# MULTIPLICATIVE STABILIZATION AND TRANSFORMATION GROUPS

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ABSTRACT. There are many pairs of nonisomorphic geometric objects whose products with a third object are isomorphic. Examples of this sort appeared frequently in the research of K. Kawakubo. This paper discusses some examples that have applications to questions about transformation groups of independent interest.

Both Katsuo Kawakubo and the second named author began their mathematical careers with special cases of the following question.

**Cancellation Problem:** *Let $A, B$ and $C$ be objects in one of the following categories:*

(1) *Topological spaces and continuous maps.*
(2) *Piecewise linear spaces and piecewise linear maps.*
(3) *Smooth manifolds and smooth maps.*

*Under what conditions are the products $A \times C$ and $B \times C$ equivalent (in the appropriate category)*?

If $\mathcal{A}$ is one of the three categories described above and $\mathcal{A}_0$ is a full small subcategory that is closed under taking finite Cartesian products, then the set $\overline{\mathcal{A}_0}$ of isomorphism classes of objects in $\mathcal{A}_0$ has a monoid structure given by the product construction, and from this viewpoint the noncancellation problem is to determine the extent to which algebraic cancellation fails to hold in $\overline{\mathcal{A}_0}$. Given objects $A$ and $B$ such that $A \times C$ and $B \times C$ are isomorphic, we shall sometimes say that $A$ and $B$ are *stably equivalent* (with respect to stabilization by $C$).

It is easy to find examples of topological spaces such that $A \times C$ and $B \times C$ are homeomorphic but $A$ and $B$ are not; in particular, if $Q$ is the Hilbert cube – a product of denumerably many copies of the unit interval – then $\{0\} \times Q$ and $Q \times Q$ are homeomorphic. Clearly one can refine the question and make it more substantial by restricting the types of objects allowed; for example, one can look at finite polyhedra or compact smooth simply connected manifolds, but after some point it is also clear that relationships with other problems are needed to motivate further study. The purpose of this article is to illustrate that stable equivalence problems arise quite frequently in various contexts from geometric topology, with particular attention to some questions about transformation groups and remarks about problems of this sort that were studied by Kawakubo.

We shall focus on three types of problems. In Section 1 we discuss some ways in which noncancellation results lend to the construction of exotic group actions in some contexts and regularity principles in others. Noncancellation phenomena played an important role in many constructions of exotic group actions from the nineteen sixties, and the earliest work of Kawakubo figured in some of these advances. Subsequent work of the first author and Kawakubo in the nineteen eighties showed some major differences between the theory of equivariant stable equivalence and its more classical nonequivariant analog. The second section of the paper deals with conceptual issues in equivariant surgery. Both ordinary surgery theory for manifolds and its generalizations to manifolds with group actions have led to enormous advances in transformation groups, but the equivariant versions of surgery theory often involve a general position condition called the Gap Hypothesis; roughly speaking, this means that if we consider the partially ordered set $\mathbf{P}$ consisting of connected components of fixed point sets for the various subgroups of the group $G$ that acts on a manifold, then the dimension of each subspace $A$ in $\mathbf{P}$ is more than twice that of every $B \in \mathbf{P}$ such that $B < A$. In Section 2 we shall describe how one can use the periodicity results of [DoS] and multiplicative stabilization to construct infinite extensions of the "rather long" but finite exact equivariant surgery sequences of [DoR] and [LM1, LM2] provided the group under consideration has odd order; the underlying idea is that multiplicative stabilization provides a way to recapture a portion of equivariant surgery theory when the Gap Hypothesis fails. Finally, in Section 3 we shall describe some applications of multiplicative stabilization and controlled topology to the symmetry properties of 4-manifolds. In particular, we shall prove that a particularly significant 4-manifold called the Chern manifold (see Freedman-Quinn [FQ]) has no effective, semifree topological action of the circle group $S^1$ (semifree means $S^1$ acts freely off the fixed point set). In a subsequent paper these methods will be used to study further restrictions on the symmetries of the Chern manifold.

Other notions of algebraic stabilization also arise naturally in algebraic and geometric topology. In particular, Kawakubo's work includes results on *connected sum stabilization*; namely, the analysis of those (isomorphism classes of) manifold triples $A, B, C$ of the same dimension such that the connected sums $A \# C$ and $B \# C$ are isomorphic (in particular, see [Kkb1]). One could write a separate article on this form of stabilization at least as long as the present paper; some relatively recent results on the equivariant version of this problem appear in [MaS1] and [MaS2]. Yet another form of stabilization that has received attention is *exponential stabilization*; specifically, if $n$ is a positive integer and $\times^n Y$ denotes the $n$-fold product of $Y$ with itself, then the problem is to determine when $\times^n A$ and $\times^n B$ are isomorphic. Results on this problem are described in [KwS5] and [KwS6].

