Cartesian powers of 3-manifolds

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Abstract

Previous results of the authors completely determine when the n-fold self-products of two 3-dimensional lens spaces are diffeomorphic; in particular, if n is odd then the fundamental group determines the diffeomorphism type. We pr4ove that for all other irreducible geometric 3-manifolds with trivial first Betti number, the n-fold products of such manifolds with themselves are homeomorphic for some $n \geq 2$ if and only if the manifolds themselves are homeomorphic and obtain partial results for other cases. The proofs use an assortment of techniques from 3-dimensional topology and group theory.

Key words: irreducible geometric 3-manifold, cartesian power, Hopfian group, large Seifert manifold, Lyndon spectral sequence, Gromov norm, prime decomposition, Kneser Conjecture

1 Introduction

In [22] the authors showed that two 3-dimensional lens spaces L and L' with isomorphic fundamental groups have diffeomorphic cartesian squares; *i.e.*, one has $L \times L \approx L' \times L'$. It should be noted that L and L' do not even have to be homotopy equivalent. The question of the existence of spaces with this

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property was originally formulated by S. Ulam [37], and as indicated in [22], lens spaces are the lowest dimensional compact examples that one can produce.

It is natural to ask what other sorts of 3-dimensional examples of this sort exist. Examples of non-compact (open) 3-manifolds W^3 such that $W^3 \not\approx \mathbb{R}^3$, but $W^3 \times W^3 = \mathbb{R}^6$ have been known for some time (*cf.* Glimm [9], Kwun [19] or McMillan [24]); one such example is the Whitehead 3-manifold that is contractible but not simply connected at infinity [42]. These suggest that one should concentrate on the existence of compact 3-manifolds A, B such that $A \times A$ is homeomorphic to $B \times B$ but A is not homeomorphic to B. The purpose of this paper is to show that no examples of this sort exist in a large class of basic 3-manifolds whose fundamental groups are either infinite or finite but noncyclic.

As in [22] one can also consider the generalization of this question in which cartesian squares are replaced by arbitrary finite powers. To streamline the notation we shall denote the n-fold product of an object A with itself by $\prod^n A$; we shall call this product the **cartesian n−th power** of A. The corresponding problem is then to find all closed 3-manifolds M, N such that $\prod^n M$ is homeomorpic or diffeomorphic to $\prod^n N$ for some integer $n \geq 2$. One step in studying the problem is to introduce a strong property of this type.

Definition 1 *Let X and Y be topological spaces. The pair X, Y is* exponentially stable *if $\prod^n X \approx \prod^n Y$, for each $n \geq 2$, implies $X \approx Y$.*

(Here $\approx$ stands for *homeomorphic*.)

Theorem 2 *Let M and N be lens spaces with isomorphic fundamental groups. Then the pair M, N is* NOT *exponentially stable if and only if M and N are homotopy equivalent but nonhomeomorphic lens spaces.*

Given the close relation between homotopy equivalence and homeomorphism for many 3-manifolds, it is natural to ask if there are any other examples of 3-manifolds that are not exponentially stable.

If Σ^3 is a closed manifold that is homotopy equivalent to S^3, then Σ^3 bounds a parallelizable 4-manifold and it is a routine exercise – either by surgery theory and the surgery exact sequence (*cf.* Wall's book [39, Chapter 10]) or by more direct and elementary considerations as in [30] – to show that $\Sigma \times \Sigma$ is diffeomorphic to $S^3 \times S^3$, and thus it is clear that a study of exponential stability for 3-manifolds is more likely to be illuminating if, for example, it avoids the Poincaré Conjecture. To achieve this we assume that all 3-manifolds considered in this paper are closed, oriented and *geometric* in the sense of W. Thurston (see [36]). This means that each prime summand of the manifold can be decomposed, by cutting along incompressible tori, into pieces which are either Seifert fibered or admit finite volume hyperbolic metrics. The well known

Geometrization Conjecture of [36] asserts that all 3-manifolds are geometric. Our main result can be stated as follows:

Theorem 3 *Let M and N be closed, connected, oriented, connected sum of geometric 3-manifolds such that $H^1(M;\mathbb{Z}) = H^1(N;\mathbb{Z}) = 0$ and one of M, N has no lens spaces in its prime decomposition. Then $\prod^n M \approx \prod^n N$ for some $n \geq 2$ if and only if $M \approx N$.*

In particular, it follows that

(1) *two closed, connected, oriented, irreducible, geometric 3-manifolds with vanishing first Betti number are exponentially stable if and only if they are not both lens spaces with isomorphic fundamental groups,*
(2) *two irreducible Seifert manifolds with vanishing first Betti number are exponentially stable if and only if they are not both lens spaces with isomorphic fundamental groups,*
(3) *two manifolds satisfying the conditions of Theorem 3 are exponentially stable.*

Remark 4 It seems that some assumptions imposed on 3-manifolds in this paper can be relaxed; for example, orientability.

Remark 5 The assumption $H^1(M;\mathbb{Z}) = H^1(N;\mathbb{Z}) = 0$ is crucial for the construction of a map $f : M \to N$ with nonzero degree (*cf.* Lemma 7 and Proposition 23). We suspect that the conclusions of Theorem 3 and Lemma 7 remain true without this restriction (compare Remark 10 below).

Remark 6 Theorems 2 and 3 and the results of [22] give complete information on the classification of cartesian powers of closed, oriented, irreducible and geometric 3-manifolds with $H^1(-;\mathbb{Z}) = 0$, and Theorem 35 also yields information for some reducible manifolds satisfying these conditions. One could also try to combine the results of H. Hendriks and F. Laudenbach [15] on splitting homotopy equivalences of 3-manifolds with surgery-theoretic considerations to analyze the reducible case in full generality, but it appears that the technical difficulties encountered in such an analysis could be monumental.

In contrast to [22], which relies heavily on higher dimensional techniques from surgery theory, the proofs in this paper involve methods and concepts from 3-dimensional topology and group theory. These include the structure and rigidity theorems for hyperbolic, Haken, and Seifert 3-manifolds, machinery from group cohomology theory, several special properties of the fundamental groups of 3-manifolds, volume related invariants of hyperbolic 3-manifolds, results of S. C. Wang on maps of 3-manifolds with nonzero degrees and the Hendriks-Laudenbach splitting theorem for homotopy equivalences of 3-manifolds.

The remainder of this paper consists of four sections. The next section contains some elementary observations and summarizes the basic rigidity theorems for 3-manifolds that play an important role in the proof. We actually need a strong form of rigidity; specifically, for suitably restricted classes of 3-manifolds, the existence of maps with degree ± 1 between one and the other *in both directions* implies that the manifolds in question are homeomorphic (see Lemma 13 and Theorem 14 below). In Section 3 we prove the main result in the case requiring the most work; namely, Seifert manifolds whose fundamental groups are large in an appropriate sense (*cf.* [28, pp. 91–92]). The remaining cases of irreducible geometric 3-manifolds are treated in Section 4. Finally, Section 5 contains the proof of the main result for connected somes of irreducible geometric 3-manifolds that do not include lens space summands.

2 Preliminaries

This section discusses some background information that will be needed; we begin with an elementary result that is important for our purposes.

