Isovariant homotopy theory and the Gap Hypothesis

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Note: *This paper was first written in 1994 and is currently awaiting revision.*

Many existence and classification questions in the topology of manifolds can be reduced to problems in algebraic topology by a collection of techniques known as *surgery theory*; a nontechnical but informative description of this subject appears in [R, pp. 375–378]. For well over three decades topologists have also known that such techniques also have far reaching implications for manifolds with group actions (*cf.* [Br1] and [R, pp. 378–379]). Many of the most striking applications of surgery theory require some assumption on the manifolds or mappings under consideration. One basic restriction on the underlying manifolds is known as the **Standard Gap Hypothesis**: *For each pair of isotropy subgroups $H \supsetneqq K$ and each pair of components $B \subset M^H$, $C \subset M^K$ such that $B \subsetneqq C$ we have* $\dim B + 1 \leq \frac{1}{2}(\dim C)$. A condition of this sort first appeared explicitly in unpublished work of S. Straus [St], and the importance and usefulness of the restriction became apparent in work of T. Petrie [P1–2] (see also [DP], [DR], and [LüMa]). Applications of surgery to group actions that do not require the Gap Hypothesis frequently assume that the underlying maps of manifolds are *isovariant* (*cf.* [BQ], [DuS], [Sc3], and [We]). The following unpublished result of Straus [St] and W. Browder [Br2] establishes a fairly strong and precise connection between isovariance and the Gap Hypothesis.

Theorem 1. *Let $f : M \to N$ be an equivariant homotopy equivalence of oriented closed smooth semifree G-manifolds that satisfy the Gap Hypothesis. Then f is equivariantly homotopic to an isovariant homotopy equivalence. Furthermore, if $M \times [0,1]$ satisfies the Gap Hypothesis then this isovariant homotopy equivalence is unique up to isovariant homotopy.*

In this context Browder also constructed degree one maps $f : M \to N$ such that the Gap Hypothesis holds but f is not equivariantly homotopic to an isovariant map.

Theorem 1 and Browder's examples lead immediately to some additional questions. Although the result is essentially a homotopy theoretic statement, the proofs in [St] and [Br2] require fairly deep results from Wall's nonsimply connected surgery theory; ideally one would like a more homotopy theoretic approach that would also provide more insight into the negative examples and yield additional insight into results of Dovermann [Do] on isovariant normal maps and the isovariance obstructions of [DuS, §5]. The main objective of this paper is to provide a criterion for deforming an equivariant degree

The author was partially supported by National Science Foundation Grants DMS 86-02543, 89-02622 and 91-02711, and also by the Max-Planck-Institut für Mathematik in Bonn.

one map into an isovariant map when the Gap Hypothesis holds, to use this criterion to provide an essentially homotopy-theoretic proof of the theorem, and to see how the criterion applies to the examples. In contrast to [St] and [Br2], our approach requires a minimum of input from geometric topology; namely, nonequivariant transversality and the existence of smooth embeddings in the general position range.

Here is a brief explanation of the criterion: The first step is to note that a degree one equivariant normal map can be approximated by a map that is isovariant near the fixed point set if the Gap Hypothesis holds; more precisely, we show that one can make the map *normally straightened* near the fixed point set in the sense of [DuS, §4]. The main result (Theorem 6) says that such a map can be equivariantly deformed into an isovariant map if an only if one can find a map h in the equivariant homotopy class that is normally straightened near the fixed point set and such that the set of *nonisovariant points*, for which the isotropy subgroups satisfy $G_x \subsetneqq G_{h(x)}$, lies in a tubular neighborhood of the fixed point set of the domain; since the necessity of the condition is trivial, Theorem 6 only deals with sufficiency. By Proposition 4 the appropriate condition holds for equivariant homotopy equivalences, and therefore Theorem 1 is an immediate consequence of Theorem 6 and Proposition 4. As noted in the discussion following Example 5, it is straightforward to verify that the criterion of Theorem 6 fails for Browder's examples.

Generalizations to nonsemifree actions. Theorem 1 remains valid if the semifreeness hypothesis is removed, but the general case requires additional notation and inductive machinery to deal with the various orbit types and related objects. A proof will be given in a sequel to [DuS].