The books of G. Bredon [Bre], T. tom Dieck [tD] and Kawakubo [Kkb4] are standard references for concepts in transformation groups that are not described explicitly in this paper.

**Acknowledgments.** In a paper of this sort it is particularly difficult to decide who deserves to be mentioned. For the sake of brevity we refer the reader to those papers of ours that are cited in the bibliography for acknowledgments, and to these one should add the coauthors of the joint papers cited here [DoS, DuS, Ksk1–2, KwS1–6, KV, MaS1–2, Sc1–5]. Beyond this, our debt to K. H. Dovermann in connection with Section 2 is clear, and we are extremely grateful to B. Hughes and S. Weinberger for numerous discussions regarding Section 3 and also for supplying copies of their preprints. The second author is also indebted to the University of Göttingen, Tulane University and Northwestern University for their hospitality during many phases of the work on the topics in Sections 2 and 3.

Last but definitely not least, the second author's contacts with Katsuo Kawakubo began in 1967, when both of us were graduate students. His impact on the research of both authors is clearly visible in our publications.

## 1. Stable equivalence

During the second half of the twentieth century, one recurrent theme in transformation groups was the construction of exotic new symmetries on familiar spaces. Many such examples arise from noncancellation phenomena. For example, consider the Bing dogbone space $X$ obtained by collapsing a wild arc in $\mathbb{R}^3$ to a point [Bi]. Although $X$ is not a topological manifold, its product with the real line is homeomorphic to $\mathbb{R}^4$. It follows that the product $S^1 \times X$ is a 4-manifold with a free action of the circle group $S^1$ given by multiplication on the first coordinate and the trivial action on $X$. One can use the methods of Freedman-Quinn [FQ] to show that $S^1 \times X$ is topologically equivalent to $S^1 \times \mathbb{R}^3$, but the group action described above is intrinsically nonsmoothable because its orbit space is the nonmanifold $X$.

During the nineteen fifties and sixties many other examples of nonisomorphic objects $X, Y$ with $X \times \mathbb{R}^k \approx Y \times \mathbb{R}^k$ were discovered (for example, see papers of B. Mazur such as [Mz1], [Mz3] or Milnor's article [Mi2]; additional examples are given by the interiors of the compact bounded manifolds constructed by Mazur [Mz2] and Poenaru [Po1], [Po2]). For our purposes one particularly noteworthy result involves the so-called *exotic spheres.* These are smooth $n$-dimensional manifolds that are homeomorphic but not diffeomorphic to the standard sphere $S^n$; the first of these were discovered by J. Milnor [Mi1] and all such manifolds in dimensions $\geq 5$ were subsequently classified up to oriented diffeomorphism by M. Kervaire and J. Milnor [KM]. In this case the results of B. Mazur [Mz1, Mz3] imply that for any exotic $n$-sphere $\Sigma^n$ the product $\Sigma^n \times \mathbb{R}^k$ is diffeomorphic to $S^n \times \mathbb{R}^k$ for all sufficiently large $k$.

In all such stable results it is natural to ask for the least possible value of $k$ such that one has an isomorphism. The earliest work of Kawakubo and the second author answered this question for exotic spheres [Kkb1, Kkb2, Kkb3], [Sc1]; these results were also obtained independently by R. DeSapio [DeS1], [DeS2]. Since the statement of the results is not entirely elementary, we shall specialize to one particular noteworthy special case (that also follows from other methods).

**Theorem 1.** *Let $n \geq 7$ and let $\Sigma^n$ be an exotic $n$-sphere that bounds a parallelizable $(n+1)$-manifold (such exotic spheres exist if $n$ is odd and $n+3$ is not a power of $2$). Then $\Sigma^n \times \mathbb{R}^3$ is diffeomorphic to $S^n \times \mathbb{R}^3$ but $\Sigma^n \times \mathbb{R}^2$ is not diffeomorphic to $S^n \times \mathbb{R}^2$. In fact, $\Sigma^n \times D^3$ is diffeomorphic to $S^n \times D^3$ where $D^3$ denotes the unit disk.*

The final statement in the theorem implies that $\Sigma^n \times S^2$ is diffeomorphic to $S^n \times S^2$. Combining this with results of W. Browder [Br1], we obtain the following application to constructing exotic group actions.

**Theorem 2.** *Let $\Sigma^n$ be as in the previous theorem. Then there is a smooth semifree $S^1$ action on some smooth homotopy sphere $M^{n+4}$ whose fixed point set is diffeomorphic to $\Sigma^n$.*

**Notes. 1.** A *homotopy sphere* is a smooth $n$-manifold $M^n$ that is homotopy equivalent to $S^n$. If $n \neq 3$ then $M^n$ is a homotopy sphere if and only if $M^n$ is homeomorphic to $S^n$.

**2.** Results of J. Levine [Lev] and the second author [Sc4] completely determine which homotopy $k$-spheres admit semifree smooth $S^1$ actions with $(k-4)$-dimensional fixed point sets.

