Example 2.6.2. Let $|\mathbb{C}|$ denote the trivial representation on $\mathbb{C}$, let Ω be a 2ℓ-dimensional complex G-representation, and consider the G-manifold $\mathbb{C}\mathbf{P}(\Omega)$ on $\mathbb{C}\mathbf{P}^{2\ell} = S(\Omega \oplus |\mathbb{C}|)/S^1$ given by factoring on the scalar multiplication of S^1 from the linear G-action on $S(\Omega\oplus|\mathbb{C}|)$.— In general $\mathbb{C}\mathbf{P}(\Omega)^G$ has many components, one of which is given by $\mathbb{C}\mathbf{P}(\Omega^G)$. The tangential representation for the latter is merely Ω, and because of this we shall call $\mathbb{C}\mathbf{P}(\Omega^G)$ the *principal component* of the fixed point set.

Example 2.6.3. Let $\mathcal{W}$ be the 3-dimensional complex representation given by

$$g \cdot (z_1, z_2, z_3) = (g^a z_1, g^b z_2, g^c z_3)$$

where a, b, c are distinct integers between 0 and p (recall that p is prime, so they are automatically relatively prime to p). Then the $\mathbb{Z}_p$-action on $S(\mathcal{W})$ determines a well defined action on the quotient $\mathbb{C}\mathbf{P}^2 = S(\mathcal{W})/S^1$ that we shall call $\mathbb{C}\mathbf{P}^2(a, b, c)$. This action has three isolated fixed points, and the tangential representations at these fixed points are $\mathbb{R}$-equivalent to direct sums of irreducible representations on $\mathbb{C}$ of the form $(g, a) \to g^r z$, where r lies in the set $\{b - a, c - a, b - c\}$.

Realizing tangential representations. One can use Theorems 2.2 and 2.5 to construct pairs

$$(M; A = \text{ component of } M^G)$$

with large equivariant inertia groups and more or less arbitrary normal representations at A; by definition, the normal representation is the nontrivial part of the tangential representation. As before let $G = \mathbb{Z}_p$ where p is an odd prime.

Notational convention. If r is an integer that is relatively prime to p (an odd prime by assumption) and Φ is a $\mathbb{Z}_p$-action on an object Y, define a new action $\psi^r\Phi$ on Y by the formula

$$\psi^r\Phi(g, y) = \Phi(g^r, y).$$

It follows immediately that the fixed point sets of the new and old action are equal; furthermore, if Φ is a smooth action on a manifold then so is $\psi^r\Phi$. This construction merely yields the usual Adams operation if Y is a linear representation.

Theorem 2.7. *Let $G = \mathbb{Z}_p$ where p is an odd prime, let the irreducible unitary representations Ω_r $\left(1 \le r \le \frac{p-1}{2}\right)$ be given by $(g, v) \longrightarrow g^r \cdot v$, and let V be an odd-dimensional G-representation of the form*

$$\mathbb{R}^k \oplus \sum m_r\Omega_r$$

$$(\textit{trivial action on } \mathbb{R}^k)$$

such that $m_r \ge \frac{p+3}{2}$ for at least one choice of r and the number of r for which m_r is odd is less than $\frac{k}{2}$. Then there is a closed oriented smooth G-manifold M such that M^G has a component A with tangential representation V and equivariant inertia group $I^G(M; A)$ satisfying $I^G(M; A) \otimes \mathbb{Q} \supset \mathrm{Image}\, T_V \otimes \mathbb{Q}$.

Note that Theorem 2.7 automatically applies if $k \ge p$ and $\dim V - \dim V^G \ge \frac{p^2+1}{2}$, for if V does not contain $\frac{p+3}{2}$ summands of any irreducible representation then V contains at most

$\frac{p+1}{2}$ summands of each irreducible type. Since there are $\frac{p-1}{2}$ different types this means that the decomposition of V_G into irreducible summands has at most $\frac{p^2-1}{4}$ irreducible factors, and since each nontrivial irreducible factor is 2-dimensional this implies that $\dim V_G \leq \frac{p^2-1}{2}$.

Proof of Theorem 2.7. Choose a specific r_0 such that $m_{r_0} \geq \frac{p+3}{2}$ and define

$$\varepsilon_r(V, r_0) = \begin{cases} 0 & \text{if } r = r_0 \\ 0 & \text{if } r \neq r_0 \text{ and } m_r \text{ is even} \\ 1 & \text{if } r \neq r_0 \text{ and } m_r \text{ is odd.} \end{cases}$$

Define $\ell(k, V, r_0) = k - 2\sum_r \varepsilon_r(V, r_0)$; for the sake of notational simplicity we shall call this ℓ. Form the product manifold

$$N^\ell \times \prod_r \mathbb{C}\mathbf{P}\left(m_r\Omega_r + \varepsilon_r(V, r_0)\,|\mathbb{C}|\right)$$

where N^ℓ is an arbitrary closed oriented smooth manifold (with trivial $\mathbb{Z}_p$-action). If A is the product of N^ℓ with the principal components of the fixed sets of the G-manifolds $\mathbb{C}\mathbf{P}(m_r\Omega_r + \varepsilon_r(V, r_0)|\mathbb{C}|)$ then the tangential representation at A is equivalent to V.

