Isovariant mappings of degree 1 and the Gap Hypothesis

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Abstract. Unpublished results of S. Straus and W. Browder state that two notions of homotopy equivalence for manifolds with smooth group actions — isovariant and equivariant — often coincide under a condition called the Gap Hypothesis; the proofs use deep results in geometric topology. This paper analyzes the difference between the two types of maps from a homotopy theoretic viewpoint more generally for degree one maps if the manifolds satisfy the Gap Hypothesis, and it gives a more homotopy theoretic proof of the Straus-Browder result.

Ever since the topological classification of surfaces, one basic theme in geometric topology has been the reduction of existence and classification questions for manifolds to problems in algebraic topology. A collection of techniques known as *surgery theory* has been particularly effective in this regard (compare [R, pp. 375–378]). For well over four decades topologists have also known that such techniques also have far reaching implications for manifolds with group actions (*cf.* [Br1] and [R, pp. 378–379]). Not surprisingly, many of the most striking applications of surgery theory require some assumption on the manifolds, mappings or structures under consideration, and for group actions the following restriction has been employed quite often:

Standard Gap Hypothesis: *For each pair of isotropy subgroups* $H \supsetneqq K$ *and each pair of components* $B \subset M^H$, $C \subset M^K$ *such that* $B \subsetneqq C$ *we have* $\dim B + 1 \leq \frac{1}{2}(\dim C)$.

A condition of this sort first appeared explicitly in unpublished work of S. Straus [St], and the importance and usefulness of the restriction became apparent in work of T. Petrie [P1–2] (see also [DP], [DR], and [LüMa]). Applications of surgery to group actions that do not require the Gap Hypothesis frequently assume that the underlying maps of manifolds are *isovariant* or *almost isovariant* (*cf.* [BQ], [DuS], [Sc3], and [We]). A mapping of $f : X \to Y$ of spaces with actions of a group G is said to be isovariant if it is equivariant — *i.e.*, $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$ — and for each x the isotropy subgroup G_x of all group elements fixing x is equal to the isotropy subgroup $G_{f(x)}$ of the image point (in general one can only say that the first subgroup is contained in the second). The general notion of almost isovariance is defined precisely in [DuS, p. 27], and the most important special case is reproduced below. For the time being, we merely note that

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(1) the sets of nonisovariant points ($G_x \neq G_{f(x)}$) for such a map may be pushed into very small pieces of the domain where they cause no problems,

(2) standard methods of homotopy theory extend directly to a suitably defined category of almost isovariant mappings (*cf.* [DuS] and [DF]),

(3) results of [DuS] show that almost isovariant homotopy and isovariant homotopy are equivalent in many important cases (including all smooth actions of finite cyclic p-groups), and a standard conjecture (believed by most workers in the area) states that the same is true for arbitrary smooth actions of finite groups.

The following unpublished result, due to Straus [St] for actions that are semifree (the group acts freely off the fixed point set) and W. Browder [Br2] more generally, implies a fairly strong and precise connection between almost isovariance and the Gap Hypothesis.

Theorem 1. *Let $f : M \to N$ be an equivariant homotopy equivalence of oriented closed smooth G-manifolds that satisfy the Gap Hypothesis. Then f is equivariantly homotopic to an almost isovariant homotopy equivalence.*

As noted ebove, in some cases the results of [DuS] allow one to replace "almost isovariant" by "isovariant" in the conclusion; in particular, this is true if the isotropy subgroups are normal and linearly ordered by inclusion.

Although Theorem 1 is a purely homotopy theoretic statement, the proofs in [St] and [Br2] require fairly deep results from Wall's nonsimply connected surgery theory [WL], which in turn depends upon other deep geometric results such as the classification theory for immersions (*cf.* [Ph]) and the Whitney process for pairs of double points that cancel algebraically (*cf.* [M]). It is natural to ask whether one can prove Theorem 1 without relying so extensively on such a large amount of auxiliary material (this is a special case of the classical scientific maxim called Ockham's Razor). In particular, since one can construct a version of obstruction theory for isovariant maps and define obstructions to finding an isovariant deformation of a given equivariant map [DuS], it is natural to search for a proof that is related to this obstruction theory. More generally, one would also like to understand the obstructions to isovariance for arbitrary equivariant mappings of degree ± 1 from one smooth manifold to another. Some basic test cases are examples of equivariant degree one mappings mentioned in [Br2] which are not equivariantly homotopic to isovariant maps; results of K. H. Dovermann on isovariant normal maps [Dov] also provide some motivation.

The main objective of this paper is to analyze the problem of deforming an equivariant degree one map into an isovariant map when the Gap Hypothesis holds, to use this criterion to provide an essentially homotopy-theoretic proof of Theorem 1, and to see how the criterion applies to equivariant homotopy equivalences and other basic examples. In contrast to [St] and [Br2], our approach requires a minimum of input from geometric topology; namely, nonequivariant transversality and standard results on smooth embeddings in the general position range. For the sake of clarity we shall restrict attention to finite group actions that are semifree in the sense described above; if G is cyclic of prime order, then all actions are semifree. We shall also discuss some applications of Theorem 1 to cyclic reduced products that were first considered in [St] and a few positive and negative results just outside the range of the Gap Hypothesis (further information on the latter will appear in sequels to this paper).