One can also prove Theorem 2 using the Brieskorn description of $\Sigma^n$ (*e.g.*, see Hirzebruch [Hi] or Hirzebruch-Mayer [HM]). However, the results of Kawakubo and the second author also yield other examples.

**Theorem 3.** *For each $k \geq 1$ there is an exotic sphere $\Sigma^{8k+2}$ such that*
*(i) $\Sigma^{8k+2}$ does not bound a spin manifold (= one whose second Stiefel-Whitney class is zero),*
*(ii) there is a smooth semifree $S^1$ action on some homotopy sphere $M^{8k+6}$ with $\Sigma^{8k+2}$ as its fixed point set.*

Once again this follows from Bredon's paper and the existence of a diffeomorphism from $\Sigma^{8k+2} \times S^2$ to $S^{8k+2} \times S^2$ as described in the example on page 322 of [Sc2].

**Note.** In the discussion of [Sc2], a crucial point is that the Adams elements $\mu_k \eta \in \pi_{8k+2} = \pi_{n+8k+2}(S^n)$ (for $n$ sufficiently large) can be desuspended to $\pi_{8k+4}(S^2)$, and the reference for this is an unpublished announcement from the early nineteen sixties. For the sake of completeness we shall explain how this result can be extracted from J. F. Adams' seminal work on systematic families of elements in the homotopy groups of spheres [A1]. Let $\mathbf{P}^m(2)$ denote the cell complex $S^m \cup_2 e^{m+1}$, which is also describable as the $(m-2)$-fold suspension of the real projective plane. A central result of [A1] is the existence of maps

$$\mathbf{p}_m : \mathbf{P}^{m+8}(2) \to \mathbf{P}^m(2)$$

(for $m$ sufficiently large)

that induce isomorphisms in $K$-theory. In fact, one can define such a map for $m = 4$ using results of H. Toda [T] and obtain similar maps for higher values of $m$ by suspension. One can form iterated compositions

$$\begin{aligned} \mathbf{p}_m^{\langle 2 \rangle} &= \mathbf{p}_m \circ \mathbf{p}_{m+8} \\ \mathbf{p}_m^{\langle 3 \rangle} &= \mathbf{p}_m \circ \mathbf{p}_{m+8} \circ \mathbf{p}_{m+16} \\ \mathbf{p}_m^{\langle k \rangle} &= \mathbf{p}_m \circ \cdots \circ \mathbf{p}_{m+8k} \end{aligned}$$

and for each of these composites the induced map in $K$-theory is again an isomorphism. The Adams elements $\mu_k \in \pi_{8k+1}$ are presented on page 68 of [A1] as composites of the form $\mu_k = \eta' \circ \mathbf{p}_m^{\langle k \rangle} \circ i$ where $\eta' : \mathbf{P}^m(2) \to S^{m-1}$ is an extension of the suspended Hopf map $\eta : S^m \to S^{m-1}$ (the existence of the extension follows because $2\eta = 0$ in stable homotopy) and $i$ denotes inclusion of the bottom cell; in particular, this is a map from $S^{m+8k}$ to $S^{m-1}$. The stable homotopy classes $\mu_k \eta$ of interest to us are given by composites of the $\mu_k$ with the suspended Hopf map from $S^{m-1}$ to $S^{m-2}$. We know that the periodicity maps desuspend to $m = 4$, and in fact one can define the map $\eta'$ on $\mathbf{P}^4(2)$ because $\pi_4(S^3) \approx \mathbb{Z}_2$. All of this combines to show that $\mu_k$ can be desuspended to $\pi_{8k+4}(S^3)$.

Since the original, unsuspended Hopf map goes from $S^3$ to $S^2$, it follows that $\mu_k\eta$ does desuspend to $\pi_{8k+4}(S^2)$ as claimed.

More generally, if $\Sigma^n$ is an exotic $n$-sphere then the existence of a smooth semifree circle action on some homotopy sphere $W^{n+2q}$ with fixed point set $\Sigma^n$ depends on whether $\Sigma^n \times \mathbb{CP}^{q-1}$ is diffeomorphic to $S^n \times \mathbb{CP}^{q-1}$, at least if $q$ is sufficiently large with respect to $n$ [Br1]. Although some general results on this question exist or can be extracted from the literature, a complete answer to this particular stable equivalence problem involves deep homotopy theoretic questions that are beyond the reach of existing techniques. On the other hand, if $q$ is very large with respect to $n$ we have the following result.