Let $W = \mathbb{R}^\ell \oplus m_{r_0}\Omega_{r_0}$, and let $U = \mathbb{R}^\ell \oplus m_{r_0}\Omega_1$. By construction $W = \psi^*_{r_0} U$, and similarly $N^\ell \times \mathbb{C}\mathbf{P}(m_{r_0}\Omega_{r_0})$ is equal to the smooth G-manifold $\psi^*_{r_0}\left(N^\ell \times \mathbb{C}\mathbf{P}(m_{r_0}\Omega_1)\right)$. By Theorem 2.2 we know that $I^G(N^\ell \times \mathbb{C}\mathbf{P}(m_{r_0}\Omega_1); N^\ell \times \{\text{pt.}\}) \otimes \mathbb{Q}$ contains $(\text{Image}\, T_V) \otimes \mathbb{Q}$, and therefore the elementary properties

(1) $M_1 \cong_G M_2 \Rightarrow \psi^r M_1 \cong_G \psi^r M_2$,
(2) $(\psi^r M_1; A_1) \# (\psi^r M_2; A_2) \cong_G (\psi^r M_1 \# \psi^r M_2; A_1 \# A_2)$,

imply that $I^G(N^\ell \times \mathbb{C}\mathbf{P}(m_{r_0}\Omega_{r_0}); N^\ell \times \{\text{pt.}\}) \otimes \mathbb{Q}$ contains $(\text{Image}\, T_V) \otimes \mathbb{Q}$. Repeated applications of Theorem 2.5 to the examples of 2.6.2 now implies that $I^G(M; A) \otimes \mathbb{Q}$ contains $\text{image}\, T_V \otimes \mathbb{Q}$.■

Realizing the Standard Gap Hypothesis. In the examples of Theorem 2.7 the Standard Gap Hypothesis (*e.g*, see [DS]) may hold for some components of M^G, but usually there are also components for which this condition does not hold. There are several ways of modifying the preceding examples to realize the Standard Gap Hypothesis. In particular, the results of [DS, §§III.1–2] suggest the following:

Theorem 2.8. *Let $G = \mathbb{Z}_p$ where p is an odd prime, and let V be an odd dimensional G-representation. Then for all sufficiently large values of t there is a closed oriented smooth G-manifold M_t such that M_t^G has a component A_t with tangential representation $V \oplus 4t\mathbb{R}[G]$ (where $\mathbb{R}[G]$ is the regular representation) and equivariant inertia group $I^G(M_t; A_t)$ satisfying $I^G(M_t; A_t) \otimes \mathbb{Q} \supset \text{Image}\, T_{V \oplus 4t\mathbb{R}[G]} \otimes \mathbb{Q}$.*

Proof. For all sufficiently large values of t the fixed point set of $V \oplus 4t\mathbb{R}[G]$ is at least 5-dimensional and every nontrivial irreducible G-rerpresentation has multiplicity $\geq \frac{p+3}{2}$ in $V \oplus 4t\mathbb{R}[G]$. Let s_0 be a specific value of this type, and use Theorem 2.7 to construct $(M; A)$ so that the tangential representation at A is $V \oplus 4s_0\mathbb{R}[G]$. By [DS, Thm. III.3.4, p. 94] there is a positive integer $s_1 \geq s_0$ such that $t \geq s_1$ implies that $M \times (\mathbb{C}\mathbf{P}^2 {\uparrow} G)^t$ satisfies the Standard Gap Hypothesis, and by Theorem 2.5 and Example 2.6.1 we then have

$$I^G(M \times (\mathbb{C}\mathbf{P}^2 {\uparrow} G)^t; A \times (\mathbb{C}\mathbf{P}^2)^t) \otimes \mathbb{Q} \supset \text{Image}\, T_{V \oplus t\mathbb{R}[G]} \otimes \mathbb{Q}$$

for all $t > 0$. Therefore one has examples of the desired type provided $t \geq s_1$.■

The preceding construction has an important disadvantage; namely, as t increases the multiplicity of every irreducible representation in $V \oplus 4t\mathbb{R}[G]$ also increases linearly with t. We would like to have examples of representations V_t such that $\dim V_t \to \infty$ but the multiplicities of most irreducible representations in V_t do not change. This can be done if one is willing to sacrifice one feature of the examples in Theorem 2.8. For the examples constructed in that proof the number of components of M^G is constant as $t \to \infty$, and the following result shows that one can exchange control over the size of $\pi_0(M^G)$ for control over the multiplicities over most of the irreducible representations that arise as summands of V_t.