Statements of main results

Suppose that M and N are compact, oriented, semifree unbounded smooth G-manifolds satisfying the Gap Hypothesis such that all components of the fixed point sets, and suppose that $f : M \to N$ is a G-equivariant map of degree 1. Let $\{N_\alpha\}$ denote the set of components of N^G where we may as well assume that α runs through the elements of $\pi_0(N^G)$, suppose that the associated map f^G of fixed point sets defines a $1-1$ correspondence between the components of M^G and N^G; for each α let

$$M_\alpha \;=\; f^{-1}(N_\alpha) \cap M^G$$

and let $f_\alpha : M_\alpha \to N_\alpha$ denote the partial map of fixed point sets determined by f. Denote the equivariant normal bundles of M_α and N_α in M and N by ξ_α and ω_α respectively, and let $S(\nu)$ generically represent the unit sphere bundle of the vector bundle ν (with the associated group action if ν is a G-vector bundle).

Theorem 2. *Suppose we are given the setting above such that* $\dim M_\alpha = \dim N_\alpha$ *for each* α.

(*i*) *If f is homotopic to an isovariant map, then the for each α the map f_α has degree ± 1, and $S(\xi_\alpha)$ is equivariantly fiber homotopy equivalent to $S(f^*\omega_\alpha)$.*

(*ii*) *If the two conditions in the preceding statement hold, then f is equivariantly homotopic to a map that is isovariant on a neighborhood of the fixed point set.*

(*iii*) *If f is isovariant on a neighborhood of the fixed point set, then f is equivariantly homotopic to an isovariant map if and only if f is equivariantly homotopic to a map f_1 for which the set of nonisovariant points of f_1 is contained in a tubular neighborhood of M^G.*

Theorem 1 will be derived as a consequence of Theorem 2 and the following result:

Theorem 3. *In the setting of the previous result, suppose that f is an equivariant homotopy equivalence. Then f is equivariantly homotopic to an isovariant homotopy equivalence.*

The results of [Br2] also include a uniqueness statement (up to isovariant homotopy) if $M \times [0,1]$ and $N \times [0,1]$ satisfy the Gap Hypothesis. One can also use the methods of this paper together with some additional homotopy theoretic input to prove such a uniquess result. The necessary machinery to do so and to generalize our results beyond semifree actions will be developed in a subsequent paper.

Overview of the paper

We shall begin Section 1 by proving that the conditions in Theorem 2 are necessary for a map f as above to be properly homotopic to an isovariant map. The proof of sufficiency in Theorem 2 splits into two steps, both of which are carried out in Section 2. To motivate the first step, observe that an equivariant map of semifree G-manifolds is automatically isovariant on the fixed point set, so a natural starting point is to determine whether the given map can be equivariantly deformed to a map that is isovariant on a neighborhood of the fixed point set. If this is possible and we have a map with this additional property, the next step is to determine whether such a map can be further deformed to another one which is isovariant everywhere. Section 3 contains the proofs of Theorems 1 and 3. Finally, in Section 4 we shall give a previously unpublished application of Theorem 1 due to Straus [Str] and discuss certain situations in which the Gap Hypothesis fails. In a few situations there are analogs of Theorem 1, but in many — probably all but a few — other cases the result does not generalize.

Implications for equivariant surgery. The methods and results of [DuS] provide a means for analyzing isovariant homotopy theory — and its relation to equivariant homotopy theory — within the standard framework of algebraic topology. Therefore Theorem 1 and the conclusions of [DuS] suggest a two step approach to analyzing smooth G-manifolds within a given equivariant homotopy type if the Gap Hypothesis does not necessarily hold; namely, the first step is to study the obstructions to isovariance for an equivariant homotopy equivalence and the second step is to study one of the versions of isovariant surgery theory from [Sc3] or [We]. This approach seems especially promising for analyzing classification questions using surgery theory and homotopy theory.

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1. Preliminary adjustments and necessity

It will be convenient to begin with some notational conventions and elementary observations in order to simplify the main discussion and the proofs.

Let P be a compact unbounded smooth G-manifold, where G is a finite group. By local linearity of the action we know that the fixed point set P^G is a union of connected smooth submanifolds; as before, denote these connected components by P_α. For each α

let $D(P_\alpha)$ denote a closed tubular neighborhood. By construction these sets are total spaces of closed unit disk bundles over the manifolds P_α, so let $S(P_\alpha)$ and denote the associated unit sphere bundles; it follows that

$$\partial[D(P_\alpha)] \;=\; S(P_\alpha)\ .$$

Suppose now that M and N are smooth semifree G-manifolds and $f : M \to N$ is an equivariant mapping.

The "only if" direction of Theorem 2

Assume that we have the setting and notation introduced in order to state Theorem 2:

- (i) M and N are compact, oriented, semifree smooth G-manifolds satisfying the Gap Hypothesis.
- (ii) $f : M \to N$ is a G-equivariant map of degree 1.
- (iii) $\{N_\alpha\}$ denotes the set of components of N^G where we may as well assume that α runs through the elements of $\pi_0(N^G)$.
- (iv) The associated map f^G of fixed point sets defines a $1-1$ correspondence between the components of M^G and N^G.
- (v) If for each α we let

$$M_\alpha \;=\; f^{-1}(N_\alpha) \cap M^G$$

  then f_α is the continuous map from M_α to N_α determined by f.
- (vi) If the equivariant normal bundles of M_α and N_α in M and N are ξ_α and ω_α respectively, and let $S(\nu)$ and $D(\nu)$ generically represent the unit sphere and disk bundle of the vector bundle ν (with the associated group action since ν is a G-vector bundle).

Not surprisingly, we shall also use the notational conventions developed previously in this section.