Implications for equivariant surgery. The methods and results of [DuS] provide a means for analyzing isovariant homotopy theory—and its relation to equivariant homotopy theory—within the standard framework of algebraic topology. Therefore Theorem 1 and the conclusions of [DuS] suggest a two step approach to analyzing smooth G-manifolds within a given equivariant homotopy type if the Gap Hypothesis does not necessarily hold; namely, the first step is to study the obstructions to isovariance for an equivariant homotopy equivalence and the second step is to study one of the versions of isovariant surgery theory from [Sc3] or [We]. This approach seems especially promising for analyzing classification questions using surgery theory and homotopy theory.

Acknowledgments. I am grateful to Bill Browder for helpful conversations and correspondence regarding his results on the questions treated here, and especially for providing a detailed account of his counterexamples. I would also like to thank Heiner Dovermann for various conversations involving his work.

1. Preliminary adjustments

The first step in the proof of Theorem 1 is essentially a combination of several elementary observations, and it shows that an equivariant homotopy equivalence $f : M \to N$ can be made homotopically transverse to N^G on a neighborhood of M^G; more precisely, we shall deform f equivariantly to a map that is *normally straightened* near the fixed point set in the sense of [DuS, §4].

Proposition 2. *Let M and N be closed smooth semifree G-manifolds (where G is a finite group), and let $f : M \to N$ be an equivariant degree one map such that (i) the stable equivariant normal bundle of M is the pullback of some stable G-vector bundle over N, (ii) f defines an equivariant homotopy equivalence from M^G to N^G. Denote the equivariant normal bundles of M^G and N^G in M and N by α_M and α_N respectively. Then f is equivariantly homotopic rel M^G to a map f' such that f' maps the sphere bundle $S(\alpha_M)$ to $S(\alpha_N)$ by an equivariant fiber homotopy equivalence and the restriction of f' to the disk bundle $D(\alpha_M)$ is the radial extension of $f|S(\alpha_M)$.*

If f is an equivariant homotopy equivalence then (ii) is automatically true and (i) holds because every G-vector bundle over M is isomorphic to the pullback of a G-vector bundle over N.

Proof. Let τ_M and τ_N be the equivariant tangent bundles of M and N. We claim that the sphere bundles of τ_M and $f^*\tau_N$ are stably equivariantly fiber homotopically equivalent. The nonequivariant version of this statement is well known (*cf.* [A]) and the proof is essentially a formal exercise in S-duality, and the equivariant cases follows from the same considerations combined with equivariant S-duality as developed in [Wi].

Consider next the restriction of the stable equivariant fiber homotopy equivalence $S(\tau_M) \sim S(f^*\tau_N)$ to M^G. The classifying maps for the two equivariant fibrations go from M^G to a space $\mathcal{B}$ such that $\pi_*(\mathcal{B}) \approx \pi^G_{*-1}$, where the latter denotes an equivariant stable homotopy group as in [Se]. On the other hand, by [Se] we also know that $\mathcal{B}$ is homotopy equivalent to the product $BF \times BF_G$ where BF classifies nonequivariant stable spherical fibrations and BF_G is defined as in [BeS]. In terms of fibrations the projections of the classifying maps $M \to \mathcal{B}$ onto BF and BF_G correspond to taking the classifying maps of the fixed point subbundles and the orthogonal complements of the fixed point subbundles respectively. Therefore it follows that the corresponding subbundles for τ_M and $f^*\tau_N$ are stably equivariantly fiber homotopy equivalent. In particular, this means that $S(\alpha_M)$ and $S((f^G)^*\alpha_N)$ are stably equivariantly fiber homotopy equivalent because they induce homotopic maps from M^G into BF_G.

Write M^G as a disjoint union of components F_β, and for each β let q_β be the codimension of F_β. Furthermore, denote the fiber representation at a point of F_β by V_β. The stabilization map $F_G(V_\beta) \to F_G$ is $(q_\beta - 2)$-connected by the considerations of [Sc1] and the Gap Hypothesis implies that $\dim F_\beta \leq q_\beta - 2$. Therefore we can destabilize the stable fiber homotopy equivalence from $S(\alpha_M)$ to $S((f^G)^*\alpha_N)$ and obtain a genuine equivariant fiber homotopy equivalence. Choose such an equivariant fiber homotopy equivalence, say Φ. It is then an elementary exercise to deform $f|D(\alpha_M)$ equivariantly rel the zero section so that one obtains the radial extension of Φ at the other end of the deformation. By the equivariant homotopy extension property one can extend this homotopy to all of M.■

If we define the set of *nonisovariant points* as in the introduction, then Proposition 2 allows us to deform an equivariant homotopy equivalence of semifree G-manifolds so that the set of nonisovariant points takes a relatively simple form.