**Theorem 4.** *Given $n \geq 7$ there is a positive integer $k(n)$ with the following properties:*
*($i$) If $\Sigma^n$ is a homotopy sphere such that $\Sigma^n \times \mathbb{CP}^q$ is diffeomorphic to $S^n \times \mathbb{CP}^q$ for $q \geq k(n)$, then $\Sigma^n$ bounds a parallelizable manifold.*
*($ii$) If $\Sigma^n$ is the fixed point set of a smooth semifree $S^1$ action on some homotopy sphere $W^{n+2q+2}$ for $q \geq k(n)$ then $\Sigma^n$ bounds a parallelizable manifold.*

These are variants of the results in [Sc3] on fixed point sets of smooth $\mathbb{Z}_p$ actions on homotopy spheres. In particular, ($ii$) follows because the fixed point set of a semifree circle action is also the fixed point set of every cyclic subgroup of $S^1$.

## Equivariant stable equivalence

Both Kawakubo and the first author have studied equivariant analogs of Mazur's stable equivalence result [Mz3], which states that a pair of closed manifolds $M^n$ and $N^n$ satisfy $M^n \times \mathbb{R}^k \cong N^n \times \mathbb{R}^k$ for sufficient large $k$ (in the appropriate category) if and only if $M^n$ and $N^n$ are tangentially homotopy equivalent (see [MTW] for more information on the latter concept). Results from [Kkb3] and [Ksk1] yield an equivariant analog for manifolds with a smooth action of a compact Lie group $G$ provided one replaces $\mathbb{R}^k$ with a judiciously chosen linear representation $V$.

It is natural to ask if one can take the representation $V$ to be the trivial representation on $\mathbb{R}^k$ for some large value of $k$. Examples of [Ksk1] show this is not possible in general, but another result of [Ksk1] implies that one can take $V$ to be $\mathbb{R}^k$ if the equivariantly tangential homotopy equivalence $f : M^n \to N^n$ is *isovariant* (compare [DuS]); in other words, for each $g \in G$ and $x \in M$ the relation $g \cdot f(x) = f(x)$ holds only if $g \cdot x = x$ (the converse follows immediately from equivariance).

If $G$ is finite and $M^n$ and $N^n$ satisfy the *standard Gap Hypothesis*

> (**GH**) *If $P^n = M^n$ or $N^n$ and $K, H$ are subgroups of $G$ such that $K \subset H$ but* $\mathrm{Fix}(H,P) \subsetneqq \mathrm{Fix}(K,P)$, *then* $\dim \mathrm{Fix}(K,P) \geq 2 \cdot \dim \mathrm{Fix}(H,P) + 1$.

then an equivariant homotopy equivalence from $M^n$ to $N^n$ can be equivariantly deformed to an isovariant homotopy equivalence, (compare Straus [St], Browder [Br3] and [Sc5]), and consequently we have the following result:

**Theorem 5.** *Suppose that $f : M^n \to N^n$ is an equivariantly tangential homotopy equivalence of closed smooth $G$-manifold and assume that $M^n$ and $N^n$ satisfy the Gap Hypothesis. Then $M^n \times \mathbb{R}^k$ is equivariantly diffeomorphic to $N^n \times \mathbb{R}^k$ for some $k$, where $G$ acts trivially on $\mathbb{R}^k$*

### Addendum: Exotic finite $H$-spaces

A somewhat different application of noncancellation from the nineteen sixties completely reshaped topologists' perspective on finite $H$-spaces; *i.e.*, finite complexes with maps $\mu : Y \times Y \to Y$ whose restrictions to

$$Y \vee Y \approx Y \times \{y_0\} \cup \{y_0\} \times Y$$

are the identity. A sequence of results from the middle third of the twentieth century suggested that all examples of finite $H$-spaces were somehow constructible from $S^7$ (with multiplication given by the Cayley numbers) and compact Lie groups. However, results of P. Hilton and J. Roitberg [HR] showed that the compact Lie group $Sp_2 \times S^3$ is diffeomorphic to $M \times S^3$ where $M$ is quite distinct from any of the previously known examples of finite $H$-spaces. The manifold $M$ is retract of the compact Lie group $Sp_2 \times S^3 \approx M \times S^3$, and $M$ is an $H$-space because a retract of a topological group is always an $H$-space. Subsequent constructions of finite $H$-spaces proceeded in quite different directions, but it seems noteworthy that the original example arose from noncancellation results.

## 2. Periodic stabilization in equivariant surgery

During the nineteen sixties and early seventies the theory of surgery on manifolds [Br2, Wa] proved to be a powerful tool for analyzing existence and classification questions about group actions on manifolds. By the mid seventies it was clear that even more striking results could be obtained using versions of surgery theory involving manifolds with, say, smooth group actions (compare Davis- Hsiang [DH] or Petrie [Pt1, Pt2, Pt3, Pt4] as well as Dovermann-Petrie [DP] and Masuda-Petrie [MP]). However, in many cases the extensions of nonequivariant surgery theory worked well only if one assumed the Gap Hypothesis (**GH**) formulated near the end of Section 1. If the group $G$ of symmetries has odd order, the results of [DoS], Ch. III, show that an arbitrary $G$-equivariant surgery problem ($f : X \to M$, other data) can be converted into a problem for which the Gap Hypothesis holds by crossing with the identity on a certain type of $r$-fold product of the form $\times^r B$ for all sufficiently large values of $r$; this is called *periodic stabilization* in [DoS]. A few illustrations of this principle are described in Section III.5 of [DoS]; as indicated there, "one would like an approximation to equivariant surgery theory that has the desirable features implied by the Gap Hypothesis, but without assuming the Gap Hypothesis itself," and on page 101 of [DoS] the usefulness of periodic stabilization for constructing such an appropriate surgery theory is discussed. In this section we shall describe the construction of a stabilized equivariant surgery theory with the desired properties.