Necessity proof for Theorem 2. Both of the basic conditions (i) and (ii) in Theorem 2 depend only on the equivariant homotopy class of a mapping of manifolds, so without loss of generality we may replace f by any map in the same proper equivariant homotopy class. In particular, since we are assuming that f is properly equivariantly homotopy to an isovariant map, we might as well assume that f itself is isovariant.

By the results of [DuS] (in particular, see Proposition 4.1 on page 27), the map f is isovariantly homotopic to a map f_0 such that for each α we have $f(\,D(M_\alpha)\,) \subset D(N_\alpha)$, $f(\,S(M_\alpha\,)) \subset S(N_\alpha)$, and

$$f\,(\;M - \cup_\alpha\ \mathrm{Int}D(M_\alpha)\;) \;\subset\; N - \cup_\alpha\ \mathrm{Int}D(N_\alpha)\ .$$

Let h_α be the associated map of pairs from $\big(D(M_\alpha), S(M_\alpha)\big)$ to $\big(D(M_\alpha), S(M_\alpha)\big)$ defined by f_0. Since the latter has degree 1, the same is true for each of the maps h_α. We have already noted that $D(-)$ and $S(-)$ are disk and sphere bundles over the appropriate components of fixed point sets, and therefore a simple spectral sequence argument implies that (a) the degrees of the maps f_α are all equal to ± 1, up to an equivariant homotopy of pairs the map h_α sends a spherical fiber in $S(M_\alpha)$ to a spherical fiber in $S(N_\alpha)$ by a map of degree ± 1. Therefore an equivariant analog of a classical result due to A. Dold [Dol] (see [Wn]) shows that there is a g-equivariant fiber homotopy equivalence from $S(\xi_\alpha)$ to $S(f_\alpha^* \omega_\alpha)$, where as before ξ_α and ω_α denoted the corresponding equivariant normal bundles for M_α and N_α. This completes the proof that Condition (i) holds. Since the set of nonisovariant points for an isovariant map is empty by definition, Condition (iii) is automatically true, so the proof in the unbounded case is complete.■

Some examples

It is not difficult to construct equivariant maps of degree 1 which satisfy the Gap Hypothesis but do not satisfy the statements in Condition (i) of Theorem 2 on degrees and equivariant normal bundles.

Example 1. Let V be a nontrivial semifree real representation of G such that $\dim V^G > 0$ and the Gap Hypothesis holds, and let S^V be the one point compactification, which is equivariantly homeomorphic to the unit sphere in $V \oplus \mathbb{R}$. It is well known that for each positive integer k there is G-equivariant m ap $h_k : S^V \to S^V$ such that $\deg h_k = 1$ and $\deg h_k^G = k|G| + 1$ (*e.g.*, this is a very special case of the equivariant Hopf Theorem stated in tom Dieck's book on the Burnside ring and equivariant homotopy theory [tD, Thm. 8.4.1, pp. 213–214]). Since the fixed point set is connected and the degree of the map on the fixed point set is not ± 1 if $|G| > 2$ or $k \neq 0$, it follows that h_k cannot be homotopic to an isovariant map. However, the map h_k does satisfy the second part of Condition (i) involving pullbacks of equivariant normal bundles because the equivariant normal bundle of $(S^V)^G$ in S^V is a product bundle.

Example 2. Let G be a cyclic group, assume that k, m and r satisfy $k \equiv 0(4)$, $k, m > 0$ and $2r > m + k$, and let γ be a complex r-dimensional vector bundle over S^k which represents a generator of $\pi_k(BU_r) \cong \mathbb{Z}$. Take M to be the associated $(2r+m)$-sphere bundle over S^k. Then G acts smoothly and fiber preservingly on M with fixed point set $S^k \times S^m$, and each point has an invariant open neighborhood on which the action of G is smoothly equivalent to the linear representation $V = \mathbb{R}^{k+m} \oplus \mathbb{C}^r$. If we collapse everything outside such a neighborhood to a point, we obtain an equivariant map of degree 1 from M to S^V. This map also has degree 1 on the fixed point sets, but we claim it does not satisfy the pullback condition for equivariant normal bundles. Since the equivariant normal bundle in M is the pullback of γ under the coordinate projection map from $S^k \times S^m \to S^m$, it will suffice to check that γ is not equivariantly fiber homotopically trivial. In fact, the underlying *nonequivariant* vector bundle is well known to be fiber homotopically *nontrivial* (compare [Ad]).

2. Normal straightening and relative isovariance obstructions

In this section we shall prove the implication of Theorem 2 in the other direction; namely, if Conditions (ii) is satisfied then one can make f isovariant near the fixed point set, and if (iii) is satisfied then the map f is equivariantly homotopic to an isovariant mapping. The first step is to examine the consequences of Condition (ii).

Proposition 2.1. *In the setting of Section 1, suppose that $f : M \to N$ is a continuous equivariant degree 1 map. Assume that each of the associated maps of fixed point components $f_\alpha : M_\alpha \to N_\alpha$ has degree ± 1 and that $S(\xi_\alpha)$ is equivariantly fiber homotopy equivalent to $S(f^*\omega_\alpha)$ for each α. Then there are closed, pairwise disjoint, equivariant tubular neighborhoods $D(M_\alpha)$ of the fixed point set components M_α and an equivariant mapping f_0 such that f_0 is equivariantly homotopic to f and for each α the restriction $f|D(M_\alpha)$ is isovariant.*

Proof. Choose equivariant fiber homotopy equivalences $h_\alpha : S(\xi_\alpha) \to S(f_\alpha^*\omega_\alpha)$, and let k_α be the composite of h_α with the canonical induced bundle mapping $S(f_\alpha^*\omega_\alpha) \to S(\omega_\alpha)$. Define $H_\alpha : D(M_\alpha) \to D(N_\alpha)$ using k_α and fiberwise radial extension. It follows that H_α is equivariantly homotopic to $f|D(M_\alpha)$ for each α, and hence by the equivariant Homotopy Extension Property we may deform f equivariantly to a map f_0 such that $f_0|D(M_\alpha) = H_\alpha$ for each α. Since each H_α is isovariant, it follows that the restriction of f_0 to a neighborhood of the fixed point set is isovariant as required.■

We shall conclude this section by proving the sufficiency of the condition in (iii). A key step in the proof of Theorem 1 will be to prove that an equivariant homotopy equivalence has this property.