Lemma 7 *Let M^3 and N^3 be closed oriented 3-manifolds such that $H^1(M;\mathbb{Z})$ and $H^1(N;\mathbb{Z})$ are both trivial, and let $f : \prod^n M \to \prod^n N$ be a homotopy equivalence for some $n \geq 2$. Then there is a map from M to N of nonzero degree.*

Remark 8 If $\prod^n M$ and $\prod^n N$ are homotopy equivalent for some $n \geq 2$, and $H_i(M;\mathfrak{D}) = 0$ when $0 < i < q$ for some principal ideal domain $\mathfrak{D}$, then $H_i(\prod^n M;\mathfrak{D})$ is also trivial by induction and the Künneth formula, and since

$$0 = H_i\left(\prod\nolimits^n M;\mathfrak{D}\right) \approx H_i\left(\prod\nolimits^n N;\mathfrak{D}\right), \qquad 0 < i < q$$

one can also apply the same results to work backwards and conclude that $H_i(N;\mathfrak{D}) = 0$ for $0 < i < q$. Furthermore, for $X = M$ or N induction and the Künneth formula also imply that the slice inclusion maps $J_i : X \to \prod^n X$ (identity on one factor, constant on the rest) induce an isomorphism

$$\mathbf{J}_X : \bigoplus^n H_q(X;\mathfrak{D}) \to H_q\left(\prod\nolimits^n X;\mathfrak{D}\right)$$

with $\mathbf{J}_X(u_1, \cdots, u_n) = \sum_i J_{i*}(u_i)$. If the projection map onto the k-th coordinate is given by $P_k : \prod^n X \to X$, then the inverse to $\mathbf{J}_X$ is the map $\mathbf{P}_X$ in the reverse direction given by the formula

$$\mathbf{P}_X(v) = (P_{1*}(v), \cdots P_{n*}(v))\,.$$

If we combine these observations with the isomorphism

$$H_q(\prod\nolimits^n M;\mathfrak{D}) \approx H_q(\prod\nolimits^n N;\mathfrak{D})$$

and the basic structure theorem for finitely generated modules over a principal ideal domain, we see that $H_q(M;\mathfrak{D})$ and $H_q(N;\mathfrak{D})$ must be isomorphic. Similar results hold in cohomology with coefficients in $\mathfrak{D}$.

Remark 9 In the special case of the preceding remark with $q = 1$, it follows that if $\prod^n M$ and $\prod^n N$ are homotopy equivalent then $H_1(M;\mathbb{Z}) \approx H_1(N;\mathbb{Z})$ and $H^1(M;\mathbb{Z}) \approx H^1(N;\mathbb{Z})$. In particular, the vanishing of either $H^1(M;\mathbb{Z})$ or $H^1(N;\mathbb{Z})$ implies the vanishing of the other.

Remark 10 Lemma 7 remains true if $H^1(M;\mathbb{Z}) \approx H^1(N;\mathbb{Z})$ is infinite cyclic; our current proof of this is elementary but tedious, and we hope to provide a more general and conceptual argument elsewhere.

Proof of Lemma 7. By the assumptions on integral cohomology and Poincaré Duality, both M and N are rational homology 3-spheres. By the argument of Remark 8 above, the projection and slice inclusion maps induce isomorphisms

$$\bigoplus^n H_3(X;\mathbb{Q}) \to H_3\left(\prod^n X;\mathbb{Q}\right)$$

where $X = M$ or N; since Z is a rational homology 3-sphere, the left hand side is isomorphic to $\mathbb{Q}^n$. The homotopy equivalence f determines an isomorphism from $H_3(\prod^n M;\mathbb{Q})$ to $H_3(\prod^n N;\mathbb{Q})$, and with respect to the previously described splittings the isomorphism determined by f corresponds to an invertible matrix in $GL(n,\mathbb{Q})$. In fact, this invertible matrix actually lies in the subgroup $GL(n,\mathbb{Z})$ because the rational homology isomorphism is given by tensoring an isomorphism of $H_3(-;\mathbb{Z})/\mathsf{Torsion}$ with the rationals (note that the slice inclusions also induce isomorphisms from $\oplus^n H_3(X;\mathbb{Z})$ to $H_3\left(\prod^n X;\mathbb{Z}\right)/\mathsf{Torsion}$ for $X = M$ or N). By invertibility the matrix has a nonzero entry, and if its (k,i) entry is nonzero then the composite $P_k \circ f \circ J_i$ is a map from M to N of nonzero degree. $\square$

Corollary 11 *Under the hypotheses of the lemma there are maps in both directions $M \to N$, $N \to M$ of nonzero degree.*

This is true because the hypothesis is symmetric in M and N. $\square$

A crucial step in our arguments will be to show that there are maps of degree ± 1 in both directions. It is well known that the existence of degree one maps from M to N and vice versa implies that M and N are homotopy equivalent if both are simply connected (such maps are split surjective in homology [3, Thm. I.2.5, pp. 8–9] so the existence of maps both ways implies that they must be isomorphisms and hence homotopy equivalences by the Whitehead Theorem for homology groups). In the nonsimply connected case things are more complicated, and the most basic issue is contained in an old conjecture due to H. Hopf (*cf.* [12, p. 333]):

Conjecture 12 *Suppose that there exist degree one maps $f : M \to N$ and $g : N \to M$ between closed manifolds M and N. Then $\pi_1(M)$ is isomorphic to $\pi_1(N)$ via f_*.*

In particular, if M and N are aspherical, then Hopf's conjecture would imply that M and N are homotopy equivalent.

There is a large class of fundamental groups for which the conjecture holds. A group G is said to be *Hopfian* if every surjective homomorphism from G to itself is an isomorphism (*cf.* Hempel's book [13, p. 175]). If G is *residually finite* — *i.e.*, all its elements are detected by the finite quotients of G [13, p. 176] — then G is Hopfian by [13, Lemma 15.17, p. 177]. For our purposes it is important to know that the fundamental groups of geometric 3-manifolds are residually finite and therefore Hopfian (see [13,14]). Therefore we have have the following:

Lemma 13 *Suppose that M and N are aspherical geometric 3-manifolds and there exist degree ± 1 maps $f : M \to N$ and $g : N \to M$ between M and N. Then M is homotopy equivalent to N via f.*

To see this, note that the composite $g \circ f : M \to M$ is also a degree ± 1 map and hence induces a surjection of fundamental groups. Since $\pi_1(M)$ is Hopfian, it follows that $g \circ f$ induces an isomorphism and thus it is a homotopy self equivalence of M by the Whitehead Theorem for homotopy groups. In particular this means that the induced map f_* on fundamental groups is injective. But since the degree of f is ± 1 it also follows that f_* is surjective; therefore f_* is an isomorphism and f is a homotopy equivalence (again) by Whitehead's Theorem. $\square$

Finally, for our purposes it is important to know when homotopy equivalent irreducible geometric 3-manifolds are homeomorphic. The following two statements summarize several important results on this question.

Theorem 14 *Let M and N be irreducible geometric 3-manifolds such that M is not a lens space, and suppose that M and N are homotopy equivalent. Then M and N are homeomorphic and in fact diffeomorphic.*

PROOF. First of all, if M is not a lens space then its fundamental group is not a finite abelian group; therefore the same is true for the fundamental group of N and hence N is also not a lens space. Since homeomorphisms of 3-manifolds can always be deformed to diffeomorphisms, it suffices to prove that M and N are homeomorphic.

There are now several cases depending upon the geometry of M.