Corollary 3. *Let f, etc. be as in Proposition 2. Then f is equivariantly homotopic to a map h such that the set of nonisovariant points $\mathcal{A}(h)$ is a union of submanifolds whose dimensions are those of the components of M^G and/or N^G, and the normal bundles of* *the components of $\mathcal{A}(h)$ are pullbacks of the normal bundles of appropriate components of N^G.*

Proof. By Proposition 2 f is equivariantly homotopic rel M^G to a map f' that is homotopy transverse to N^G on a tubular neighborhood, say V, of M^G. Since G acts freely on $M - V$ we may equivariantly deform $f'|M - V$ to a map h_0 that agrees with f' near the frontier of V and is transverse to the components of N^G. By definition the set of nonisovariant points of h_0 is the transverse inverse image of N^G. Extend h_0 to a map h on M by setting $h|V = f'$ and extend the homotopy from the restriction of f' to h_0 by taking the stationary homotopy $H_t = f'$ on $V \times [0, 1]$. It follows immediately that h has the desired properties.■

Engulfing the nonisovariant points

The following observation is a crucial step in the proof of Theorem 1.

Proposition 4. *Let $f : M \to N$ be an equivariant homotopy equivalence of closed smooth semifree G-manifolds, and suppose that f is homotopy transverse to N^G near M^G in the sense of Proposition 2. Assume further that the Gap Hypothesis holds and that $f|M - M^G$ is transverse to N^G. Then there is a tubular neighborhood V_1 of M^G such that the set of nonisovariant points of f lies in V_1.*

If we knew that f mapped V_1 into a tubular neighborhood of N^G then we could conclude that f was isovariant by the results of [DuS, §4]. However, we do not know this and therefore some additional work will be necessary.

Proof. As before let $\mathcal{A}(f)$ denote the set of nonisovariant points, and let $f_1 : \mathcal{A}(f) \to N^G$ be the map defined by f. Next let h^* be a homotopy inverse to f; it follows immediately that the composite $h^* f_1$ is homotopic to the inclusion of $\mathcal{A}(f)$ in M. Approximate $h^* f_1$ nonequivariantly by a smooth embedding into a tubular neighborhood of M^G, and similarly approximate the homotopy between this embedding and the inclusion by a smooth isotopy. The Gap Hypothesis ensures that such embeddings and isotopies can be constructed. It follows that the submanifold $\mathcal{A}(f) \subset M$ can be isotopically deformed to lie in a tubular neighborhood of M^G, and the conclusion of the proposition follows directly from this and the Isotopy Extension Theorem.■

Complement to Proposition 4. *If $M^G = \cup_j F_j$ is the decomposition of M^G into components and $N^G = \cup_j F'_j$ is the corresponding decomposition such that $f(F_j) = F'_j$, then one can can choose the tubular neighborhood V_1 so that $f^{-1}(F'_j)$ lies in the component of V_1 containing F_j for all j.*■

Notation. We shall say that the set of nonisovariant points is *regularly engulfed* in a tubular neighborhood of the fixed point set if the condition in the preceding result is satisfied.

The following examples of W. Browder show that the conclusion of Proposition 4 and Theorem 1 both fail for degree one maps, even if the conditions of Proposition 2 are satisfied.

Examples 5. Let k and q be distinct positive integers such that q is even and G has a free q-dimensional linear representation. Let $N = S^k \times S^q$ with trivial action on the first coordinate and the one point compactification of the free linear action on the second, let M_0 be the disjoint union of N and two copies of $G \times S^k \times S^q$ (where G acts by translation on itself and trivially on the other two coordinates), and define an equivariant map $f_0 : M_0 \to N$ by taking the identity on M, the unique equivariant extension of the identity map on $S^k \times S^q$ over one copy of $G \times S^k \times S^q$, and the unique equivariant extension of an orientation reversing self diffeomorphism of $S^k \times S^q$ over the other copy. By construction this map has degree one, and one can attach 1-handles equivariantly to M_0 away from the fixed point set to obtain an equivariant cobordism of maps from f_0 to a map f on a connected 1-manifold M that is nonequivariantly diffeomorphic to a connected sum of $2 \cdot |G| + 1$ copies of $S^k \times S^q$. Since the fixed point sets of M and N are k-dimensional and the manifolds themselves are $(k+q)$-dimensional, it follows that the Gap Hypothesis holds if we impose the stronger restriction $q \geq k+2$. By construction the map f satisfies the homotopy transversality condition near the fixed point sets that was introduced in Proposition 2.