Proposition 2.2. *Let M and N be as before, and suppose that $f : M \to N$ is a continuous equivariant map that is isovariant on a neighborhood of the fixed point set. If there is a system of closed tubular neighborhoods W_α of M^G such that the set of non-isovariant points lies in the interiors of the sets W_α, then f is equivariantly homotopic to an isovariant map.*

Note that we make no assumption about the images of the sets W_α, and in particular we do not assume that they lie in the tubular neighborhoods of the components of the fixed point set of N.

Proof. By assumption f is already isovariant on the closed complement of a submanifold T of the form $\cup_\alpha S(M_\alpha) \times [\frac{1}{2}, 1]$ where as usual $S(M_\alpha)$ denotes the boundary of a tubular neighborhood. Let T_α be the portion lying over M_α and denote the boundary components corresponding to $S(M_\alpha) \times \{\frac{1}{2}\}$ and $S(M_\alpha \times \{1\}$ by $\partial_0 T_\alpha$ and $\partial_1 T_\alpha$ respectively. Let

$$B_\alpha \;=\; M \;-\; \cup_{\beta\neq\alpha} M_\beta \qquad\qquad C_\alpha \;=\; N \;-\; \cup_{\beta\neq\alpha} N_\beta$$

let S_α be the spherical fiber of $S(M_\alpha)$, and let A_α correspond to the annulus $S_\alpha \times [\frac{1}{2}, 1]$.

For each value of j the map f determines a map of triads

$$f_\alpha : (T_\alpha; \partial_0 T_\alpha, \partial_1 T_\alpha) \to (C_\alpha; D(N_\alpha), N - N^G)$$

and by the results of [DuS, §4] it suffices to show that each such map of triads can be compressed equivariantly rel $\partial_1 T_\alpha \cup A_\alpha$ into $(N - N^G; S(N_\alpha), N - N^G)$. The methods of [DuS, §§1,5] imply that the obstructions to compression lie in diagram-theoretic Bredon equivariant cohomology groups of the form ${}_{BR}H^i(\mathbf{T}; \Pi_i)$, where $\mathbf{T}$ is the diagram associated to the triad $(T_\alpha; \partial_0 T_\alpha, \partial_1 T_\alpha \cup A_\alpha)$ and Π_i is the following diagram of abelian groups:

$$\pi_i(D(N_\alpha), S(N_\alpha)) \xrightarrow{\partial_0^\#} \pi_i(C_\alpha, N - N^G) \xleftarrow{\partial_1^\#} \{0\}$$

If $q_\alpha = \dim M - \dim F_\alpha$ then $(D(N_\alpha), S(N_\alpha))$ is $(q_\alpha - 1)$-connected by the identity $\pi_s(D(N_\alpha), S(N_\alpha)) \cong \pi_{s-1}(S_\alpha)$, and a standard general position argument shows that $(C_\alpha, N - N^G) = (C_\alpha, C_\alpha - N_\alpha)$ is also $(q_\alpha - 1)$-connected. Therefore the Blakers-Massey Theorem (*e.g.*, see [Gr, p. 143]) implies that the map $\partial_0^\#$ is bijective if $i \leq 2q_\alpha - 3$ and surjective if $i = 2q_\alpha - 2$. In particular, this means that the equivariant diagram cohomology groups ${}_{BR}H^i(\mathbf{T}; \Pi_i)$ reduce to ordinary Bredon cohomology groups ${}_{BR}H^i(T_\alpha, \partial_1 T_\alpha; \pi_i(C_\alpha, N - N^G))$ if $i \leq n-1$ or if $i = n$ and $\dim F_\alpha + 3 \leq q_\alpha$. Since $T_\alpha \cong \partial_1 T_\alpha \times [0,1]$ it follows immediately that the relative cohomology groups vanish in all such cases. Since $\dim T_\alpha = \dim M = \dim N$, this implies that the isovariance obstructions vanish in all cases except perhaps when $i = n = 2q_\alpha - 2$. In such cases the value group fits into the following exact sequence, which arises by restricting diagram-theoretic cochains in $C(X' \to X; \pi' \to \pi)$ to ordinary cochains in $C(X'; \pi')$:

$$(\star) \qquad H^{n-1}\left(\partial_0 T_\alpha; \pi_n(D(\alpha'_\alpha), S(\alpha'_\alpha))\right) \xrightarrow{\Delta} H^n(T_\alpha, \partial T_\alpha; \pi_n(N_\alpha, N - N^G)) \to {}_{BR}H^n(\mathbf{T}; \Pi_n) \to H^n\left(\partial_0 T_\alpha; \pi_n(D(\alpha'_\alpha), S(\alpha'_\alpha))\right)$$