One of the main features of ordinary surgery theory is the so-called long exact surgery sequence of D. Sullivan and C.T.C. Wall [Wa], Section 10. For a closed smooth $n$-manifold $M^n$ with $n \geq 5$, the sequence in the smooth category terminates on the right with

$$L^x_{n+1}(\pi_1(M^n), w_1) \to \mathcal{S}^{DIFF,x}(M^n) \to [M^n, F/O] \to L^x_n(\pi_1(M^n), w_1)$$

and it extends infinitely to the left with pieces of the following form:

$$L^x_{n+k+1}(\pi_1(M^n), w_1) \to \mathcal{S}^{DIFF,x}(D^k \times M^n, \partial) \to [S^k M^n \vee S^k, F/O] \to L^x_{n+k}(\pi_1(M^n), w_1)$$

*Explanation of notation:* In the exact sequences $\mathcal{S}^{DIFF,x}(\ldots)$ refers to $x$-restricted smooth homotopy structures represented by pairs $(f, M)$ where $f$ is a homotopy equivalence $X \to M$ with $X$ a compact manifold and the Whitehead torsion of $f$ [Mi3] is regulated by $x$. If $\partial M$ is nonempty we assume $f$ is a homotopy equivalence of pairs from $(X, \partial X)$ to $(M, \partial M)$, and in this case $\mathcal{S}^{DIFF,x}(M, \partial M)$ refers to homotopy structures that are diffeomorphisms on the boundary. The group $[M, F/O]$ is a homotopy theoretic object that classifies degree 1 normal maps into $M$ ($=$ *surgery problems*) up to a suitable equivalence relation, and $L^x_{n+\varepsilon}(\pi_1(M^n), w_1)$ is the Wall surgery obstruction group with twisting $w_1 : \pi_1(M^n) \to \mathbb{Z}_2$ determined by the first Steifel-Whitney class [Wa], a group that can be defined purely algebraicially. The sequence is an exact sequence of abelian groups except near the end, where it behaves like the final terms of the homotopy exact sequence of a fibration; as noted below, this is not a coincidence.

In [DoR] K.H. Dovermann and M. Rothenberg constructed a partial equivariant surgery sequence that terminated on the left after finitely many steps, and this construction was refined and generalized by W. Lück and I. Madsen [LM1, LM2]. If we assume that $N \times D^{k+1}$ satisfies a strong version of the Gap Hypothesis then one has an analogous exact sequence with the following replacements:

**Surgery groups:** $L^x_{n+\varepsilon}(\pi_1(M^n), w_1(M^n)) \longmapsto \mathcal{L}^\kappa_{n+\varepsilon}(\pi^G(M),$ other data). ($\mathcal{L}$ denotes the Lück-Madsen equivariant $L$-group, $\kappa$ is an equivariant analog of $x$, and $\pi^G(M)$ is a tabulation of the fundamental groups of the components of fixed point sets of subgroups of $G$)

**Structure sets:** $\mathcal{S}^{DIFF,x}(N, \partial N) \longmapsto \mathcal{S}^T_G(N, \partial N)$ ( $=$ $T$-restricted equivariant homotopy structure)

**Normal maps:** $[N/\partial N, F/O] \longmapsto \mathcal{N}^T_G(N, \partial N)$ ( $=$ suitably defined equivariant surgery problems)

The sequence terminates near $\mathcal{L}^\kappa_{n+q}(-)$ where $q$ is the largest positive integer such that $M^n \times D^q$ satisfies the Gap Hypothesis (note that $M^n \times D^\ell$ only satisfies this condition for finitely many values of $\ell$).

So long as the Gap Hypothesis holds the equivariant Wall groups satisfy a fourfold periodicity relation

$$\mathcal{L}^\kappa_{n+\varepsilon}(\pi^G(M), -) \cong \mathcal{L}^\kappa_{n+\varepsilon+4}(\pi^G(M), -)$$

given geometrically by crossing with the complex projective plane $\mathbb{CP}^2$ as in ordinary surgery theory. Therefore one usually can define $\mathcal{L}^\kappa_{n+\varepsilon}(\pi^G M, -)$ formally for all $\varepsilon \geq 0$. Also, one can define the set of (equivalence classes of) equivariant surgery problems

$$\mathcal{N}^T_G(D^k \times M^n, S^{k-1} \times M^n)$$

even if the Gap Hypothesis does not hold. Therefore it is natural to ask if one can define homomorphisms

$$\mathcal{N}^T_G(D^k \times M^n, S^{k-1} \times M^n) \to \mathcal{L}^\kappa_{n+k}(\pi^G M, -)$$

for *all* $k > 0$ and embed them into an exact sequence which ends with the sequence of [LM2] on the right but extends infinitely to the left.