Assertion. *It is not possible to deform f equivariantly so that the set of nonequivariant points lies in a tubular neighborhood of the fixed point set. In particular, it is also not possible to deform f equivariantly to an isovariant map.*

To prove the assertion, assume that one has a map h equivariantly homotopic to f with the stated property, and let U be a tubular neighborhood of M^G that contains the set of nonisovariant points. Let X be a submanifold of the form $\{g\} \times \{v\} \times S^q$ in M that arises from one of the copies of $G \times S^k \times S^q$ in M_0. Although X and U may have points in common, by the uniqueness of tubular neighborhoods we can always isotop X into a submanifold X' that is disjoint from U. By the hypotheses on h we know that $h(X')$ is disjoint from $N^G = S^k \times S^0$, and therefore $h(X')$ is also disjoint from a standardly embedded copy of S^k in $S^k \times S^q$. Standard considerations involving intersection numbers now imply that the image of the fundamental class of X in $H_*(N)$ under h_* is trivial. On the other hand, h is homotopic to f, and by construction the image of the fundamental class of X in $H_*(N)$ under f_* is nontrivial. This contradiction shows that the map h cannot exist.■

2. Isovariance obstructions

In view of the results of Section 1 the obvious next step is to consider an equivariant homotopy equivalence $f : M \to N$ of closed smooth (semifree) G-manifolds such that (i) the map f is normally straightened near the fixed point set, (ii) the set of nonisovariant points lies in a tubular neighborhood of M^G and the additional conditions in the complement to Proposition 4 also hold. If α is the componentwise equivariant normal bundle of M^G, then by hypothesis the map is already equivariant on the closed complement of a submanifold T of the form $S(\alpha) \times [0,1]$ where $S(-)$ denotes the unit sphere bundle. Therefore the proof of Theorem 1 would follow if one could deform $f|T : T \to N$ equivariantly rel ∂T into $N - N^G$. Although it is not always possible to find such deformations, there is a slightly weaker sufficient condition that we shall verify by means of isovariance obstructions similar to those of [DuS, §5].

In order to discuss these isovariance obstructions we need to split M^G as a union of components F_j as in the Complement to Proposition 4, and a treatment of this sort requires some additional notation. If $M = \cup_j F_j$ is given as in the Complement to

Proposition 4 let $N^G = \cup_j F'_j$ be the corresponding decomposition such that $f(F_j) = F'_j$, let

$$M_j = M - \cup_{i \neq j} F_i \qquad\qquad N_j = N - \cup_{i \neq j} F'_i$$

let T_j be the portion of T that lies over F_j, and let f'_j be the factorization of $f_j|T_j$ through N_j (this uses the extra condition in the Complement to Proposition 4). Furthermore, let $S(\alpha_j)$ and $D(\alpha_j)$ be the unit sphere and disk bundles for the equivariant normal bundle of F_j, let $S(\alpha'_j)$ and $D(\alpha'_j)$ be the corresponding bundles for F'_j, let S_j be a fiber of $S(\alpha_j)$, let $\partial_0 T$ and $\partial_1 T$ be the boundary components corresponding to $S(\alpha) \times \{0\}$ and $S(\alpha) \times \{1\}$ respectively, and define $\partial_i T_j = \partial_i T \cap T_j$.

We are now ready to state the main technical result of this paper on approximating equivariant degree one maps by isovariant maps.