Theorem 2.9. *Let $G = \mathbb{Z}_p$ where $p \geq 7$ is prime. Then in all sufficiently large odd dimensions n there are closed smooth 1-connected G-manifolds M with the following properties:*

1. *All components of M^G are simply connected and at least 5-dimensional.*
2. *The Standard Gap Hypothesis holds.*
3. *If $\mathcal{F}$ is a finite set of G-representations that consists of the tangential representations from all components of M^G, then there are at most three inequivalent nontrivial irreducible representations of G that have nontrivial multiplicities in some representation $V \in \mathcal{F}$.*
4. *If A is an arbitrary component of M^G and V is the tangential representations at points of A, then $I^G(M; A) \otimes \mathbb{Q}$ contains the image of $T_V \otimes \mathbb{Q}$.*

Proof. Take one of the G-manifolds $S^k \times \mathbb{C}\mathbf{P}^m$ from Theorem 2.2 and form the product

$$X_r = S^k \times \mathbb{C}\mathbf{P}(m\Omega_1) \times \mathbb{C}\mathbf{P}^2(1,2,3)^r$$

for an arbitrary integer $r \geq 1$. If $k \geq 5$ it follows immediately that (1) holds, and the construction also implies that (3) holds. To prove that the Standard Gap Hypothesis holds if r is sufficiently large, notice that all components of X_r^G have dimensions equal to k or $k + 2m - 2$ and that $\dim X_r = k + 2m + 4r$; this implies the Standard Gap Hypothesis provided

$$r \geq \frac{k + 2m - 1}{4}.$$

Thus the manifolds satisfy (1)–(3), and in addition their dimensions are all sufficiently large integers congruent to $k + 2m \mod 4$ if r satisfies the inequality. Thus everything reduces to proving that (4) holds for the manifolds X_r.

Let p_0 be one of the three isolated fixed points in $\mathbb{C}\mathbf{P}^2(1,2,3)$, and let W be the tangential representation at the component $S^k \times \{p_0\}^r \subset X_r$. If we apply Theorems 2.2 and 2.5 to Example 2.6.3, it follows that $I^G(X_r; S^k \times \{p_0\}^r) \otimes \mathbb{Q}$ contains $\operatorname{Image} T_W \otimes \mathbb{Q}$. This proves (4) for at least one component of X_r^G. The following result will allow us to extend this to other components.

Lemma 2.10. *Let M be a closed smooth 1-connected G-manifold, where $G = \mathbb{Z}_p$ ($p \geq 3$ prime), and suppose that each component of M^G is at least 5-dimensional with $\dim M - \dim M^G \geq 4$. Let A and B be components of M^G with tangential representations V and W*

respectively. Then the following diagram is commutative:

$$\begin{array}{ccc}
\widetilde{L}^s_{n+1}(G) & \xrightarrow{T'_V} & \mathcal{S}^{s,BQ}_G(D(V),S(V)) \\
{\scriptstyle T'_W}\big\downarrow & & \big\downarrow{\scriptstyle (M;A)\#} \\
\mathcal{S}^{s,BQ}_G(D(W),S(W)) & \xrightarrow{(M;B)\#} & \mathcal{S}^{s,BQ}_G(M)
\end{array}$$

Proof that Lemma 2.10 implies Theorem 2.9. Let W be given as in the paragraph before the statement of Lemma 2.10. Then the validity of (4) for $B = S^k \times \{p_0\}^r$ implies that $\widetilde{L}^s_{n+1}(G)$ contains a subgroup E_W of finite index such that $T_W(E_W) \subset I^G(M;B)$.

Let $I^G_0(M;B)$ is the subgroup of $I^G(M;B)$ described in [MSc2, Prop. 2.4], which consists of all Σ for which there is an oriented equivariant diffeomorphism $M\#_x\Sigma \to M$ that induces the same map of fixed point components as a certain canonical equivariant homeomorphism $M\#_x\Sigma \to M$. By the previously cited result and the finite generation of $\Theta^{G,s}_W$ there is a subgroup $E'_W \subset E_W$ of finite index such that $T_W(E'_W) \subset I^G_0(M;B)$.

Let $\alpha \in E'_W$ be arbitrary, let Σ_W represent $T'_W(\alpha) \in \mathcal{S}^{s,BQ}_G(D(W),S(W))$, let Σ_V represent $T'_V(\alpha) \in \mathcal{S}^{s,BQ}_G(D(V),S(V))$, and let $t(\Sigma_W)$ and $t(\Sigma_V)$ be canonical almost diffeomorphisms as defined in [MSc2, §2]. By Lemma 2.10 there is a G-orientation preserving equivariant diffeomorphism

$$f : (M;B)\#\Sigma_W \to (M;A)\#\Sigma_V$$

such that $t(\Sigma_W)$ and $t(\Sigma_V)\circ f$ induce the same map from $\pi_0\left(((M;B)\#\Sigma_W)^G\right)$ to $\pi_0(M^G)$.

On the other hand, since $T_W(\alpha)$ lies in $I^G_0(M;B)$ there is a G-orientation preserving equivariant diffeomorphism $h : (M;B)\,\#\Sigma_W \to M$ such that $t(\Sigma_W)$ and h induce the same map from $\pi_0\left(((M;B)\,\#\Sigma_W)^G\right)$ to $\pi_0(M^G)$. Combining these, we obtain a G-orientation preserving equivariant diffeomorphism $h\circ f^{-1} : (M;A)\,\#\Sigma_V \to M$ such that $t(\Sigma_V)$ and $h\circ f^{-1}$ induce the same map from $\pi_0\left(((M;A)\#\Sigma_V)^G\right)$ to $\pi_0(M^G)$; the last assertion follows from a diagram chase. Therefore $T_V(E'_W)$ is contained in $I^G_0(M;A)$.