The map Δ in this sequence is given by combining the coefficient homomorphism for the map $\delta_0^\#$ in dimension n with the suspension isomorphism $H^{n-1}(\partial_0 T_\alpha; \pi) \to H^n(T_\alpha, \partial T_\alpha; \pi)$. Therefore the Blakers-Massey Theorem, the $(n-1)$-dimensionality of $\partial_0 T_\alpha$, and the Bockstein exact sequence for the short exact sequence

$$0 \longrightarrow \text{Kernel} \longrightarrow \pi_n(D(\alpha'_\alpha), S(\alpha'_\alpha)) \xrightarrow{\partial_0^\#} \pi_n(C_\alpha, N - N^G) \longrightarrow 0$$

imply that Δ is onto. But the last object in $(\star)$ is zero because $\dim \partial_0 T_\alpha = n - 1$, and it follows by exactness that ${}_{BR}H^n(\mathbf{T}; \Pi_n)$ is also trivial. Therefore the isovariance obstructions always vanish.■

The following examples due to Browder [Br2] show that it is not always possible to deform an equivariant degree 1 map so that it is isovariant near the fixed point set and the set of nonisovariant points lies in a tubular neighborhood of the fixed point set.

Examples. Let k and q be distinct positive integers such that q is even and G has a free q-dimensional linear representation. Let $N = S^k \times S^q$ with trivial action on the first coordinate and the one point compactification of the free linear action on the second, let M_0 be the disjoint union of N and two copies of $G \times S^k \times S^q$ (where G acts by translation on itself and trivially on the other two coordinates), and define an equivariant map $f_0 : M_0 \to N$ by taking the identity on M, the unique equivariant extension of the identity map on $S^k \times S^q$ over one copy of $G \times S^k \times S^q$, and the unique equivariant extension of an orientation reversing self diffeomorphism of $S^k \times S^q$ over the other copy. By construction this map has degree one, and one can attach 1-handles equivariantly to M_0 away from the fixed point set to obtain an equivariant cobordism of maps from f_0 to a map f on a connected 1-manifold M that is nonequivariantly diffeomorphic to a connected sum of $2 \cdot |G| + 1$ copies of $S^k \times S^q$. Since the fixed point sets of M and N are k-dimensional and the manifolds themselves are $(k+q)$-dimensional, it follows that the Gap Hypothesis holds if we impose the stronger restriction $q \geq k+2$. By construction the map f determines a homotopy equivalence of fixed point sets and is isovariant on a neighborhood of the fixed point set.

Assertion. *It is not possible to deform f equivariantly so that the set of nonisovariant points lies in a tubular neighborhood of the fixed point set. In particular, it is also not possible to deform f equivariantly to an isovariant map.*

To prove the assertion, assume that one has a map h equivariantly homotopic to f with the stated property, and let U be a tubular neighborhood of M^G that contains the set of nonisovariant points. Let X be a submanifold of the form $\{g\} \times \{v\} \times S^q$ in M that arises from one of the copies of $G \times S^k \times S^q$ in M_0. Although X and U may have points in common, by the uniqueness of tubular neighborhoods we can always isotop X into a submanifold X' that is disjoint from U. By the hypotheses on h we know that $h(X')$ is disjoint from $N^G = S^k \times S^0$, and therefore $h(X')$ is contained in

$$N - N^G \;\cong\; S^k \times S^{q-1} \times \mathbb{R}$$

so that the image of the generator of $H_q(X') = \mathbb{Z}$ maps trivially into $H_q(N)$. However, h is supposed to be homotopic to a map which is nontrivial on the latter by construction, so we have a contradiction, and therefore it is not possible to find an isovariant map h that is equivariantly homotopic to f.■

A refinement of the preceding argument shows that if Y is a subset of M such that the image of $H_q(Y)$ in $H_q(M)$ is equal to the image of $H_q(X)$, then Y must contain some nonisovariant points of any equivariant map that is equivariantly homotopic to f.

Remarks. By construction, Browder's examples are normally cobordant to the identity; an explicit normal cobordism frome the identity to f_0 is given by

$$W \;=\; N \times [0,1] \amalg G \times S^k \times S^q \times [0,1]$$

where $\partial_- W = N \times \{0\}$ and $\partial_+ W$ is the remaining $2|G|+1$ components of the boundary, and one can obtain a normal cobordism to f by adding 1-handles equivariantly along the top part of the boundary. More generally, results of K. H. Dovermann [Dov] imply that one can always construct equivariant normal cobordisms to equivariant normal maps if the Gap Hypothesis holds and the map is already an equivariant homotopy equivalence on the singular set as in Browder's examples.

However, it is also possible to construct examples like Browder's that are not cobordant to highly connected maps. It suffices to let $k \equiv 0(4)$ and replace $G \times S^k \times S^q$ by $G \times S(\gamma)$, where the latter is the sphere bundle of a fiber homotopically trivial vector bundle γ over S^k with nontrivial rational Pontryagin classes; one must also replace the equivariant folding mapfrom $G \times S^k \times S^q$ to N be its composite with the identity on G times a fiber homotopy equivalence from $S(\gamma)$ to $S^k \times S^q$. Characterixtic number arguments imply the map obtained in this fashion is not cobordant to a k-connected map. Of course, a degree 1 map of this type does not have the bundle date required for a normal map in the sense of equivariant surgery theory.

3. Equivariant homotopy equivalences

In this section we shall show that an equivariant homotopy equivalence can be deformed to satisfy the conditions in parts (*ii*) and (*iii*) of Theorem 2 and thus must be equivariantly homotopic to an isovariant map, which we shall prove must be an isovariant homotopy equivalence.