Theorem 6. *Let M and N be closed oriented smooth semifree G-manifolds that satisfy the Gap Hypothesis, and let $f : M \to N$ be an equivariant degree one map such that the map of fixed point sets f^G is a homotopy equivalence and f is normally straightened near the fixed point set. Then f is G-homotopic to an isovariant map if the set of nonisovariant points is regularly engulfed in a tubular neighborhood of M^G.*

Proof. We begin by formulating the weak sufficient condition that was mentioned in the first paragraph of this section. For each value of j the map f determines a map of triads

$$f_j : (T_j; \partial_0 T_j, \partial_1 T_j) \to (N_j; D(\alpha'_j), N - N^G)$$

and by the results of [DuS, §4] it suffices to show that each such map of triads can be compressed equivariantly rel $\partial_1 T_j \cup S_j \times [0,1]$ into $(N - N^G; S(\alpha'_j), N - N^G)$. The methods of [DuS, §§1,5] imply that the obstructions to compression lie in diagram-theoretic Bredon equivariant cohomology groups of the form ${}_{BR}H^i(\mathbf{T}; \mathit{\Pi}_i)$, where $\mathbf{T}$ is the diagram associated to the triad $(T_j; \partial_0 T_j, \partial_1 T_j \cup S_j \times [0,1])$ and $\mathit{\Pi}_i$ is the following diagram of abelian groups:

$$\pi_i(D(\alpha'_j), S(\alpha'_j)) \xrightarrow{\ \partial_0^\# \ } \pi_i(N_j, N - N^G) \xleftarrow{\ \partial_1^\# \ } \{0\}$$

If $q_j = \dim M - \dim F_j$ then $(D(\alpha'_j), S(\alpha'_j))$ is $(q_j - 1)$-connected by the identity $\pi_s(D(\alpha'_j), S(\alpha'_j)) \cong \pi_{s-1}(S_j)$, and a standard general position argument shows that $(N_j, N - N^G) = (N_j, N_j - F'_j)$ is also $(q_j - 1)$-connected. Therefore the Blakers-Massey Theorem (*e.g.*, see [Gr, p. 143]) implies that the map $\partial_0^\#$ is bijective if $i \leq 2q_j - 3$ and surjective if $i = 2q_j - 2$. In particular, this means that the equivariant diagram cohomology groups ${}_{BR}H^i(\mathbf{T}; \mathit{\Pi}_i)$ reduce to ordinary Bredon cohomology groups ${}_{BR}H^i(T_j, \partial_1 T_j; \pi_i(N_j, N - N^G))$ if $i \leq n - 1$ or if $i = n$ and $\dim F_j + 3 \leq q_j$. Since $T_j \cong \partial_1 T_j \times [0,1]$ it follows immediately that the relative cohomology groups vanish in all such cases. Since $\dim T_j = \dim M = \dim N$, this implies that the isovariance obstructions vanish in all cases except perhaps when $i = n = 2q_j - 2$. In such cases the value group fits into the following exact sequence, which arises by restricting diagram-theoretic cochains in $C(X' \to X; \pi' \to \pi)$ to ordinary cochains in $C(X'; \pi')$:

$$(\star) \qquad \begin{array}{ccc} H^{n-1}\left(\partial_0 T_j; \pi_n(D(\alpha'_j), S(\alpha'_j))\right) & \xrightarrow{\ \Delta\ } & H^n(T_j, \partial T_j; \pi_n(N_j, N - N^G)) \\ & & \downarrow \\ & & {}_{BR}H^n(\mathbf{T}; \mathit{\Pi}_n) \\ & & \downarrow \\ & & H^n\left(\partial_0 T_j; \pi_n(D(\alpha'_j), S(\alpha'_j))\right) \end{array}$$

The map Δ in this sequence is given by combining the coefficient homomorphism for the map $\delta_0^{\#}$ in dimension n with the suspension isomorphism $H^{n-1}(\partial_0 T_j; \pi) \to H^n(T_j, \partial T_j; \pi)$. Therefore the Blakers-Massey Theorem, the $(n-1)$-dimensionality of $\partial_0 T_j$, and the Bockstein exact sequence for the short exact sequence

$$0 \longrightarrow \text{Kernel} \longrightarrow \pi_n(D(\alpha'_j), S(\alpha'_j)) \xrightarrow{\ \partial_0^{\#}\ } \pi_n(N_j, N - N^G) \longrightarrow 0$$

imply that Δ is onto. But the last object in $(\star)$ is zero because $\dim \partial_0 T_j = n - 1$, and it follows by exactness that ${}_{BR}H^n(\mathbf{T}; \mathit{\Pi}_n)$ is also trivial. Therefore the isovariance obstructions always vanish.■