Finally, since E'_W has finite index in E_W and $I^G_0(M;A)$ has finite index in $I^G(M;A)$ [MSc2, Prop. 2.4], it follows that $I^G(M;A)\otimes\mathbb{Q}$ contains the image of $T_V\otimes\mathbb{Q}$. Thus (4) also holds for every other component of X^G_r.■

Proof of Lemma 2.10. In Proposition 2.4 we constructed commutative diagrams

$$\begin{array}{ccc}
\widetilde{L}^s_{n+1}(G) & \xrightarrow{j_1} & L^{s,BQ}_{n+1}(M) \\
\big\downarrow & & \big\downarrow{\scriptstyle \Delta_0(M)} \\
\mathcal{S}^{s,BQ}_G(D(U),S(U)) & \xrightarrow{(M;C)\#} & \mathcal{S}^{s,BQ}_G(M)
\end{array}$$

in which j_1 comes from the Orbit Sequence for $L^{s,BQ}_*(M)$ and $\{U,C\}$ can denote either $\{W,B\}$ or $\{V,A\}$. The lemma follows by splicing the two diagrams along the composite map $\mathrm{CLS}\circ\Delta_0(M)\circ j_1$.■

Remark. Lemma 2.10 can also be used to extend the conclusions of Theorems 2.7 and 2.9 to other components of the fixed point sets in most cases.

Connected fixed point sets. Since the results of [RT2] and [MSc2] on classification up to finite ambiguity assume that the G-manifolds in question have connected fixed point sets, the examples of Theorems 2.7–2.8 and 3.4.1–3.4.2 are not included in the setting of [RT2, bottom of p. 762]. However, results of B. Fine [F] and constructions of Dovermann and Rothenberg [DR] strongly suggest that the program of [RT2] goes through if one merely assumes that all fixed point sets M^H are unions of 1-connected manifolds, where H runs through all subgroups of G except $\{1\}$; in particular, it should be possible to extend the equivariant simple rational homotopy theory of [RT1] to objects with disconnected fixed point sets using the combinatorial machinery of [DR].

Of course, given an odd-dimensional G-representation V (as usual, $G = \mathbb{Z}_p$), it would also be enlightening to have examples of closed, equivariantly 1-connected smooth G-manifolds M such that V is equivalent to the tangent space at fixed points and $I^G(M; M^G)$ has finite index in Γ^G_V. In general this appears to be difficult, and the ultimate solution seems likely to require an understanding of cobordism of equivariant diffeomorphisms along the lines of [Kr]. On the other hand, if V is an odd dimensional G-representation such that $k = \dim V^G$ and every nontrivial irreducible representation has multiplicity at least $\frac{k+1}{2}$ in V, then the reduced G-signature defect for the set of equivariantly V-framable G-manifolds (*cf.* [MSc2, definition before Prop. 4.8]) is trivial [MSc2, (1.1) and Cor. 1.2B]. In such cases [MSc2, Prop. 3.2] implies that $\Gamma^G_V \otimes \mathbb{Q}$ has dimension 0 or 1 depending on whether k is congruent to 1 or 3 mod 4. It follows that in such cases $I^G(M; M^G)$ has finite index in Γ^G_V if $k \equiv 1 \mod 4$; furthermore, if $k \equiv 3 \mod 4$ and the orientation class of M^G maps to zero in $H_k(M; \mathbb{Q})$, then [MSc2, Thm. 4.6] implies that $I^G(M; M^G)$ likewise has finite index in Γ^G_V. For smooth G-manifolds satisfying these conditions and the other assumptions of [RT2] there are only finitely many equivariantly oriented diffeomorphism classes of smooth G-manifolds that are equivariantly orientation preservingly almost diffeomorphic to a given example, and therefore in such cases the invariants of [RT2] are a complete set of oriented equivariant diffeomorphism classification invariants up to finite ambiguity.

3. Implications for generalized Atiyah-Singer invariants

As indicated in the introduction, the examples of this paper show that one cannot form a nontrivial theory of generalized Atiyah-Singer invariants unless some restrictions are placed on the class of manifolds under consideration (see Theorem 3.1); in fact, this is true even when the value group for the invariants of [MSc2, §1] is highly nontrivial. The crucial difference between the examples of Section 2 and the class of manifolds treated in [MSc2] is that the former have nontrivial rational characteristic classes while the latter have trivial rational characteristic classes if one restricts to invariant tubular neighborhoods of the fixed point set.