Case 1 *M is a spherical spaceform with a finite nonabelian fundamental group.*

It follows that N must also have a finite nonabelian fundamental group and since N is geometric it must also be a spherical spaceform with a finite nonabelian fundamental group. Since such manifolds are determined up to homeomorphism by their fundamental groups (*cf.* Seifert-Threlfall [32]; also see [21, p. 737, Case 1]), it follows that M and N must be homeomorphic. $\square$

Case 2 *M is hyperbolic.*

In this case we may use the hyperbolization result of D. Gabai, R. Meyerhoff and N. Thurston [8] to conclude that N is also hyperbolic and that every homotopy equivalence from M to N is homotopic to a homeomorphism; other proofs may be given using G. Mostow's rigidity theorem (*cf.* [26,27]; see also the discussion in [21, p. 738]), but a reference to [8] is the most efficient. $\square$

Case 3 *M has infinite fundamental group but is neither Seifert fibered nor hyperbolic.*

In this case the classification of geometric 3-manifolds implies that M must be Haken. We claim the same is true for N; it suffices to check that N is neither Seifert fibered nor hyperbolic. The result of [8] implies that N cannot be hyperbolic because we are assuming that M is not hyperbolic, and similarly a result of P. Scott [31] shows that N cannot be Seifert fibered (again we are assuming M is not Seifert fibered, and the fundamental group of M is assumed to be infinite, so [31] applies). Since M and N are both Haken and they are homotopy equivalent, results of F. Waldhausen [38] show that M and N are homeomorphic. $\square$

Case 4 *M has infinite fundamental group and is Seifert fibered.*

In this case the main result of [31] implies that M and N are homeomorphic. $\square$

It is well known that Theorem 14 does not hold if M is a lens space, but in such cases one has the following conclusion:

Theorem 15 *Let M and N be irreducible geometric* 3*-manifolds such that M is a lens space. Then M and N are diffeomorphic if and only if they are homotopy equivalent by a* **simple** *homotopy equivalence.*

Once again, if M is a lens space and N is an irreducible geometric 3-manifold that is homotopy equivalent to M then N must also be a lens space. The conclusion then follows because the topological and simple homotopy classifications of lens spaces are identical (*cf.* M. Cohen's book [5]). □

3 Proof of Theorem 3 for large Seifert 3-manifolds

We shall consider first the case that requires the most work.

Seifert 3-manifolds are circle bundles over 2-manifolds with certain specific types of singularities (*cf.* Orlik's book [28] or the expository article by K. B. Lee and F. Raymond [23]). In particular the construction of a Seifert 3-manifold M yields an associated projection $M \to B$ for some surface B such that the inverse image of each point is a circle. We shall call the map from M to B a *Seifert fibering* and say that B is the *associated base space*.

The classification scheme for Seifert manifolds in [28, Thm. 2, p. 88] lists six different types. Since we only consider Seifert structures for which the underlying 3-manifold M is orientable, only two of these types play a role in this work.

(1) **Type** o_1: These are the cases where both M and B are orientable, and the have circle actions.
(2) **Type** n_2: These are the cases where both M is orientable but B is nonorientable. Typical examples are given by certain nonorientable circle bundles over nonorientable surfaces for which the total spaces are orientable (*i.e.*, the first Stiefel-Whitney classes of B and the circle bundle are equal).

As in [28, pp. 91–92] it is convenient to split the class of Seifert 3-manifolds into two classes – **small** and **large**. The class of small manifolds is summarized in [28, pp. 91–91] and described explicitly in [28, Section 5.4, pp. 99–102]. Most of these are total spaces of fiber bundles over S^1 with connected fibers (which are surfaces), and for such examples we know that $H^1(M, \mathbb{Z})$ is nonzero. The remaining possibilities (given by (i), (ii) and (vii) in Orlik's list), which are the ones of primary interest to us here, are given by spherical space forms (including lens spaces) and the reducible 3-manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$.

For our purposes the necessary properties of large Seifert 3-manifolds are as follows:

(A) *Large Seifert* 3*-manifolds are irreducible and aspherical* (see [28, Prop. 3, p. 93]).

(B) *The fundamental group* π *of a large Seifert* 3*-manifold contains a unique maximal infinite cyclic normal subgroup* (see [28, pp. 91–92]).

We note that the first property leads to a very simple criterion for distinguishing small and large orientable Seifert 3-manifolds satisfying $H^1(M;\mathbb{Z}) = 0$.

Theorem 16 *Let M be an orientable Seifert 3-manifold such that $H^1(M;\mathbb{Z}) = 0$. Then M is large if and only if it is aspherical. If in addition M is irreducible, then M is large if and only if it* **is not** *a spherical spaceform (hence M is small if and only if it* **is** *a spherical spaceform).*

This is true because the irreducible small Seifert manifolds with $H^1(M;\mathbb{Z}) = 0$ are spherical spaceforms and the only other example satisfying the cohomological restriction is $\mathbb{RP}^3 \# \mathbb{RP}^3$. The universal coverings of the spaceforms are homeomorphic to S^3, and the universal covering of the final example is homeomorphic to $S^2 \times \mathbb{R}$. Since we have seen that the large examples are aspherical, it follows that the latter condition holds if and only if the Seifert manifold is large. The final assertion in the theorem follows from the second sentence of this paragraph. □

3.1 Seifert structures given by circle actions

In [28, Thm. 2, p. 88] Seifert fiberings are separated into six types depending upon the orientability properties of the base and the fibering itself. Types o_1 and o_2 correspond to examples where the base manifold is orientable and Types n_i for $1 \leq i \leq 4$ correspond to examples where the base space is not orientable; as noted before, the associated Seifert 3-manifold is orientable precisely in cases o_1 and n_2.

The most familiar examples of Seifert 3-manifolds are those admitting fixed point free smooth actions of the circle group S^1; for these examples the base is orientable if and only if the 3-manifold itself is orientable. Since we are dealing with orientable manifolds in this paper, these examples are given by Type o_1 in the terminology of [28, Thm. 2, p. 88], and in fact they include *all* examples of Type o_1. In addition to being the most important case, Type o_1 is also the easiest to analyze and the other orientable cases (Type n_2 in [28, Thm. 2, p. 88]) are relatively simple to study once we dispose of the Type o_1 case.

In the Type o_1 case properties (A) and (B) of the maximal infinite cyclic normal subgroup can be strengthened as follows:

Property 17 *For Seifert 3-manifolds of Type o_1, the unique maximal infinite cyclic normal subgroup of π is the class generated by a generic orbit of the circle action and is central.*

This follows from the discussion in the first paragraph of in [28, Section 5.3, pp. 90–91]. □

Property 18 *For Seifert* 3*-manifolds of Type* o_1*, the unique maximal infinite cyclic normal subgroup of* π *is the center of* π*.*

A self-contained way of seeing this without checking generators and relations from [28, p. 91] is as follows: If the center C is cyclic then by the maximality property it it must be the subgroup in question, but if C were not cyclic then C would have to contain a free abelian subgroup of rank 2, and in this case there could not be a unique maximal infinite cyclic normal subgroup. □

If M is a large Seifert 3-manifold of Type o_1 with fundamental group π, let $\mathcal{C} : \mathbb{Z} \to \pi$ denote the inclusion of the central subgroup in (B) and let Γ be the quotient $\pi/\mathcal{C}(\mathbb{Z})$; it follows that Γ is a planar crystallographic group.

The homology and cohomology of $K(\Gamma, 1)$ over the rationals can be determined fairly easily because a model for the latter is given by the "homotopy orbit space" or Borel construction M_{S^1}, which is the associated fiber bundle with fiber M for the universal principal S^1 bundle over $\mathbb{CP}^\infty$ (*cf.* Bredon's book [2, p. 369]). There is a canonical map φ from M_{S^1} to the orbit space M/S^1 and if S^1 acts without fixed point on M it follows that φ is an isomorphism over the rationals (*cf.* the Borel seminar notes [1, Application IV.3.4.(b), p. 54] and the discussion preceding the latter). Since M/S^1 is a surface that is part of the data describing the Seifert structure on M, the rational information about $K(\Gamma, 1)$ is easy to retrieve. In cases where $H^1(M;\mathbb{Z}) = 0$ the orbit space is known to be homeomorphic to S^2.

An analysis of the homology and cohomology of $K(\Gamma, 1)$ requires more effort but is still relatively easy. The central extension $\mathcal{E}(\pi)$ given by the diagram

$$0 \to \mathbb{Z} \to \pi \to \Gamma \to 1$$

gives rise to a Lyndon-Hochschild-Serre spectral sequence (also called simply the *Lyndon spectral sequence*) in group cohomology and homology; the cohomology sequence is presented in Hilton-Stammbach [16, Theorem 9.5, p. 303] and the analogous homology sequence is discussed in [16, Remark (ii), p. 304, and Exercise 9.4 on p. 305]. Given an integral domain A this spectral sequence has

$$E^2_{s,t} = H_s\left(\Gamma; H_t(\mathbb{Z}; A)\right)$$

and abuts to $H_{s+t}(\pi; A) \approx H_{s+t}(M; A)$.