Proof of existence in Theorem 1. The results of Section 1 imply that the given equivariant homotopy equivalence can be equivariantly deformed to a map with all the properties in the hypothesis of Theorem 6, including the condition on the set of nonisovariant points. Therefore f is equivariantly homotopic to an isovariant map f' by Theorem 6. By the isovariant Whitehead Theorem of [DuS, §4] the map f' is an isovariant homotopy equivalence if f' defines a homotopy equivalence from $M - M^G$ to $N - N^G$. General position considerations imply that f' induces an isomorphism of fundamental groups, and therefore it suffices to check that f' defines an isomorphism in homology with twisted coefficients in the group ring of the fundamental group. Exact sequence and excision arguments show that the latter holds if f' induces homotopy equivalences from M to N, from M^G to N^G, and from $\coprod S(\alpha_j)$ to $\coprod S(\alpha'_j)$. The first two of these follow because f' is an equivariant homotopy equivalence. To prove the third property first note that for each j the homotopy fibers of $S(\alpha_j) \subset D(\alpha_j)$ and $S(\alpha'_j) \subset D(\alpha'_j)$ are simply the fibers of the sphere bundles; since each $D(\alpha_j)$ maps to $D(\alpha'_j)$ by a homotopy equivalence, it suffices to know that a fiber of $S(\alpha_j)$ maps to a fiber of $S(\alpha'_j)$ with degree ± 1. This follows directly from the construction of the isovariant map; the first step was to make an equivariant homotopy equivalence normally straightened near the fixed point set, and the equivariant deformation in Theorem 6 is constant near some fiber of $S(\alpha_j)$.■

Implications in a special case. In his thesis [St] Straus used his version of Theorem 1 to obtain the following result:

Theorem 7. *Let M and N be closed smooth manifolds of dimension ≥ 2, let p be an odd prime, and suppose that M and N are homotopy equivalent. Let $\mathbb{Z}_p$ act smoothly on the p-fold self products $\Pi^p M$ and $\Pi^p N$ (where $\Pi^p X = X \times \cdots \times X$, p factors) by* *cyclically permuting the coordinates, and let* $\mathbf{D}^p(M)$*,* $\mathbf{D}^p(N)$ *be the invariant subsets sets given by removing the diagonals from* $\Pi^p M$ *and* $\Pi^p N$*. Then the deleted cyclic products* $\mathbf{D}^p(M)/G$ *and* $\mathbf{D}^p(N)/G$ *are homotopy equivalent.*

As also noted in [St], this result does not extend to compact bounded manifolds, and in fact the closed unit disks of various dimensions provide simple counterexamples. See [LöMa] and [Sc2] for related results.

Sketch of proof. If $f : M \to N$ is a homotopy equivalence then $\Pi^p f : \Pi^p M \to \Pi^p N$ is an equivariant homotopy equivalence of closed smooth $\mathbb{Z}_p$-manifolds. All actions of $\mathbb{Z}_p$ are semifree if p is prime, so this condition holds automatically; the Gap Hypothesis also holds because $p > 2$. Therefore Theorem 1 implies that $\Pi^p f$ equivariantly homotopic to an isovariant homotopy equivalence, and the latter in turn yields an equivariant homotopy equivalence from $\mathbf{D}^p(M)$ to $\mathbf{D}^p(N)$. The induced map of orbit spaces is the desired homotopy equivalence from $\mathbf{D}^p(M)/G$ to $\mathbf{D}^p(N)/G$.■

3. Relative statements and uniqueness preperties

Theorem 1 extends in an obvious fashion to manifolds with boundary provided the Gap Hypothesis holds on the boundary. This result and the uniqueness property are both direct direct consequences of the following relative statement.

Theorem 8. *Let* M *and* N *be compact bounded smooth semifree* G*-manifolds that satisfy the Gap Hypothesis, and let* $f : (M, \partial M) \to (N, \partial N)$ *be an equivariant homotopy equivalence that is an isovariant homotopy equivalence on the boundary. Then* f *is equivariantly homotopic rel* ∂M *to an isovariant homotopy equivalence.*

Before proving this we shall derive the results mentioned above.