The precise statements of results require some notation from [MSc2]. If G is a finite group then $\mathcal{R}(G)$ is the ring of complex valued functions on $G - \{1\}$, and $\mathcal{R}_\pm(G)$ is the complex subspace of functions satisfying $f(g^{-1}) = \pm f(g)$. Suppose now that $G = \mathbb{Z}_p$ where p is an odd prime, and let V be an odd-dimensional G-representation. Then $\mathbf{J}(V) \subset \mathcal{R}_\pm(G)$ is the complex subspace spanned by the equivariant signature defects of the sphere $S(V \oplus \mathbb{R})$ with all possible choices of partial equivariant framings (complete descriptions of these notions appear in [MSc2, §1]). If $\mathcal{F}$ is a family of odd-dimensional G-representations with the same dimension, then $\mathbf{J}(\mathcal{F})$ is the sum of the subspaces $\mathbf{J}(V)$ over all $V \in \mathcal{F}$.

The following result provides an abstract nonexistence criterion for generalized Atiyah-Singer invariants on certain families of compact smooth G-manifolds; we shall use the examples of Section 2 to show that the abstract condition holds in several different cases.

Theorem 3.1. *Let $G = \mathbb{Z}_p$ where p is an odd prime, let $\mathcal{F}$ be a finite family of n-dimensional G-representations where n is odd, and let $\mathcal{M}$ be a family of closed oriented smooth G-manifolds such that $\mathcal{F}$ consists of the tangent space representations at fixed points for all manifolds representing classes in $\mathcal{M}$. Suppose further that $\mathcal{M}$ contains a closed smooth G-manifold N such that $I^G(N;A)$ has finite index in Γ_U^G, where A is a component of N^G and U is the tangent space representation at points of A. Then there is no proper subspace $J^* \subset \mathcal{R}_\pm(G)$ for which one can define an invariant of Atiyah-Singer type from $\mathcal{M}$ to $\mathcal{R}_\pm(G)/J^*$.*

As in [MSc2, (1.3.A)–(1.3.B)] an invariant of Atiyah-Singer type is understood to be additive with respect to disjoint unions and connected sums along fixed point sets and to satisfy the following additional condition: *If $M_0 \in \mathcal{M}$ and $(W; M_0, M_1)$ is an equivariantly oriented G-cobordism such that the fixed point sets satisfy $W^G \approx M_0^G \times I$, then $f(M_1) = f(M_0) + \widetilde{\mathrm{sgn}_G}(W)$, where $\widetilde{\mathrm{sgn}_G}(W)$ denotes the image of the G-signature mod J^*.*

Proof. Let $G = \mathbb{Z}_p$ and suppose that one has an invariant of the type described. If for each $V \in \mathcal{F}$ the linear sphere $S(V \oplus \mathbb{R})$ belongs to $\mathcal{M}$ then by [MSc2, paragraph following Proposition 1.1] we know that J^* contains $\mathbf{J}(\mathcal{F})$; in fact, an elaboration of this argument implies the same conclusion even if these spheres do not belong to $\mathcal{F}$ provided each $V \in \mathcal{F}$ is equivalent to a tangent space representation for some manifold in $\mathcal{M}$. Since the latter is assumed, it follows that J^* contains $\mathbf{J}(\mathcal{F})$ in all cases.

Let N, A and U be given as in the statement of the theorem. The assumption on $I^G(N;A)$ and the results of [MSc2, §3] imply the existence of classes $[\Sigma_1], \cdots, [\Sigma_r]$ in $\Theta_U^{G,s}$ such that their reduced G-signature defects span $\mathcal{R}_\pm(\mathbb{Z}_p)/\mathbf{J}(U)$ over the complex numbers and $[\Sigma_j] \in I^G(N;A)$ for all j.

If f is the invariant of Atiyah-Singer type whose existence is assumed, then

$$N \# \Sigma_j \cong N \qquad \text{(all } j\text{)}$$

implies that

$$f(N) + f(\Sigma_j) = f(N \# \Sigma_j) = f(N)$$

so that $f(\Sigma_j)$ must be trivial in $\mathcal{R}_\pm(G)/J^*$ for all j. On the other hand, since $J^* \supset \mathbf{J}(U)$ it follows that the set $\{f(\Sigma_1), \cdots, f(\Sigma_r)\}$ spans $\mathcal{R}_\pm(G)/J^*$ over the complex numbers, and this in turn implies that the quotient is zero.■

Examples where $\mathcal{R}_\pm(G)/\mathbf{J}(\mathcal{F})$ is large. Of course Theorem 3.1 has nontrivial content only if J^* is a proper subspace of $\mathcal{R}_\pm(G)$ and there are closed equivariantly oriented smooth G-manifolds satisfying the conditions of the theorem. Therefore we shall describe two classes of examples for which $\dim \mathcal{R}_\pm(G)/\mathbf{J}(\mathcal{F}) > 0$. In the second class of examples the Standard Gap Hypothesis holds. In both cases the verification of the dimension inequality is an elementary exercise. As usual $\mathcal{F}$ will denote a finite set of G-representations that consists of tangential representations from all components of M^G where M is given in the context.