The Lyndon spectral sequence is natural with respect to morphisms of central extension diagrams [16, Exc. 9.7, p. 305].

For the remainder of the discussion for Type o_1 *we shall assume that* $H^1(M;\mathbb{Z})$ *is trivial.* — It follows that $H_1(\pi : \mathbb{Z}) \approx H_1(M;\mathbb{Z})$ is finite, and in this case the spectral sequence yields a great deal of information about the homology of Γ with integral coefficients.

Claim 19 *The differential* $d^2_{2,0}$ *maps injectively to* $E^2_{0,1} \approx \mathbb{Z}$ *and thus* $H_2(\Gamma) \approx \mathbb{Z}$.

By Poincaré duality we know that $H_2(\pi;\mathbb{Z}) = H_2(M;\mathbb{Z}) = 0$, and the claim describes the only possible map consistent with this fact. □

Claim 20 *The differential* $d^2_{3,0}$ *maps surjectively to* $E^2_{1,1} \approx H_1(\Gamma;\mathbb{Z})$ *and the kernel is a finite cyclic group (possibly trivial).*

This is the only possibility consistent with the vanishing of H_2 and the fact that $H_3(\pi;\mathbb{Z}) = H_3(M;\mathbb{Z})$ is infinite cyclic. □

NOTATION. The order of the kernel of $d^2_{3,0}$ will be denoted by $e^\infty(M)$. It follows that $e^\infty(M)$ is also the order of the image of $H_3(M;\mathbb{Z})$ in $H_3(\Gamma;\mathbb{Z})$.

Claim 21 *The differential* $d^2_{4,0}$ *maps trivially to* $E^2_{2,1} \approx H_2(\Gamma;\mathbb{Z}) \approx \mathbb{Z}$.

The rational calculation $H_i(\Gamma;\mathbb{Q}) \approx 0$ for $i \geq 3$ implies that $E^2_{4,0} \approx H_4(\Gamma;\mathbb{Z})$ is torsion, and consequently the only possible homomorphism is the trivial one. □

Claim 22 *We have* $E^2_{2,1} = E^\infty_{2,1} \subset H_3(\pi;\mathbb{Z}) \approx H_3(M;\mathbb{Z})$ *and* $e^\infty(M)$ *is also the index of this subgroup.*

We know that $E^2_{2,1} = E^\infty_{2,1}$ because $d^2 = 0$ and there are no higher differentials into or out of $E^r_{2,1}$ for $r \geq 3$. The inclusion in $H_3(\pi)$ follows because $E^2_{s,t} = 0$ for $t \geq 2$, and the statement about $e^\infty(M)$ follows from edge homomorphism considerations in the spectral sequence. □

Suppose now that we have a homeomorphism $f : \prod^n M \to \prod^n M'$ where M and M' are large Seifert 3-manifolds of Type o_1 and $n \geq 2$. Let $\pi' := \pi_1(M')$, let $\mathcal{C}' : \mathbb{Z} \to \pi'$ be the inclusion of the center in $\pi_1(M')$, and let $\Gamma' := \pi'/\mathcal{C}'(\mathbb{Z})$ so that we have a central extension $\mathcal{E}(\pi')$

$$0 \to \mathbb{Z} \to \pi' \to \Gamma' \to 1$$

analogous to $\mathcal{E}(\pi)$ above. The following sharpening of Lemma 3 is the key observation for comparing M and M'.

Proposition 23 *In the situation above, there is a map* g *from* M *to* M' *such that the associated map of fundamental groups sends the center* $\mathcal{C}(\mathbb{Z}) \subset \pi$ *isomorphically to* $\mathcal{C}'(\mathbb{Z}) \subset \pi'$ *and the degree of* g *satisfies*

$$|\deg(g)| \cdot e^\infty(M) = e^\infty(M')$$

where e^∞ *is defined as above.*

To prove Proposition 23, let $f_* : \pi^n \to (\pi')^n$ be the associated isomorphism of fundamental groups. This map sends the center of the source onto the center of the target, so denote the associated automorphism of $\mathbb{Z}^n$ by h_*. Passage to quotients then defines an isomorphism of quotient groups

$$k_* : \Gamma^n \to (\Gamma')^n$$

that is associated to a homotopy equivalence $k : \prod^n K(\Gamma, 1) \to \prod^n K(\Gamma', 1)$. Consequently, if $\prod^n \mathcal{E}(\pi)$ and $\prod^n \mathcal{E}(\pi')$ denote the n-fold products of the extensions $\mathcal{E}(\pi)$ and $\mathcal{E}(\pi')$ with themselves, it follows that f_*, h_* and k_* determine an isomorphism of diagrams from $\prod^n \mathcal{E}(\pi) = \mathcal{E}(\pi^n)$ and $\prod^n \mathcal{E}(\pi') = \mathcal{E}\left((\pi')^n\right)$, and this in turn yields an isomorphism between the corresponding Lyndon spectral sequences.

Consider the isomorphism

$$\widehat{k_*} : H_2\left(\prod^n K(\Gamma, 1); \mathbb{Z}\right)/\mathsf{Torsion} \to H_2\left(\prod^n K(\Gamma', 1); \mathbb{Z}\right)/\mathsf{Torsion}$$

Free generators for the free abelian groups

$$H_2\left(\prod^n K(\Gamma, 1); \mathbb{Z}\right)/\mathsf{Torsion} \qquad \text{and} \qquad H_2\left(\prod^n K(\Gamma', 1); \mathbb{Z}\right)/\mathsf{Torsion}$$

are given by $\{J_{i*}\gamma \mid 1 \le i \le n\}$ and $\{J_{i*}\gamma' \mid 1 \le i \le n\}$ respectively. The previously determined properties of $\widehat{k_*}$ may be restated as follows:

Property 24 *There is a permutation ℓ of $\{1, \cdots, n\}$ such that $\widehat{k_*}\left(J_{i*}\gamma\right) = \pm J_{\ell(i)*}\gamma'$ for all i.*

By the naturality of the Lyndon spectral sequence we know that

$$d^2_{2,0}\left[\mathcal{E}\left(\Pi'\right)^n\right)\right] \circ E^2_{2,0}[k_*] = E^2_{1,0}[\ell_*] \circ d^2_{2,0}\left[\mathcal{E}(\Pi')^n)\right]$$

and therefore if σ generates $H_1(\mathbb{Z}; \mathbb{Z}) \approx \mathbb{Z}$ we also have the following:

Property 25 *For the same permutation ℓ as in Property 24 we have $h_* J_{i*}\sigma = \pm J_{\ell(i)*}\sigma$ for all i.*

This implies that that the composite maps $F_{\ell i} := P_\ell \circ f \circ J_i$ from $M \simeq K(\pi, 1)$ to $M' \simeq K(\pi', 1)$ all have the property that the induced maps in fundamental groups $F_{\ell i*}$ send the centers of the sources isomorphically to the centers of the targets.

It remains to prove the assertions about the degree of $F_{\ell i}$. The preceding considerations show that $F_{\ell i}$ induces an isomorphism from $E^2_{2,1}[\mathcal{E}(\pi)]$ to $E^2_{2,1}[\mathcal{E}(\pi')]$. On the other hand, by Claim 21 above we know that $E^2_{2,1} = E^\infty_{2,1}$ for both $\mathcal{E}(\pi)$ to $\mathcal{E}(\pi')$. By Claim 20 above we know that $E^\infty_{2,1}[\mathcal{E}(\pi)]$ has index $e^\infty(M)$

in $H_3(\pi) \approx H_3(M)$ and likewise $E^\infty_{2,1}[\mathcal{E}(\pi')]$ has index $e^\infty(M')$ in $H_3(\pi') \approx H_3(M')$, where all homology groups are over the integers. In particular, all of this implies the formula

$$F_{\ell i *}(e^\infty(M)[M]) = \pm e^\infty(M')[M'].$$

The formula for the degree of $F_{\ell i}$ is an immediate consequence, and this complete the proof of Proposition 23. $\square$

Corollary 26 *The degree of the map in Proposition* 23 *is* ± 1.