The results of [DoS] imply that *such an infinite exact sequence exists in most cases if* $G$ *has odd order.* To avoid some technical complications we shall restrict our attention to the case $G = \mathbb{Z}_p$ where $p$ is an odd prime. As indicated at the beginning of this section, the process for constructing the infinite extension is the periodic stabilization of [DoS], Section III.5.

Given a finite group $G$, let $|G|$ denote its order. If $X$ is a topological space, then $X \uparrow G$ is defined to be the product of $|G|$ copies of $X$ with $G$ acting by permuting the coordinates in the standard fashion; if $X$ is a smooth manifold, then the action of $G$ on $X \uparrow G$ (viewed as a smooth product) is a smooth action.

Assume now that $G = \mathbb{Z}_p$ where $p$ is an odd prime, and also assume that we are considering equivariant surgery problems with nonempty fixed point sets (if the fixed sets are empty, then the actions are free and everything reduces to ordinary surgery over the orbit spaces).

Earlier in this section we mentioned that the long exact surgery sequence behaved like the long exact homotopy sequence of a fibration and that this was more than coincidental. In fact, the methods of Rourke [Ro] or Quinn [Q1] show that one can construct $\Delta$-sets (essentially simplicial sets without degeneracies); see Rourke-Sanderson [RS] for more information) so that the surgery sequence is equal to the homotopy sequence of a Kan fibration up to homotopy

$$\mathcal{S}_\bullet^{DIFF,x}(M^n) \to \mathcal{F}_\bullet(M^n, F/O) \to L_\bullet^x(\pi_1(M), w_1(M^n))$$

where a $k$-simplex in $\mathcal{S}_\bullet^{DIFF,x}(M^n)$ is represented by a suitable homotopy equivalence of manifold $n$-ads into the standard $n$-ad given by $\Delta_k \times M$ (this is Wall's terminology [Wa]), the set $\mathcal{F}_\bullet(M^n, F/O)$ is the simplicial function set as defined in May [My] (see Definition I.6.4, p. 17) or Goerss-Jardine [GJ] (see p. 20), and a $k$-simplex of $L_\bullet^x(-)$ is represented by a suitable surgery problem of manifold $n$-ads.

This extends to the equivariant surgery sequences of [LM2] as follows. First of all, there is no problem defining a $\Delta$-set $\mathcal{N}_{G_\bullet}^T(M^n)$ whose $q$-th homotopy group is

$$\mathcal{N}_G^T(D^q \times M^n, S^{q-1} \times M^n).$$

Furthermore, if $k$ is so small that $D^{k-n} \times M^n$ satisfies the appropriate form of the Gap Hypothesis, then one can define $k$-simplices of $\mathcal{L}_\bullet^\kappa(\pi^G(M), -)$ in analogy with the nonequivariant case; for our purposes it will not matter how we define simplices in higher dimensions. For sufficiently low values of $k$ there is an equivariant surgery obstruction map of $\Delta$-sets

$$\sigma : \mathcal{N}_{G\bullet}^T(M) \to \mathcal{L}_\bullet^\kappa(\pi^G(M), -)$$

such that the surgery sequence of [LM2] is the homotopy exact sequence of the Kan fibration associated to $\sigma$. In analogy with the nonequivariant case, the homotopy fiber of $\sigma$ is homotopy equivalent to the $\Delta$-set $\mathcal{S}_{G\bullet}^T(M^n)$ in the appropriate range of dimensions.

The key to extending the finite equivariant surgery sequence into an infinite one is to construct an enhancement of $\sigma$, and a crucial step in this process is to replace $\mathcal{L}_\bullet^\kappa(\pi^G(M), -)$ with a new $\Delta$-set whose homotopy groups behave in the right fashion. This can be accomplished by taking products with $\mathbb{CP}^2 \uparrow G$. The latter is equivariantly 1-connected, and consequently $\pi^G(M)$ is isomorphic to $\pi^G(M \times (\mathbb{CP}^2 \uparrow G))$. Since the product manifold $D^{k-n+1} \times M \times (\mathbb{CP}^2 \uparrow G)$ satisfies the Gap Hypothesis if $D^{k-n} \times M$ does, the good range of dimensions for $\mathcal{L}^{\kappa}_{\bullet}\big((\pi^G(M \times (\mathbb{CP}^2 \uparrow G)), -\big)$ is greater than the analog for $\mathcal{L}^{\kappa}_{\bullet}(\pi^G(M), -)$. Let

$$P_1 : \mathcal{L}^{\kappa}_{\bullet}(\pi^G(M), -) \to \mathcal{L}^{\kappa}_{\bullet}(\pi^G(M \times (\mathbb{CP}^2 \uparrow G)))$$

be the (partial) map of $\Delta$-sets given by taking products with $\mathbb{CP}^2 \uparrow G$.