(3.2) *In the setting of Theorem 2.7, if only d types of nontrivial irreducible representations appear as direct summands of V, where $d \cdot 2^{d+1} < p-1$ and M is the example constructed for V in the proof of Theorem 2.7, and $\mathcal{F}$ is given as above, then $\mathcal{R}_{\pm}(G)/\mathbf{J}(\mathcal{F})$ is nontrivial.*■

(3.3) *In the setting of Theorem 2.9, if $p-1 > 4 \cdot 3^{r+1}$, and X_r is one of the examples $S^k \times \mathbb{C}\mathbf{P}(m\Omega_1) \times \mathbb{C}\mathbf{P}^2(1,2,3)^r$ constructed in that theorem, and let $\mathcal{F}$ is given as above, Then $\mathcal{R}_{\pm}(G)/\mathbf{J}(\mathcal{F})$ is nontrivial.*■

In (3.3) the Standard Gap Hypothesis holds if r is greater than some linear expression in k and m, so for each choice of k and m this yields an infinite set of primes for which (3.3) applies and the Standard Gap Hypothesis holds.

If we combine the preceding observations with the results of Section 2 (and especially with Examples 2.6.1–2.6.3), we obtain very specific classes of smooth G-manifolds for which $\dim \mathcal{R}_{\pm}(G)/\mathbf{J}(\mathcal{F}) > 0$:

Example 3.4.1. Let $\mathcal{F}$ be a finite family of n-dimensional $\mathbb{Z}_p$-representations that are restrictions of semifree S^1-representations, and suppose that $\frac{p-1}{2}$ is greater than the cardinality of $\mathcal{F}$. Let $\mathcal{M}$ be a family of closed oriented smooth $\mathbb{Z}_p$-manifolds such that $\mathcal{F}$ contains the tangent space representations at fixed points for all manifolds representing classes in $\mathcal{M}$ and $\mathcal{M}$ contains one of the examples $S^k \times \mathbb{C}\mathbf{P}^m$, where $k \geq 5$, $m \geq 2$, and $m \geq \frac{p+3}{2}$. Then by Theorem 2.2 all the conditions of the theorem hold except perhaps the nontriviality of $\mathcal{R}_{\pm}/\mathbf{J}(\mathcal{F})$. To see the latter, observe that the formula for $L(g,\xi)$ shows that $\mathbf{J}(V_\alpha)$ is 1-dimensional if V_α is the restriction of a semifree S^1-representation. Since $\mathbf{J}(\mathcal{F}) = \sum \mathbf{J}(V_\alpha)$ it follows that $\dim \mathbf{J}(\mathcal{F})$ cannot exceed the number of representations in $\mathcal{F}$, and therefore the hypothesis implies that $\mathbf{J}(\mathcal{F})$ is a proper subspace of $\mathcal{R}_{\pm}(G)/\mathbf{J}(\mathcal{F})$.

Example 3.4.2. Suppose that $\mathcal{F}$ satisfies the assumptions of (3.2) or (3.3) and that $\mathcal{M}$ contains the examples described in these results. Then the theorem applies and the subspaces $\mathbf{J}(\mathcal{F})$ are proper subspaces of $\mathcal{R}_{\pm}(G)$.

The results and examples of [MSc2] and this paper lead naturally to the following

Question. *Let M be an equivariantly oriented closed smooth G-manifold, and let $\mathcal{F}$ be a family of representations giving the equivalence classes of all tangent space representations at fixed points of M, let A be a component of M^G, and let V be the tangent space representation at points of A. To what extent can one describe the image of $I^G(M;A)$ in $\mathcal{R}_{\pm}(G)/\mathbf{J}(\mathcal{F})$ in terms of the equivariant characteristic classes of M?*

The techniques of [MSc2] show that the image is zero if all such classes vanish on a tubular neighborhood of the fixed point set, but the examples of this paper show that the image is the entire codomain in some cases where the rational characteristic classes do not vanish. In particular, it would be interesting to know if the image could be a nonzero proper subspace.

4. Nonequivariant inertia groups and semifree circle actions

Theorem 1.3 implies that the conclusion of [Sc6, Prop. 3.5 and Thm. II] in the 9-dimensional case is incorrect, and the purpose of this section is to correct both the statement and the proof in that case. The statements and proofs of the results in [Sc6] for all remaining cases are unaffected.

If $\mathcal{A}_V^G$ denotes the subgroup of Θ_V^G represented by G-manifolds Σ such that $\Sigma^G \cong S^q$ and the equivariant normal bundle of Σ^G in Σ is equivariantly trivial (*cf.* [BP]), then the following result is an immediate consequence of Theorem 1.5:

Proposition 4.1. *Let* $\mathrm{res}_1 : \Theta_V^G \longrightarrow \Theta_{\dim V}$ *be the homomorphism defined by forgetting the group action, where* G *and* V *are given as in Theorem* 1.5. *Then for every closed oriented smooth manifold* N^q *the nonequivariant inertia group* $I(\Lambda\mathbf{P}^m \times N^q)$ *contains* $\mathrm{res}_1(\mathcal{A}_V^G)$.■