Proposition 3.1. *Let $f : M \to N$ be a homotopy equivalence of compact, oriented, unbounded, smooth, semifree G-manifolds which satisfy the Gap Hypothesis such that all components of all fixed point sets are also orientable. Then f is equivariantly homotopic to a map that is isovariant on a neighborhood of the fixed point set.*

Proof. We shall prove that f satisfies the conditions in part (*ii*) of Theorem 2. Since f defines a homotopy equivalence of fixed point sets, it follows immediately that for each component M_α of M_G the restriction f_α of f defines a homotopy equivalence from M_α to N_α and hence has degree ± 1. In order to apply part (*ii*) of Theorem 2, we also need to verify the homotopy pullback condition on the equivariant normal bundles of the fixed point set components.

Let τ_M and τ_N be the equivariant tangent bundles of M and N. We claim that the sphere bundles of τ_M and $f^*\tau_N$ are stably equivariantly fiber homotopically equivalent. The nonequivariant version of this statement is well known (*cf.* [At]) and the equivariant case is due to K. Kawakubo [Ka].

Consider next the restriction of the stable equivariant fiber homotopy equivalence $S(\tau_M) \sim S(f^*\tau_N)$ to M^G. The classifying maps for the two equivariant fibrations go from M^G to a space $\mathcal{B}$ such that $\pi_*(\mathcal{B}) \approx \pi^G_{*-1}$, where the latter denotes an equivariant stable homotopy group as in [Se]. On the other hand, by [Se] we also know that $\mathcal{B}$ is homotopy equivalent to the product $BF \times BF_G$ where BF classifies nonequivariant stable spherical fibrations and BF_G is defined as in [BeS]. In terms of fibrations the projections of the classifying maps $M \to \mathcal{B}$ onto BF and BF_G correspond to taking the classifying maps of the fixed point subbundles and the orthogonal complements of the fixed point subbundles respectively. Therefore it follows that the corresponding subbundles for τ_M and $f^*\tau_N$ are stably equivariantly fiber homotopy equivalent. In particular, this means that $S(\alpha_M)$ and $S((f^G)^*\alpha_N)$ are stably equivariantly fiber homotopy equivalent because they induce homotopic maps from M^G into BF_G.

As usual, write M^G as a disjoint union of components M_α, and for each α let q_α be the codimension of M_α. Furthermore, denote the fiber representation at a point of M_α by V_α. The stabilization map $F_G(V_\alpha) \to F_G$ is $(q_\alpha - 2)$-connected by the considerations of [Sc1], and the Gap Hypothesis implies that $\dim M_\alpha \leq q_\alpha - 2$. Therefore we can destabilize the stable fiber homotopy equivalence from $S(\xi_\alpha)$ to $S((f^G)^*\omega_\alpha)$ and obtain a genuine equivariant fiber homotopy equivalence. Choose such an equivariant fiber homotopy equivalence, say Φ. It is then an elementary exercise to deform $f | \cup_\alpha D(M_\alpha)$ equivariantly relative the zero section so that one obtains the radial extension of Φ at the other end of the deformation. By the equivariant homotopy extension property one can extend this homotopy to all of M.■

Our choice of fiber homotopy equivalences was arbitrary, but it is possible to find a canonical choice up to homotopy using equivariant S-duality [Wi] and Kawakubo's result; in fact, one must work with the latter to prove a uniqueness result for isovariant deformations as in [Br2], and we shall explain this in a subsequent article.

The preceding result and part *(iii)* of Theorem 2 reduce the proof of Theorems 1 and 3 to the following two results:

Proposition 3.2. *Suppose that f satisfies the conditions of the previous result, including the condition that f is isovariant on a neighborhood of the fixed point set. Then f is homotopic to an almost isovariant map.*

Proposition 3.3. *If f as above is isovariant, then f is an isovariant homotopy equivalence.*

We shall prove these results in order.

Proof of Proposition 3.2 The first step is to construct an equivariant homotopy from f to a mapping f_1 such that the homotopy is fixed on a neighborhood of the fixed point set and f_1 is smoothly equivariantly transverse to the fixed point set; there are no problems with equivariant transversality obstructions because the relevant part of the domain has a free G-action. By construction the transverse inverse image of the fixed point set is the set of nonisovariant points, and it splits into a union of smooth submanifolds $V_\alpha = f_1^{-1}(N_\alpha)$. Note that $\dim V_\alpha = \dim N_\alpha = \dim M_\alpha$ which is less than half the dimensions of M and N.

By construction the image of $f_1|V_\alpha$ is contained in N_α, so let k_α be the associated map from V_α to N_α; if $h_\alpha : N_\alpha \to M_\alpha$ is determined by a homotopy inverse to f_1, then the map $h_\alpha \circ k_\alpha$ is equivariantly homotopic to the inclusion of V_α in M. By general position it follows that the latter is also equivariantly homotopic to a map into $D(M_\alpha) - M_\alpha$ and in fact can be approximated by a smooth equivariant embedding e_α; in fact, the numerical condition in the Gap Hypothesis is strong enough to guarantee that e_α is equivariantly isotopic to the inclusion. Since the image of e_α is contained in a tubular neighborhood of M_α, the Equivariant Isotopy Extension Theorem implies the inclusion is isotopic to a smooth equivariant embedding of V_α in a tubular neighborhood and hence the image of the inclusion itself must also be contained in some tubular neighborhood. Since this is truc for every α, it is also true for the entire set of nonisovariant points.■

By Theorem 2 and the preceding propositions we know that f is equivariantly homotopic to an almost isovariant mapping, and by [DuS] it is also equivariantly homotopic to an isovariant mapping.