*Notational convention.* We shall denote $\mathbb{CP}^2 \uparrow G$ by $\mathbf{B}$ for the sake of conciseness.

The reason for taking products with $\mathbf{B} = \mathbb{CP}^2 \uparrow G$ is the following: If $\ell$ is the largest positive integer such that $D^{\ell-n} \times M^n$ satisfies the Gap Hypothesis, then the partial map of $\Delta$-sets given by $P_1 \circ \sigma$ is only defined for simplices of dimension $\leq \ell$. In this range there is a commutative diagram

$$\begin{array}{ccc} \mathcal{N}^T_{G\bullet}(M) & \xrightarrow{Q_1} & \mathcal{N}^T_{G\bullet}(M \times \mathbf{B}) \\ \downarrow \sigma & & \downarrow \sigma' \\ \mathcal{L}^{\kappa}_{\bullet}(\pi^G(M), -) & \xrightarrow{P_2} & \mathcal{L}^{G}_{\bullet}(\pi^G(M \times \mathbf{B}), -) \end{array}$$

in which $Q_1$ is given by taking products with the identity on $\mathbf{B}$ and $\sigma'$ is a surgery obstruction. On the other hand, the composite $\sigma' \circ Q_1$ can be defined for simplices of dimension $\leq \ell + 1$, so we have an extension of $P_1 \circ \sigma$ to simplices of (at least) one higher dimension. More generally, if $P_r$ and $Q_r$ denote products with $\mathbf{B}^r$, then the same considerations yield an extension of $P_r \circ \sigma$ to the $(\ell + r)$-simplices of $\mathcal{N}^T_{G\bullet}(M)$, and consequently we can pass to (co)limits and obtain a well defined map of $\Delta$-sets

$$\sigma[\infty] : \mathcal{N}^T_{G\bullet}(M) \to \mathcal{L}^{\kappa}_{\bullet}(\pi^G(M), -)[\infty] := \mathrm{colim}_{r\to\infty} \mathcal{L}^{\kappa}_{\bullet}(\pi^G(M \times \mathbf{B}^r), -)$$

where the connecting maps

$$\mathcal{L}^{\kappa}_{\bullet}\big(\pi^G(M) \times \mathbf{B}^r), -\big) \to \mathcal{L}^{\kappa}_{\bullet}\big(\pi^G(M) \times \mathbf{B}^{r+1}), -\big)$$

are defined using multiplication by $\mathbf{B}$.

There is also no problem in defining a $\Delta$-set $\mathcal{S}^T_{G_\bullet}(M^n)$ whose $q$-th homotopy group is

$$\mathcal{S}^T_G(D^q \times M^n, S^{q-1} \times M^n)$$

and there is also a map of $\Delta$-sets

$$\eta_\bullet : \mathcal{S}^T_{G_\bullet}(M^n) \to \mathcal{N}^T_{G_\bullet}(M^n)$$

which essentially views an element in the source as a "solved equivariant surgery problem" over $M^n$. The composite $\sigma[\infty] \circ \eta_\bullet$ is nullhomotopic by construction and hence $\eta_\bullet$ factors up to homotopy through the homotopy fiber of $\sigma[\infty]$, which we shall denote by

$$\widehat{\eta}_\bullet : \widehat{\mathcal{S}}^T_{G_\bullet}(M^n) \to \mathcal{N}^T_{G_\bullet}(M^n).$$

Take $j_\bullet : \mathcal{S}^T_{G_\bullet}(M^n) \to \widehat{\mathcal{S}}^T_{G_\bullet}(M^n)$ to be a map of $\Delta$-sets such that $\widehat{\eta}_\bullet j$ is homotopic to $\eta_\bullet$.

We claim that $\sigma[\infty]$ is the desired modification of $\sigma$. Among other things, this means that $\sigma$ and $\sigma[\infty]$ should essentially be the same map in homotopy for small values of $\ell$ and that the homotopy groups of $\mathcal{L}^{\kappa}_{\bullet}(\pi^G(M), -)[\infty]$ should be equivariant surgery obstruction groups in the sense of [DoR] or [LM1, LM2]. These properties are implicit in the following result, which is an immediate consequence of the periodicity theorems for equivariant surgery groups in [DoS], Ch III.