PROOF. By Proposition 23 we know that $e^\infty(M)$ divides $e^\infty(M')$. However, the hypothesis is symmetric in M and M', so it also follows that $e^\infty(M')$ divides $e^\infty(M)$, which means that $e^\infty(M) = e^\infty(M')$. The degree formula now implies $d = \pm 1$. $\square$

The preceding discussion yields the following ultimate conclusion for Seifert manifolds of Type o_1 (*i.e.*, the orientable examples with fixed point free circle actions).

Theorem 27 *If M and M' are large Seifert 3-manifolds of Type o_1, then Theorem* 3 *is valid for M and M'.*

This follows directly from Lemma 13, Theorem 14, Proposition 23 and Corollary 26. $\square$

3.2 Seifert manifolds with solvable fundamental groups

According to the list on pages 91 and 92 of [28], a Seifert manifold of Type o_1 is either large or else its fundamental group is finite or solvable (and possibly both). In fact, it is not difficult to show that the conclusion of Theorem 3 holds if M and M' are small of Type o_1. The cases with finite fundamental groups are treated in the next section (see Case 4.1). Since we need the result for cases with infinite solvable fundamental groups we shall verify it here.

Proposition 28 *The conclusion of Theorem* 3 *is valid if M and M' are irreducible Seifert manifolds of Type o_1 with infinite solvable fundamental groups and arbitrary first Betti number.*

In order to prove this result, it is necessary to describe certain Seifert manifolds using the classification invariants defined in [28, Thm. 3, p. 90]. These data are given by lists of the form

$$\{b\ ;\ (\varepsilon, g)\ ;\ (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$$

where

(1) b is a generalization of the (oriented or nonoriented) Euler class for a circle bundle,
(2) ε is one of the six types o_i and n_i mentioned previously,
(3) g is the genus of the base space (which is a surface),
(4) r is some nonnegative integer (possibly zero),
(5) each (α_j, β_j) is an ordered pair of relatively prime positive integers describing the structure of the Seifert fibering near the singular points (hence r can be any nonnegative integer). If the Seifert fibering is the orbit space projection of a circle action, then α_j is the order of the isotropy group for a singular orbit and β_j describes the slice representation [2] for that orbit.

For the orientable Seifert manifolds of Types o_1 and n_2 the Euler class b is an integer and the pairs (α_j, β_j) satisfy $0 < \beta_j < \alpha_j$.

Proof of Proposition 28. As noted in [28, Thm. 1, pp. 142–143] (also see Evans-Moser [6]), the irreducible manifolds of Type o_1 with solvable infinite fundamental groups are those whose Seifert invariants have one of the following descriptions:

- **Subclass** (i) $\{b; (o_1, 1)\}$
- **Subclass**(ii) $\{b; (o_1, 0)\,; (2, 1), (2, 1), (2, 1), (2, 1)\}$
- **Subclass**(iv) Some examples with $g = 0$ and $r \leq 2$.

Subclass (iii) from [28, Thm. 1, pp. 142–142] has been omitted because the fundamental groups in these cases are finite, and the only examples in Subclass (iv) with infinite fundamental groups are homeomorphic to $S^1 \times S^2$.

It is a routine exercise to compute the first homology groups of these manifolds over the integers using the presentations on page 91 of [28], and the results are given below; in cases where we write $\mathbb{Z}_d$ and d turns out to be zero, the group in question is merely $\mathbb{Z}$.

- **Subclass** (i) The first integral homology group is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{|b|}$.
- **Subclass** (ii) The first integral homology group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{|8+4b|}$.
- **Subclass** (iv) If the fundamental group is infinite, then the first integral homology group is $\mathbb{Z}$.

The only ways in which two different sets of data can yield the same first homology are two examples in subclass (i) with $b = -b'$ or two examples in

Subclass (ii) with $(8 + 4b) = -(8 + 4b')$. However, as noted in [28, p. 90, paragraph preceding Section 5.3] the two sets of data in such cases represent the same Seifert manifold with different orientations. In particular, nonhomeomorphic examples have nonisomorphic first integral homology groups.

Suppose now that M and M' satisfy the hypotheses of the proposition and $\prod^n M$ is homeomorphic to $\prod^n M'$ for some $n \geq 2$. As in Remark 8 above, it follows that $H_1(\prod^n M; \mathbb{Z}) \approx \oplus^n H_1(M; \mathbb{Z})$ is isomorphic to $H_1(\prod^n M'; \mathbb{Z}) \approx \oplus^n H_1(M'; \mathbb{Z})$. But the structure theorem for finitely generated abelian groups implies that two such groups A, B are isomorphic if $\oplus^n A \approx \oplus^n B$, and therefore $H_1(M; \mathbb{Z})$ must be isomorphic to $H_1(M'; \mathbb{Z})$. By the conclusion of the previous paragraph this means that M and M' must be homeomorphic. $\square$

3.3 The remaining large Seifert manifolds

We have already noted that the classification of Seifert 3-manifolds as in [28, Thm. 2, p. 88] categorizes them into six types depending on orientation invariants, and aside from Type o_1 the only other type yielding orientable 3-manifolds is Type n_2, which includes certain nonorientable circle bundles over nonorientable surfaces. For the manifolds of Type n_2 one has the following strengthening of general property (B):

Property 29 *The fundamental group π contains a normal subgroup K of index 2 such that K is the centralizer of the unique maximal cyclic normal subgroup Ω. Furthermore, the group K is the fundamental group of a Seifert 3-manifold of Type o_1.*

PROOF. The presentation for π in [28, p. 91] shows that the action of π on $\Omega \approx \mathbb{Z}$ by conjugation determines a nontrivial homomorphism from π to $\mathrm{Aut}(\Omega) \approx \mathbb{Z}_2$. If K denotes the kernel of this map, then clearly K has all the required properties except perhaps the last one (*i.e.*, its realizability as the fundamental group of a Seifert manifold). A fast way of realizing K as the fundamental group of a Seifert manifold is to apply the result of A. Casson and D. Jungreis [4] to the double cover $\widehat{M}$ of M associated to the subgroup K; note that $\widehat{M}$ is orientable because it is a covering space of an orientable manifold. The result of [4] is applicable because $\widehat{M}$ is irreducible (its universal cover is the universal cover of M, which is $\mathbb{R}^3$) and $K \approx \pi_1(\widehat{M})$ contains the subgroup Ω as a central cyclic subgroup, so it follows that $\widehat{M}$ is a Seifert 3-manifold, and since it contains a central infinite cyclic subgroup it must be of Type o_1 (associated to a circle action; *cf.* the discussion preceding Property 17). $\square$

In the proof of Theorem 3 for large Seifert manifolds of Type o_1 we concluded that the subgroup $\mathcal{C}(\mathbb{Z}^n) \subset \pi^n$ was a characteristic subgroup because it was the center. We shall also need to know this subgroup is characteristic for Type n_2, but this requires more detailed information about the subgroup's properties for large Seifert 3-manifolds.