Proof of Proposition 3.3. By [DuS, Prop. 4.1, p. 27], the map f is isovariantly homotopic to a map f_0 such that for each α we have $f(\,D(M_\alpha)\,) \subset D(N_\alpha)$, $f(\,S(M_\alpha\,)) \subset S(N_\alpha)$, and

$$f\,(\,M - \cup_\alpha \,\mathrm{Int}D(M_\alpha)\,)\ \subset\ N - \cup_\alpha \,\mathrm{Int}D(N_\alpha)\ .$$

Furthermore, using [DuS, Theorem 4.4, pp. 29–31] one can further deform this map to some f_1 that is fiber preserving on the tubular neighborhoods and maps disk fibers to disk fibers by cones of maps over the boundary spheres (*i.e.*, the map is *normally straightened* in the sense of [DuS, p. 31]). It will suffice to prove that f_1 is an isovariant homotopy equivalence, so without loss of generality we might as well assume that f itself is normally straightened.

By the isovariant Whitehead Theorem of [DuS, §4] the map f is an isovariant homotopy equivalence if f defines a homotopy equivalence from $M - M^G$ to $N - N^G$. General position considerations imply that f induces an isomorphism of fundamental groups, and therefore it suffices to check that f defines an isomorphism in homology with twisted coefficients in the group ring of the fundamental group. Exact sequence and excision arguments show that the latter holds if f induces homotopy equivalences from M to N, from M^G to N^G, and from $\coprod S(\xi_\alpha)$ to $\coprod S(\omega_\alpha)$. The first two of these follow because f is an equivariant homotopy equivalence. To prove the third property first note that for each α the homotopy fibers of $S(\xi_\alpha) \subset D(\xi_\alpha)$ and $S(\omega_\alpha) \subset D(\omega_\alpha)$ are simply the fibers of the sphere bundles; since each $D(\xi_\alpha)$ maps to $D(\omega_\alpha)$ by a homotopy equivalence, it suffices to know that a fiber of $S(\xi_\alpha)$ maps to a fiber of $S(\omega_\alpha)$ with degree ± 1. This follows directly from the construction of the isovariant map; the first step was to make an equivariant homotopy equivalence normally straightened near the fixed point set, and the equivariant deformation in part (*iii*) of Theorem 2 is constant near some fiber of $S(\xi_\alpha)$.■

4. Applications and generalizations

In his thesis [St] Straus used his version of Theorem 1 to obtain the following result:

Theorem 4. *Let M and N be closed smooth manifolds of dimension ≥ 2, let p be an odd prime, and suppose that M and N are homotopy equivalent. Let $\mathbb{Z}_p$ act smoothly on the p-fold self products $\Pi^p M$ and $\Pi^p N$ (where $\Pi^p X = X \times \cdots \times X$, p factors) by cyclically permuting the coordinates, and let $\mathbf{D}^p(M)$, $\mathbf{D}^p(N)$ be the invariant subsets sets given by removing the diagonals from $\Pi^p M$ and $\Pi^p N$. Then the deleted reduced cyclic products $\mathbf{D}^p(M)/\mathbb{Z}_p$ and $\mathbf{D}^p(N)/\mathbb{Z}_p$ are homotopy equivalent.*

As also noted in [St], this result does not extend to compact bounded manifolds, and in fact the closed unit disks of various dimensions provide simple counterexamples. The results of [Sc2] imply that the result extends to simply connected manifolds if $p = 2$, but recent results of R. Longoni and P. Salvatore [LS] imply that Theorem 4 does not extend to 3-dimensional lens spaces when $p = 2$. Further results on the relationship between $\mathbf{D}^2(M)/\mathbb{Z}_2$ and $\mathbf{D}^2(N)/\mathbb{Z}_2$ for homotopy equivalent manifolds appear in [LöMa].

Sketch of proof. If $f : M \to N$ is a homotopy equivalence then $\Pi^p f : \Pi^p M \to \Pi^p N$ is an equivariant homotopy equivalence of closed smooth $\mathbb{Z}_p$-manifolds. All actions of $\mathbb{Z}_p$ are semifree if p is prime, so this condition holds automatically; the Gap Hypothesis also holds because $p > 2$. Therefore Theorem 1 implies that $\Pi^p f$ is equivariantly homotopic to an isovariant homotopy equivalence, and the latter in turn yields an equivariant homotopy equivalence from $\mathbf{D}^p(M)$ to $\mathbf{D}^p(N)$. The induced map of orbit spaces is the desired homotopy equivalence from $\mathbf{D}^p(M)/\mathbb{Z}_p$ to $\mathbf{D}^p(N)/\mathbb{Z}_p$.■

Extending Theorem 1 to other cases

Since the Gap Hypothesis was used at several crucial points in the proof of our main theorems, one might reasonably expect that these results do not necessarily hold if the Gap Hypothesis fails. Despite this, there are some situations in which one can prove analogs of Theorem 1, particularly when G is cyclic of prime order, $\dim M = 2 \dim M^G$, and there is only one component with the maximal dimension. If $|G| = 2$ and M is simply connected, this is established in [Sc2], and if $|G|$ is an odd prime and M is simply connected this will be shown in a forthcoming paper by K. H. Dovermann and the author. On the other hand, the results of Longoni and Salvatore imply that the analog of Theorem 1 does not necessarily hold if $|G| = 2$ and M is not simply connected.