In this section we are mainly interested in Γ_V^G when $G = S^1$ and $V = \mathbb{R}^3 \oplus \mathbb{C}^2$; in this case the formal analog of the exact sequence of [MSc2, Prop. 3.7] is the following:

$$\cdots \longrightarrow hS_4(\mathbb{C}\mathbf{P}^2) \xrightarrow{\gamma_G} \Gamma_V^G \longrightarrow \pi_3\left(F_G\left(\mathbb{C}^2\right), U_2\right) \longrightarrow hS_3\left(\mathbb{C}\mathbf{P}^2\right)$$

Strictly speaking the exactness of this sequence does not follow from the statement of the result in [MSc2], but the proof extends to the case under consideration, the most notable exceptions being that Γ_V^G must be viewed as mapping into $\Gamma_3 \oplus \pi_3\left(F_G\left(\mathbb{C}^2\right), U_2\right)$ rather than $\Theta_3 \oplus \pi_3\left(F_G\left(\mathbb{C}^2\right), U_2\right)$ because the fixed point set must be a twisted 3-sphere; since Γ_3 is trivial, it follows that we can forget about it in the exact sequence.

There is an obvious forgetful homomorphism res_1 as noted before, and if $\mathcal{A}_V^G \subseteq \Gamma_V^G$ is defined as in the paragraph preceding Proposition 4.1, then results of [Da, Thm. 3.3, p. 69] (see also [Ms1]) imply that res_1 maps $\mathcal{A}_V^G$ onto Θ_7. If we combine this with Proposition 4.1, we obtain the following conclusion:

Corollary 4.2. *If* N^3 *is a closed oriented* 3*-manifold, then* $I(\mathbb{C}\mathbf{P}^2 \times N^3) = \Theta_7$.■

Correction to [Sc6]. The preceding result is inconsistent with one case of [Sc6, Prop. 3.5 and Thm II], so we shall indicate the repairs needed for the latter. For reasons of space we shall not attempt to discuss the entire background here; needless to say, this is all covered in [Sc6]. We begin by stating the correct version of [Sc6, Prop. 3.5].

Proposition 4.3. *Let* Ψ *be a homotopy self equivalence of* $S^3 \times \mathbb{C}\mathbf{P}^k (k \geq 2)$ *induced by an element of* $\pi_3(F_G(\mathbb{C}^{k+1}))$ *that is not in the image of* $\pi_3(U_{k+1})$. *Then* Ψ *is not homotopic—in fact, not normally bordant—to the identity if* $k \geq 3$. *On the other hand, if* $k = 2$ *and* M^7 *generates* $\Theta_7 \approx \mathbb{Z}_{28}$, *then the composite of* Ψ *with the canonical homotopy equivalence* $f_M : S^3 \times \mathbb{C}\mathbf{P}^2 \# M \longrightarrow S^3 \times \mathbb{C}\mathbf{P}^2$ *is homotopic to a diffeomorphism* $S^3 \times \mathbb{C}\mathbf{P}^2 \# M \longrightarrow S^3 \times \mathbb{C}\mathbf{P}^2$.

As noted in [Sc6] the group $\pi_3(F_G(\mathbb{C}^{k+1}), U_{k+1})$ is isomorphic to $\mathbb{Z}_2$ if $k \geq 2$, and the canonical map from $\pi_3(F_G(\mathbb{C}^{k+1}))$ is split surjective.

Proof of Proposition 4.3. (*Sketch*) The argument of [Sc6] for $k \geq 3$ is correct; the mistake arises in analyzing the case $k = 2$. Corollary 4.2 implies the existence of some homotopy equivalence $\widetilde{h}$ of $S^2 \times \mathbb{C}\mathbf{P}^2$ such that $\widetilde{h} \circ f_M$ is homotopic to a diffeomorphism. On the other hand, a result of L. Taylor implies that f_M itself is not homotopic to a diffeomorphism (*cf.* [Sc6, Thm. 2.1]). Finally, a case by case analysis of all homotopy self equivalences of $S^3 \times \mathbb{C}\mathbf{P}^2$ shows that either $\Psi \circ f_M$ is homotopic to a diffeomorphism or else there is no self equivalence h such that $h \circ f_M$ is homotopic to a diffeomorphism. The only consistent alternative is that $\Psi \circ f_M$ is homotopic to a diffeomorphism.■

This correction forces a corresponding modification of [Sc6, Thm. II]. Before stating the corrected result, we recall that the *Rochlin invariant* (= Eells-Kuiper invariant or μ-invariant) of an integral homology 3-sphere Σ^3 is given by expressing $\Sigma^3 = \partial W^4$ for some parallelizable

4-manifold W^4 and taking the signature of $W \mod 16$ (the signature of W is always divisible by 8; *cf.* [HNK]).

Theorem 4.4. (*i*) *Suppose that $q \geq 5$ is odd and the Rochlin invariant of Σ^3 is nonzero. Then Σ^3 is not the fixed point set of a semifree differentiable S^1-action on a homotopy $(2q+3)$-sphere.*

(*ii*) *If the Rochlin invariant of Σ^3 is nonzero, then Σ^3 is the fixed point set of a semifree differentiable S^1-action on a homotopy 9-sphere. Every such homotopy 9-sphere bounds a spin manifold but does not bound a parallelizable manifold.*

The first statement is identical to [Sc6, Thm. II] for the cases listed, and the proof in [Sc6] is correct as written, so it is only necessary to prove the second part of the theorem.