Proposition 30 *Let π be the fundamental group of a large Seifert 3-manifold, and let $\Omega \subset \pi$ denote the unique maximal (infinite) cyclic normal subgroup. Then $\Omega^n \subset \pi^n$ is the unique maximal normal subgroup of π^n that is free abelian of rank n and expressible as a direct product of n cyclic normal subgroups.*

Corollary 31 *In the preceding notation, Ω^n is a characteristic subgroup of π^n. Furthermore if the Seifert manifold is of Type n_2, and $K \subset \pi$ is the subgroup of index 2 described in Property 29 then the n-fold product K^n is also characteristic in the n-fold product π^n.*

Proof of Corollary 31. The subgroup Ω^n is characteristic because an isomorphism must send a subgroup with the stated properties to another of the same type. The assertion about K^n follows because it is the centralizer of Ω^n and any isomorphism must preserve the centralizers of subgroups satisfying the same properties as Ω^n. $\square$

Proof of Proposition 30. Let C be an arbitrary infinite cyclic normal subgroup of π^n, let j be an integer such that $1 \leq j \leq n$, and consider the projection $P_j(C)$ of C onto the j-th coordinate of π^n. This is a cyclic normal subgroup of π and as such is contained in Ω. Since j is arbitrary it follows that $C \subset \Omega^n$. If L is an arbitrary normal subgroup that is a product of the cyclic normal subgroups $L_1, \cdots L_m$ (where m need not equal n), it follows that $L \subset \Omega^n$. This shows that Ω^n is the unique normal subgroup with the stipulated properties.

3.4 Completion of the proof of Theorem 3 for large Seifert 3-manifolds

Let $f : \prod^n M \to \prod^n M'$ be a homeomorphism and let $f_* : \pi^n \to (\pi')^n$ be the associated isomorphism of fundamental groups. One way of distinguishing between manifolds of Types o_1 and n_2 is that the fundamental groups for Type o_1 have nontrivial centers while those of Type n_2 do not. This and the isomorphism f_* imply that both M and M' are either of Type o_1 or of Type n_2. Since we have already proven the result if M and M' are of Type o_1, we shall assume they are both of Type n_2 henceforth.

If $K \subset \pi$ and $K' \subset \pi'$ are the centralizers of the maximal cyclic normal subgroups, then the reasoning of Corollary 31 implies that f_* maps K^n onto $(K')^n$. The proof now splits into cases depending upon whether $\widehat{M}$ is large or $\widehat{M}$ has a solvable fundamental group (note that these cases are not disjoint, but this is not important for our purposes).

Suppose first that K is solvable; since K has index 2 in π it follows that the latter is also solvable. Furthermore, since a group G is solvable if and only if G^n is solvable, it follows that both K' and π' are also solvable. If $\widehat{M}$ and $\widehat{M'}$ are the double coverings associated to K and K', then by the previous paragraph f lifts to a homeomorphism from $\prod^n \widehat{M}$ to $\prod^n \widehat{M'}$. By Proposition 28 it follows that $\widehat{M} \approx \widehat{M'}$ and $K \approx K'$. Since π and π' are infinite solvable groups and M and M' are irreducible, by [28, Thm. 1, p. 142] these manifolds must have Seifert invariants of the form

$$\{b \ ; \ (n_2, 1) \ ; \ (2,1), (2,1)\}$$

where b is an arbitrary integer. As noted in theorem cited in the previous sentence these manifolds have double coverings that are Seifert manifolds of Type o_1, and the corresponding invariants for the double coverings are

$$\{2b \ ; \ (o_1, 0) \ ; \ (2,1), (2,1), (2,1), (2,1)\}$$

(note that the first numerical invariant doubles when one passes to the double covering). By construction the fundamental groups of these double coverings are the centralizers of the maximal normal cyclic subgroup in the fundamental groups of the original manifolds. Choose integers b and b' such that M has first numerical invariant b and M' has first numerical invariant b'. The hypothesis $\prod^n M \approx \prod^n M'$ then implies that the fundamental groups of the double coverings are isomorphic, and by the proof of Proposition 28 this means that either $2b' = 2b$ or $2b' = -2b - 4$; *i.e.*, either $b' = b$ or $b' = -b - 2$. On the other hand, we can once again apply [28, p. 90, paragraph preceding Section 5.3] to conclude that the two sets of data yield the same manifold with different orientations. Therefore we have $M \approx M'$ if the group K is solvable. $\square$

Assume now that K is not solvable; since K is infinite it follows that $\widehat{M}$ must be large, and as in the previous paragraph we also know that K' is also not solvable and $\widehat{M'}$ must also be large. By the last sentence in the proof of Corollary 31, we know that the map of fundamental groups determined by f must send $K^n \approx \pi_1(\prod^n \widehat{M})$ to $(K')^n \approx \pi_1(\prod^n \widehat{M'})$, and therefore it follows that f lifts to a homeomorphism $\widehat{f}$ from $\prod^n \widehat{M}$ to $\prod^n \widehat{M'}$. There are now two subcases.

Subcase 3.1 $H^1(\widehat{M}; \mathbb{Z}) = 0$.

In this case Proposition 23 and Corollary 26 imply the existence of integers

i and ℓ such that $P_\ell \circ \widehat{f} \circ J_i$ is a map of degree ± 1 and in fact is a homotopy equivalence. This composite is a lifting of a corresponding map $F_{\ell i} = P_\ell \circ \widehat{f} \circ J_i$ from M to M', and a diagram chase shows that the degree of the map $F_{\ell i}$ must also be ± 1. Therefore we have shown the existence of a map $M \to M'$ of degree ± 1. Using the symmetry of the hypothesis in M and M' we see that there must also be a map of degree ± 1 in the opposite direction, and therefore we can again use Lemma 13 to conclude that M and M' are homotopy equivalent and thus homeomorphic by Theorem 14.

Subcase 3.2 $H^1(\widehat{M}; \mathbb{Z}) \neq 0$.

In this case we claim that $H^1(\widehat{M}; \mathbb{Z}) \approx \mathbb{Z}$ and the restriction to an orbit of the associated circle action induces a monomorphism in cohomology. This is particularly important because it implies that $\widehat{M}$ fibers over a circle [28, Cor. 5, p. 122]. We begin by noting that the base B of the original Seifert structure on M must be $\mathbb{RP}^2$; since B is nonorientable it is a connected sum of (say) g copies of $\mathbb{RP}^2$, and the presentation of $\pi_1(M)$ on [28, p. 91] shows that its abelianization $H_1(M; \mathbb{Z})$ is finite only if $g = 1$. The double covering $\widehat{M}$ can be realized geometrically by taking the pullback of M under the double covering $S^2 \to \mathbb{RP}^2$. If $L = K/\Omega$, then we again have a Lyndon spectral sequence that abuts to the homology of K (which is just $H_*(M)$, and we also know that $H_*(L; \mathbb{Q}) \approx H_*(S^2; \mathbb{Q})$ as before. The only way that the rational spectral sequence can abut to $H_*(M; \mathbb{Q})$ if $H_1 \neq 0$ is if $E^2 = E^\infty$, and in this case the inclusion of an orbit $S^1 \to M$ determines an isomorphism in one dimensional homology. This yields the claims made at the beginning of the paragraph.

Let K, *etc.* be defined as in Subcase 3.1 and let $L' = K'\Omega'$. Since $\widehat{f}_*$ maps Ω^n to Ω'^n it follows that f passes to a homotopy equivalence

$$\varphi : \prod^n K(L, 1) \to \prod^n K(L', 1).$$

As in earlier arguments we know that these products have the same rational cohomology rings as $\prod^n S^2$, and if P_ℓ and J_i denote the canonical projection and injection maps then there is a permutation σ of $\{1, \cdots, n\}$ such that for each i the map $\varphi_{\ell i} = P_\ell \circ \varphi \circ J_i$ induces a nontrivial map in 2-dimensional rational cohomology if and only if $\ell = \sigma(i)$. In fact, if we replace f with the composite $S(\sigma^{-1}) \circ f$, where $S(\sigma^{-1})$ shuffles coordinates via the permutation σ^{-1}, then we can simplify things so that the permutation becomes the identity. *We shall assume this for the remainder of the proof.*

Passing back to integral coefficients we see that the maps associated to $\varphi_{\ell i}$ in either the functorially modified homology groups $H_2(-; \mathbb{Z})/\mathsf{Torsion}$ or the functorially modified cohomology groups $H^2(-; \mathbb{Z})/\mathsf{Torsion}$ are trivial if $\ell \neq i$ and isomorphisms if $\ell = i$.