In a sequel to this papaer we shall use equivariant function spaces as in [Sc1] and [BeS] to construct systematic families of equivariant homotopy equivalences that are not homotopic to isovariant maps in situations where the Gap Hypothesis fails. In particular, we shall construct connected examples where G is cyclic of prime order, $\dim M = 2 \dim M^G$, and there are two components with the maximal dimension.

As noted at the beginning of this paper, it is natural to ask whether Theorem 1 generalizes to nonsemifree actions in situations where the Gap Hypohtesis holds. Group actions with linearly ordered isotropy structure are natural test cases. In order to analyze the problem for such actions, it is necessary to introduce some equivariant homotopy theoretic functors like the free G-vector bundles of [Sc4]; we plan to do this in a subsequent paper.

References

[Ad] J. F. Adams, *On the groups J(X) — I*, Topology **2** (1963), 181–195.

[At] M. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310.

[BeS] J. C. Becker and R. E. Schultz, *Equivariant function spaces and stable homotopy theory*, Comment. Math. Helv. **49** (1974), 1–34.

[Br1] W. Browder, *Surgery and the theory of differentiable transformation groups*, Proceedings of the Conference on Transformation Groups (Tulane, 1967), Springer, Berlin-Heidelberg-New York, 1968, pp. 3–46.

[Br2] ———, *Isovariant homotopy equivalence*, Abstracts Amer. Math. Soc. **8** (1987), 237–238.

[BQ] W. Browder and F. Quinn, *A surgery theory for G-manifolds and stratified sets*, Manifolds–Tokyo, 1973, (Conf. Proc., Univ. of Tokyo, 1973), University of Tokyo Press, Tokyo, 1975, pp. 27–36.

[Dol] A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. (2) **78** (1963), 223–255.

[Dov] K. H. Dovermann, *Almost isovariant normal maps*, Amer. J. Math. **111** (1989), 851–904.

[DP] K. H. Dovermann and T. Petrie, *G-Surgery II*, Memoirs Amer. Math. Soc. **37** (1982), No. 260.

[DR] K. H. Dovermann and M. Rothenberg, *Equivariant Surgery and Classification of Finite Group Actions on Manifolds*, Memoirs Amer. Math. Soc. **71** (1988), No. 379..

[DuS] G. Dula and R. Schultz, *Diagram cohomology and isovariant homotopy theory*, Memoirs Amer. Math. Soc. **110** (1994), No. 527..

[Gr] B. Gray, *Homotopy Theory: An Introduction to Algebraic Topology*, Pure and Applied Mathematics Vol. 64, Academic Press, New York, 1975.

[Ka] K. Kawakubo, *Compact Lie group actions and fiber homotopy type*, J. Math. Soc. Japan **33** (1981), 295–321.

[LöMi] P. Löffler and R. J. Milgram, *The structure of deleted symmetric products*, Braids (Santa Cruz, Calif., 1986), Contemp. Math. **78** (1988), pp. 415–424.

[LS] R. Longoni and P. Salvatore, *Configuration spaces are not homotopy invariant*, Topology **44** (2005), 375–380.

[LüMa] W. Lück and I. Madsen, *Equivariant L-theory I*, Math. Zeitschrift **203** (1990), 503–526.

[M] J. W. Milnor, *Lectures on the h-cobordism Theorem*, Princeton Mathematical Notes No. 1, Princeton University Press, Princeton, N. J., 1965.

[P1] T. Petrie, *Pseudoequivalences of G-manifolds*, Proc. AMS Sympos. Pure Math **32 Pt. 1** (1978), 169–210.

[P2] ———, *One fixed point actions on spheres I–II*, Adv. Math. **46** (1982), 3–14, 15–70.

[Ph] A. V. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206.

[R] M. Rothenberg, *Review of the book, "Equivariant Surgery Theories and Their Periodicity Properties," by K. H. Dovermann and R. Schultz*, Bull. Amer. Math, Soc. (2) **28** (1993), 375–381.

[Sc1] R. Schultz, *Homotopy decomposition of equivariant function spaces I*, Math. Zeit. **131** (1973), 49–75.

[Sc2] ———, *A splitting theorem for manifolds with involution and two applications*, J. London Math. Soc. (2) **39** (1989), 183–192.

[Sc3] ———, *Isovariant homotopy theory and differentiable group actions*, Proc. KAIST Math. Workshops **7** (1992), 81–148.

[Sc4] ———, *Spherelike G-manifolds with exotic equivariant tangent bundles*, Studies in Algebraic Topology, (Adv. in Math. Supplementary Studies Vol. 5), Academic Press, New York, 1979, pp. 1–39.

[Se] G. Segal, *Equivariant stable homotopy theory*, Actes, Congrès Internat. de Math. (Nice, 1970), T. 2, Gauthier-Villars, Paris, 1971, pp. 59–63.

[Str] S. H. Straus, *Equivariant codimension one surgery*, Ph. D. Dissertation, University of California, Berkeley, 1972.

[WL] C. T. C. Wall, *Surgery on Compact Manifolds*, (Second edition. Edited and with a foreword by A. A. Ranicki. Mathematical Surveys and Monographs, 69, American Mathematical Society, Providence, RI, 1999.

[Wn] S. Waner, *Equivariant classifying spaces and fibrations*, Trans. Amer. Math. Soc. **258** (1980), 385–405.

[We] S. Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994.

[Wi] K. Wirthmüller, *Equivariant S-duality*, Arch. Math. (Basel) **26** (1975), 427–431.

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