Proof of uniqueness in Theorem 1 (*assuming Theorem* 8)*.* Suppose that f_0 and f_1 are equivariantly homotopic isovariant homotopy equivalences from M to N, and let $h : M \times [0,1]$ be an equivariant homotopy joining them. If $H : M \times [0,1] \to N \times [0,1]$ is the map $H(x,t) = (h(x,t), t)$ then H defines an equivariant homotopy equivalence of manifolds with boundary that is isovariant on the boundary. Apply Theorem 8 to deform H equivariantly rel boundary to an isovariant homotopy equivalence H', and take h' to be the composite of H' with projection onto N. Then h' defines an isovariant homotopy from f_0 to f_1 because H' is isovariant and the projection onto N is also isovariant.■

Corollary 9. *Let* M *and* N *be compact bounded smooth semifree* G*-manifolds whose boundaries satisfy the Gap Hypothesis, and let* $f : (M, \partial M) \to (N, \partial N)$ *be an equivariant homotopy equivalence. Then* f *is homotopic as a map of pairs to an isovariant homotopy equivalence.*

A further refinement of our techniques shows that the isovariant equivalence in Corollary 9 is unique up to isovariant homotopy of pairs if $M \times [0,1]$ satisfies the Gap Hypothesis; verification of this is left to the reader.

Proof of Corollary 9. Assume Theorem 8 is true, and let ∂f be the induced map of boundaries. By the existence statement of Theorem 1 ∂f is equivariantly homotopic to an isovariant homotopy equivalence, and by the G-homotopy extension property we can use f and the homotopy of ∂f to construct an equivariant homotopy equivalence h that is homotopic to the original map (in the category of pairs) and is isovariant on the boundary. Application of Theorem 8 to h yields an isovariant homotopy equivalence that is homotopic to the original map of pairs.■

Proof of Theorem 8. Much of the argument is analogous to the proof of the existence statement in Theorem 1, so we shall focus on points where new ideas are needed. Since it is elementary to deform the original equivalence of pairs to a map that is isovariant on a neighborhood of the boundary, we shall assume this in the discussion below.

First of all, the methods of Proposition 2 imply that we can make the original map normally straightened on all components of M^G that are disjoint from the boundary. Let D be a closed tubular neighborhood of these components.

Extend D to a closed equivariant tubular neighborhood E of M^G that meets the boundary nicely and such that $f(E)$ lies in a similar tubular neighborhood E^* of N^G, let $\partial_1 E$ and $\partial_1 E^*$ be the sphere bundles for these tubular neighborhoods, and let U be another closed tubular neighborhood so that $E \subset U - \partial_1 U$ (∂_1 again denotes the boundary sphere bundle). An elementary transversality argument shows that there is an equivariant approximation f_1 to f rel $D \cup \partial M$ so that $f_1|\partial_1 E$ is equivariantly transverse to N^G on $\partial_1 E$. We can further approximate to obtain a map f_2 that is also transverse to N^G on $M - E_0$, where $E_0 = E - \partial_1 E$. As in the proof of Proposition 4 we can now isotop $\mathcal{A}(f_2) \cap M - E_0$ into U; in fact, we claim this can be done by an isotopy rel E. This follows immediately if there is a deformation of the inclusion $\mathcal{A}(f_2) \cap M - E_0 \subset M - M^G$ into $U - E_0$ rel $\partial_1 E$, and the existence of such a deformation follows from (i) the existence of deformations into U by the arguments arising in the proof of Proposition 4, (ii) the fact that the connectivity of $(M, M - M^G)$ is sufficiently large with respect to the dimension of $\mathcal{A}(f_2) \cap M - E_0$ (*i.e.*, general position considerations).

It follows that f_2 defines a map of triads $(M; E, M - M^G) \to (N; E^*, N)$ such that the set of nonequivariant points lies in the intersection of the interior of M with a tubular neighborhood of M containing E. One can now use the techniques of Theorem 6 to show that f_2 is equivariantly homotopic rel boundary to an isovariant map. The argument at the end of the existence proof will show this map is an equivariant homotopy equivalence provided we know that the induced map from $\partial_1 E$ to $\partial_1 E^*$ is a homotopy equivalence. As in the existence proof for Theorem 1, it suffices to verify that for each j a spherical fiber of $S(\alpha_j)$ maps to a spherical fiber of $S(\alpha'_j)$ with degree ± 1. If $F_j \cap \partial M = \varnothing$ this follows exactly as in the existence proof for Theorem 1. On the other hand, if $F_j \cap \partial M \neq \varnothing$ then it suffices to show that the induced map from $S(\alpha_j | F_j \cap \partial M)$ to $S(\alpha'_j | F'_j \cap \partial N)$ has this property, and in this case the desired property follows because the boundary map ∂f is an isovariant homotopy equivalence by hypothesis.■

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