By Lemma 7 there is a map $P_s \circ f \circ J_t$ from M to M' of nonzero degree, and this is covered by a map $P_s \circ \widehat{f} \circ J_t$ of the same degree.

Claim 32 *If* $\deg(P_s \circ f \circ J_t) \neq 0$ *then* $s = t$.

PROOF. Suppose $s \neq t$; it suffices to prove that $\deg(P_s \circ \widehat{f} \circ J_t) = 0$; to simplify notation we shall set $\lambda_{st} = P_s \circ \widehat{f} \circ J_t$. Let $q : \widehat{M} \to S^1$ and $q' : \widehat{M'} \to S^1$ be the fiberings described above, and denote their fibers by Σ and Σ'. Both q and q' induce isomorphisms in 1-dimensional integral cohomology, and thus there is a self-map ψ of $H^1(S^1)$ such that $\psi \circ q^* = (q')^* \lambda_{st}^*$. Since S^1 is a $K(\mathbb{Z}, 1)$, there is a self map Ψ of S^1 that induces ψ on integral cohomology and satisfies $\Psi \circ q \simeq q' \circ \lambda_{st}$, and this in turn yields a corresponding map of fibers $e : \Sigma \to \Sigma'$ such that

$$\left(\text{Inclusion}\,[\Sigma' \subset \widehat{M'}]\right) \circ e \simeq \lambda_{st} | \widehat{M}.$$

Earlier observations and elementary considerations involving the cohomology of mapping tori (*cf.* [25, Lemma 8.4, p. 67]) imply that the maps $\Sigma \subset \widehat{M}$, $\widehat{M} \to K(L, 1)$ and their counterparts $\Sigma' \subset \widehat{M'}$, $\widehat{M'} \to K(L', 1)$ all induce isomorphisms in $H^2(-;\mathbb{Z})/\mathsf{Torsion}$. Choose δ_{st} so that $(P_s \circ \varphi \circ J_t)^*$ maps a generator of

$$H^2(K(L', 1); \mathbb{Z})/\mathsf{Torsion} \approx \mathbb{Z}$$

to δ_{st} times a generator of

$$H^2(K(L', 1); \mathbb{Z})/\mathsf{Torsion} \approx \mathbb{Z}.$$

A diagram chase then shows that

$$|\deg(\lambda_{st})| = |\deg(\delta_{st})| \cdot |\deg(\Psi)|.$$

However, we had previously noted that $\deg(\delta_{st}) = 0$ if $s \neq t$, and this establishes the claim. $\square$

Corollary 33 *We also have* $\deg(\lambda_{tt}) = \pm 1$.

This follows immediately because $(\lambda_{tt})_*$ induces isomorphisms in homology. $\square$

We have thus established the existence of a map $M \to M'$ of degree ± 1 in Subcase 3.2, and the rest of the argument for this subcase proceeds as for Subcase 3.1. $\square$

Remark 34 There is an analog of Theorem 3 for planar crystallographic groups: *Given two such groups G, H of this type, relatively elementary considerations like those of* [43, p. 120, and Cor. 1.10.8 on p. 136] *imply that $G \approx H$ if $G^n \approx H^n$ for some $n \geq 2$.*

4 Proof of Theorem 3 for other irreducible 3-manifolds

As the title suggests, this section proves Theorem 3 for irreducible 3-manifolds that are not large Seifert 3-manifolds. Not surprisingly, the proof of splits into subcases depending upon the geometry of the manifold M.

Subcase 4.1 *M is a spherical spaceform with a finite nonabelian fundamental group.*

In this case we have $[\pi_1(M)]^n \approx [\pi_1(N)]^n$ for some $n \geq 2$. Since $\pi_1(M)$ is finite it follows that $[\pi_1(M)]^n \approx [\pi_1(N)]^n$ must also be finite, which in turn implies that $\pi_1(N)$ is finite. Furthermore, it follows that $\pi_1(N)$ must be nonabelian (otherwise $[\pi_1(M)]^n \approx [\pi_1(N)]^n$ would also be abelian and likewise for $\pi_1(M)$), and since N is geometric it follows that N must also be a spherical spaceform with a finite nonabelian fundamental group.

Since M and N are different from lens spaces, then as noted previously in Theorem 14 they are determined up to homeomorphism by their fundamental groups (once again see [32] and also [21, p. 737, Case 1]), and therefore the proof of the theorem in Subcase 4.1 reduces to checking that if G and H are finite groups such that $G^n \approx H^n$ for some $n \geq 2$ then $G \approx H$. Applying this to $\pi_1(M)$ and $\pi_1(N)$ and combining it with the first sentence of this paragraph, we may then conclude that M and N must be homeomorphic.

If G and H are finite groups that are not isomorphic to direct products of nontrivial subgroups (*i.e.*, not *directly decomposable*), then the group-theoretic statement is merely the Wedderburn–Remak–Schmidt Theorem (see [11, p. 130]). The general case is an easy consequence of this result, but since it is not easy to find a specific reference we shall include a proof for the sake of completeness: By finiteness one can write G and H as direct products $G \approx \prod_i G_i$ and $H \approx \prod_j H_j$ where each of the groups G_i and H_j is nontrivial but not directly decomposable. Let $L_1, ..., L_k$ be the isomorphism types of the direct factors G_i; *i.e.*, each G_i is isomorphic to exactly one of the groups L_α. Then the Wedderburn–Remak–Schmidt Theorem implies that each H_j is also isomorphic to exactly one of the groups L_α.

Given a finite group K such that L_α is a direct factor of K, define the multiplicity $\mu(\alpha, K)$ such that K is isomorphic to a direct product of $\mu(\alpha, K)$ copies of L_α together with other subgroups that are not isomorphic to L_α and also are not directly decomposable. The Wedderburn–Remak–Schmidt Theorem implies that this multiplicity is well defined and that we have the following identities:

(1) $G \approx \prod_\alpha L_\alpha^{\mu(\alpha,G)}$
(2) $H \approx \prod_\alpha L_\alpha^{\mu(\alpha,H)}$)

(3) $\mu(\alpha, G^n) = n \cdot \mu(\alpha, G)$ for all α.
(4) $\mu(\alpha, H^n) = n \cdot \mu(\alpha, H)$ for all α.

If $G^n \approx H^n$ then the quantities in the last two identities are equal, and this implies that $\mu(\alpha, G) = \mu(\alpha, H)$ for all α. Combining this with the first two identities, we see that G and H must be isomorphic. As noted in the second paragraph above, this completes the proof in Subcase 4.1. $\square$

We are now left with three cases where $\pi_1(M)$ is infinite, depending on whether M is hyperbolic, Seifert fibered but not large, or neither (in which case M is Haken).

Subcase 4.2 *M is hyperbolic.*

In this case we claim that there are maps of degree ± 1 from M to N and vice versa. Lemma 7 shows that there are maps $h_1 : M \to N$ and $h_2 : N \to M$ of nonzero degree, and their composite $h = h_2 \circ h_1 : M \to M$ also has nonzero degree. We claim that the degree of h is ± 1. To see this, consider the Gromov norm $\|M\|$ of M (see [10]); in this case $\|M\|$ is the normalized volume of M given by

$$\|M\| = \frac{\mathrm{vol}\,(M)}{v_3}$$

where v_3 is the volume of a regular ideal simplex in the standard hyperbolic space $\mathbf{H}^3$. One of the basic properties of $\|M\|$ is that for $f : M \to M$ one has $\|M\| \geq |\deg(f)| \cdot \|M\|$. This implies that h has degree ± 1, and it follows that the degrees of h_1 and h_2 are also ± 1. Therefore by Lemma 13 we know that each of h, h_1 and h_2 is a homotopy equivalence, and as before each of them is homotopic to a homeomorphism by Mostow's rigidity theorem [26,27] (see also the discussion in [21], p. 738). $\square$

Subcase 4.3 *M is neither Seifert fibered nor hyperbolic.*

We first claim that N is also *neither* Seifert fibered *nor* hyperbolic. Suppose to the contrary that one of these is true. Since the hypotheses of the theorem are symmetric in M and N, by the previously established cases we can conclude that M should be Seifert fibered if N were Seifert fibered, and likewise M would be hyperbolic if N were hyperbolic. Hence, as noted before, the classification of geometric 3-manifolds implies that both M and N are Haken. The fundamental decomposition of M into pieces bounded by incompressible tori (*cf.* Jaco-Shalen [18] and Johannson [17]) decomposes M as

$$M \quad = \quad X \bigcup_{\Sigma} Y$$

where Σ is a family of essential tori in M. Moreover, each component of X is Seifert fibered and each component of Y has a complete hyperbolic structure with finite volume. The existence of such hyperbolic structures is just W.