Proof of 4.4(*ii*)**.** As noted in the proof of [Sc6, (3.1)] there is a cobordism W_1 such that

1. $\partial W_1 \cong \Sigma^3 \times \mathbb{C}\mathbf{P}^2 \coprod S^3 \times \mathbb{C}\mathbf{P}^2 \# M^7$,
2. the inclusions of $\Sigma^3 \times \mathbb{C}\mathbf{P}^2$ and $S^3 \times \mathbb{C}\mathbf{P}^2 \# M^7$ induce isomorphisms in integral homology,
3. W_1 is simply connected.

Let $h : S^3 \times \mathbb{C}\mathbf{P}^2 \# M^7 \to S^3 \times \mathbb{C}\mathbf{P}^2$ be a diffeomorphism homotopic to $\Psi \circ f_M$, as in the previous proposition, let W_1' be the principal S^1 bundle over W_1 whose first Chern class generates $H^2(W_1)$, let h' be an S^1 equivariant diffeomorphism of principal S^1 bundles covering h, and form the smooth semifree S^1-manifold

$$X^9 = \left(\Sigma^3 \times D^6\right) \cup_\varphi W_1' \cup_{h'} \left(D^4 \times S^5\right)$$

where $\varphi : \Sigma^3 \times S^5 \to \partial_0 W_1' = \Sigma^3 \times S^5$ is the identity. It follows immediately that X^9 is a homotopy sphere, and by construction the fixed point set is Σ^3.

To determine the differential structure on the exotic sphere X^9 we shall use the methods of [Sc3]. The latter associates a *knot invariant* to the pair (X^9, semifree S^1-action) with values in $\pi_3(F_G(\mathbb{C}^3), U_3) \cong \mathbb{Z}_2$ and the explicit construction of X^9 shows that the knot invariant is the nontrivial class. By the results of [Sc3,§3] for $G = S^1$ the differential structure on X is given implicitly as follows: Take the suspension of the nontrivial class in $\pi_3(F_G(\mathbb{C}^3))/\operatorname{Image}\pi_3(U_3)$, form its suspension in $\pi_3(F_G(\mathbb{C}^4))/\operatorname{Image}\pi_3(U_4)$ and let Ψ' be the an associated homotopy self equivalence of $S^3 \times \mathbb{C}\mathbf{P}^3$. Then $\Psi' \circ f_X$ is homotopic to a diffeomorphism (in this connection also see [Ms3]). By the proof of [Sc6, (5.7b)] the image of X^9 under the Pontryagin-Thom map $\Theta_9 \to \pi_9^S/\operatorname{Image} J \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the nonzero class represented by $\nu^3 \in \pi_9$. It follows that X^9 bounds a spin manifold but does not bound a parallelizable manifold.

Suppose now that Y^9 is a homotopy sphere supporting a semifree differentiable S^1 action with Σ^3 as its fixed point set, where the Rochlin invariant of Σ^3 is 1. General considerations as in [Sc1] show that Y bounds a spin manifold. It remains to show that Y cannot bound a parallelizable manifold. The first step is to show that the knot invariant cannot be trivial. If it were, then the results of [Sc3, §4] imply that the image of Y under the composite $\Theta_9 \to \pi_9^S/\operatorname{Image} J \subseteq \pi_9(F/O)$ must belong to the image of the map

$$w^* : [\mathbf{S}^4 \mathbb{C}\mathbf{P}^2, F/O] \to [\mathbf{S}^4 S^5, F/O] = \pi_9(F/O)$$

induced by the fourth suspension of the usual orbit space projection $w : S^5 \to \mathbb{C}\mathbf{P}^2$. Since F/O is an infinite loop space and stably w is given by the composite $S^5 \xrightarrow{2\nu} S^2 \subseteq \mathbb{C}\mathbf{P}^2$ (*cf.*

[Sc3, (4.15)]) it follows that the image of ν^3 in $\pi_9(F/O)$ does not lie in the image of w^*. Similarly, if the knot invariant is nontrivial then the images of X^9 and Y^9 in $\pi_9(F/O)$ differ by an element of the image of w^*, and the same computations, show that this difference cannot be ν^3; but this means that Y^9 cannot bound a parallelizable manifold.■

The problem considered above is a special case of an extremely recalcitrant question in surgery theory.

Problem 4.5. *Let $f : X \to X$ be a simple homotopy equivalence on a closed manifold X such that f is normally cobordant to the identity. Is f homotopic to a homeomorphism? In the smooth category, if f is smoothly normally cobordant to the identity, is f homotopic to a diffeomorphism?*

Proposition 4.3 and other known results (*e.g.*, see [CS], [KS]) suggest that the answers to such questions are often unpredictable. In view of the successful use of M. Weiss' visible surgery theory [We] to study problems of this type in [CS] and [KS], it would be enlightening to have an alternate proof of Proposition 4.3 with a similar approach.

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