Thurston's hyperbolization theorem for Haken manifolds (*cf.* [36]). Let us assume $Y \neq \varnothing$. By Gromov's Cutting Theorem [10] (see also the discussion in [34, pp. 141–144]), we have

$$\|M\| = \|Y\| \neq 0 \ .$$

Since $\pi_1(M)$ is again residually finite (*cf.* [14]) in this case, one can now proceed as in Subcase 4.2 to find degree one maps $M \to N$ and $N \to M$, and this proves Subcase 4.3 of Theorem 3 when $Y \neq \varnothing$.

Suppose now that $Y = \varnothing$. It follows that M is a "graph manifold" (see [28, p. 131]) with a nontrivial torus decomposition. Consider, once again, the composite $g \circ f$ where $f : M \to N$ and $g : N \to M$. Then the degree of $g \circ f$ is nonzero and hence by [41, Lemma 4.2, p. 186] the composite degree is ± 1. This implies the degrees of f and g are ± 1, and as before the existence of degree one maps both ways suffices to prove Subcase 4.3 of Theorem 3 when Y is empty (once again we use Theorem 14). □

Subcase 4.4 *M is Seifert fibered but not a spherical spaceform (hence has infinite fundamental group).*

The previous considerations (Subcases 4.1–4.3) show that without loss of generality we can assume that N is Seifert fibered as well and that both M and N have infinite fundamental groups. By Theorem 16 it follows that M and N are both large, and consequently the results of the previous section imply that M and N are homeomorphic. □

5 Reducible 3-manifolds

One can combine the methods of this paper together with those of S. C. Wang [41, specifically, Remark 3.7 on p. 185] and the Kneser Conjecture (*cf.* [13, p. 66]) to obtain the following version of Theorem 3 for reducible manifolds:

Theorem 35 *Let M and N be closed, connected, oriented, reducible, geometric 3-manifolds such that $H^1(M;\mathbb{Z})$ and $H^1(N;\mathbb{Z})$ are trivial and $\prod^n M \approx \prod^n N$ for* some $n \geq 2$. *Then if M or N does not contain a lens space in its prime decomposition we have $M \approx N$.*

PROOF. Here is an outline of the argument: Let

$$M = M_1 \# \cdots \# M_k \# M_{k+1} \# \cdots N_{k+r}$$

$$N = N_1 \# \cdots \# N_s \# N_{s+1} \# \cdots N_{s+p}$$

be prime decompositions of M and N with the summands indexed so that the first k summands of M have finite fundamental groups and likewise for the first s summands of N. Lemma 7 shows the existence of nonzero degree maps $f : M \to N$ and $g : N \to M$. Let us assume that neither M nor N is homeomorphic to $\mathbb{RP}^3 \# \mathbb{RP}^3$. It follows then from [41, Remark 3.7, p. 185] that the self maps $(gf)_*$ of $\pi_1(M)$ and $(fg)_*$ of $\pi_1(N)$ induce isomorphisms, and therefore that f and g induce isomorphisms of fundamental groups. Therefore, unless r and p are both zero in the decompositions above, it follows that the bijectivity of f and g on fundamental groups implies that the maps are homotopy equivalences by a result of G. A. Swarup [35, Cor. 2.3]. Combining these with the Splitting Theorem of H. Hendriks and F. Laudenbach [15, Théorème de scindement, p. 203] and the validity of the Kneser Conjecture, we conclude that the respective summands of M and N are homotopy equivalent. The absence of lens spaces in the prime decompositions and Theorem 14 imply that these summands are homeomorphic.

Assume now that r and p are both zero. The isomorphism of fundamental groups and the validity of Kneser's conjecture imply that $k = s$, and that the corresponding summands have the same fundamental groups. The result is then completed by using some of the previous arguments.

Finally, suppose that $M = \mathbb{RP}^3 \# \mathbb{RP}^3$, so that $\pi_1(M)$ is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, which is isomorphic to the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$ of $\mathbb{Z}$ and $\mathbb{Z}_2$ with nontrivial twisting homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}) \approx \{\pm 1\}$. Note that the commutator group corresponds to $2\,\mathbb{Z}$ in the semidirect product and the copy of $\mathbb{Z}$ is its centralizer. We shall prove that $\pi_1(M) \approx \pi_1(N)$ using a variant of the proof of Theorem 3 for Seifert manifolds. Since $(\mathbb{Z}_2 * \mathbb{Z}_2)^n$ and $[\pi_1(N)]^n$ are isomorphic, it follows that their abelianizations are also isomorphic. Since the abelianization of the free product is equal to $\mathbb{Z}_2 \times \mathbb{Z}_2$ it follows that the abelianization of $[\pi_1(N)]^n$ is isomorphic to $(\mathbb{Z}_2)^{2n}$, and by the structure theorems for finite abelian groups we conclude that the abelianization of $\pi_1(N)$ must also be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Similarly the commutator subgroup of $[\pi_1(N)]^n$ is isomorphic to the commutator subgroup of $(\mathbb{Z}_2 * \mathbb{Z}_2)^n$, which is a free abelian group of rank n. Furthermore, the centralizer of this commutator subgroup is also free abelian of rank n and the inclusion of the commutator subgroup in its centralizer corresponds to the standard inclusion of $2\,\mathbb{Z}^n$ in $\mathbb{Z}^n$. Since the commutator subgroup of a product (*resp.*, the centralizer of this subgroup) is just the product of the commutator subgroups (*resp.*, their centralizers), it follows that the centralizer C of the commutator subgroup of $\pi_1(N)$ is infinite cyclic, the same is true for its centralizer, and the commutator subgroup has index two in its centralizer. Finally, C must have index 2 in $\pi_1(N)$ and hence must be a normal subgroup.

Note that $\pi_1(N)$ cannot be abelian, for if it were then one could conclude that $\pi_1(M) \approx \mathbb{Z}_2 * \mathbb{Z}_2$ was also abelian. Therefore the action of $\pi_1(N)/C \approx \mathbb{Z}_2$

on $C \approx \mathbb{Z}$ by inner automorphisms must be nontrivial; otherwise the group extension $\pi_1(N)$ would be abelian. This means that the group extension is given by an element of the twisted cohomology group $H^2(\mathbb{Z}_2; \mathbb{Z}^-)$, where $\mathbb{Z}^-$ denotes the integers with the nontrivial action of $\mathbb{Z}_2$. It is well known that this cohomology group vanishes, and therefore it follows that $\pi_1(N)$ must be a semidirect product and hence must be isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. One can now use the validity of the Kneser Conjecture together with the geometrization hypothesis on N and Theorem 15 to conclude that the fundamental group for each irreducible summand of N must be isomorphic to $\mathbb{Z}_2$ and hence that N must be diffeomorphic to $M = \mathbb{RP}^3 \# \mathbb{RP}^3$. $\square$

As noted before, the case where M and N both have lens spaces in their prime decompositions would require a combination of the results from Hendriks-Laudenbach [15] with surgery theoretic considerations as in the authors' work on squares of lens spaces [22], but it is not clear if adequate machinery currently exists to study this effectively.

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