**Theorem 6.** *Suppose that $D^\ell \times M^n$ satisfies the Gap Hypothesis. Then the partial maps of $\Delta$-sets*

$$P_1 : \mathcal{L}^\kappa_\bullet\left(\pi^G(M \times \mathbf{B}^r), -\right) \to \mathcal{L}^\kappa_\bullet\left(\pi^G(M \times \mathbf{B}^{r+1}), -\right)$$
$$P_\infty : \mathcal{L}^\kappa_\bullet\left(\pi^G(M \times \mathbf{B}^r), -\right) \to \mathcal{L}^\kappa_\bullet\left(\pi^G(M), -\right)[\infty]$$

*are $(\ell + r - 1)$-connected.*

This means that the $q$-th homotopy group of $\mathcal{L}^\kappa_\bullet(\pi^G(M), -)[\infty]$ is isomorphic to the previously considered group $\mathcal{L}^\kappa_{n+q}(\pi^G(M), -)$, where the latter are defined by fourfold periodicity if the Gap Hypothesis fails for, say, $D^q \times M^n$. We can now state the main result.

**Theorem 7.** *Suppose that $D^\ell \times M^n$ satisfies the Gap Hypothesis. Then there is an infinite exact sequence with pieces of the form*

$$\mathcal{L}^\kappa_{n+q+1}(\pi^G(M), -) \xrightarrow{\Delta_\infty} \widehat{\mathcal{S}}^T_G(D^q \times M, \partial) \xrightarrow{\hat{\eta}} \mathcal{N}^T_G(D^q \times M, \partial) \to \mathcal{L}^\kappa_{n+q}(\pi^G(M), -)$$

*that coincides with the surgery sequence of* [LM2] *when $q \le \ell - 1$. Furthermore, there are canonical maps*

$$j : \mathcal{S}^T_G(D^q \times M, \partial) \to \mathcal{L}^T_G(D^q \times M, \partial)$$

*such that*
*(i) the composite $\widehat{\eta} \circ j$ is the normal invariant map,*
*(ii) the map $j$ is bijective if $q \le \ell - 1$,*
*(iii) $\Delta_\infty$ has a canonical factorization of the form $j \circ \Delta_{BQ}$.*

The long exact sequence is merely the exact homotopy sequence of a Kan fibration associated to $\sigma[\infty]$, and as noted before the map $j$ arises from the homotopy triviality of $\sigma[\infty] \circ \eta$. Properties (i) and (ii) for $j$ follow immediately, and the factorization in (iii) is essentially given by the transverse linear, isovariant surgery theory developed by Browder and Quinn [BQ] (*cf.* Chapter II of [DoS]).

One important feature of the long exact sequence in ordinary surgery theory is that the two thirds of the groups can be described by other means. Specifically, the Wall groups have purely algebraic definitions (and can be analyzed quite effectively by algebraic means if the strata of the $G$-manifold $M$ are all simply connected), and the terms of the form $[-, F/O]$ are intrinsically homotopy theoretic. Similarly, the groups $\mathcal{N}^T_G(D^q \times M, \partial)$ can be computed by homotopy theory (although this is substantially more difficult than in the nonequivariant case), and the (stabilized) equivariant Wall groups $\mathcal{L}^\kappa_*(\pi^G(M), -)$ can be studied by means of spectral sequences whose $E^1$ terms are ordinary Wall groups (in fact, since $G$ is of odd order the equivariant groups split into sums of the nonequivariant groups by [LM2], Thm. 2.11). Therefore, in analogy with the nonequivariant case, the stabilized equivariant structure sets of Theorem 7 fit into long exact sequences in which the adjacent terms can all be studied by other means.

**Note.** In [LM2] an infinite long exact sequence for the normal invariant map $\eta = \widehat{\eta} \circ j$ is constructed. One major difference between the formal geometric analogs of the surgery obstruction groups in these sequences and the standard surgery obstruction groups appearing in Theorem 7 is that the former depend upon more than the data needed to define the latter. This reflects the relationship between the Gap Hypothesis and the extendibility of the $\pi - \pi$ theorem of surgery theory (see [Wa], Ch. 3) to equivariant surgery theory. Specifically, there is an equivariant $\pi - \pi$ theorem if the Gap Hypothesis holds (see [DoR], [LM1] and [LM2]) but examples of Rothenberg and Weinberger show that such a result does not necessarily hold if the Gap Hypothesis fails (see [DoS], Section I.6).

*Extensions to other groups*

If $G$ is an arbitrary group of odd order, the preceding discussion generalizes directly provided the $G$-manifold $M^n$ has strongly saturated orbit structure in the sense of [DoS], p. 87; in other words, if $x \in M^n$ has isotropy subgroup $G_x$, then every subgroup of $G_x$ can be realized as an isotropy subgroup in arbitrarily small neighborhoods of $x$. This condition is needed to ensure that $M \times \mathbf{B}$ does not have more isotropy subgroups than $M$, which in turn is needed to guarantee that $\mathcal{L}_*^\kappa(\pi^G(M), -)$ and $\mathcal{L}_*^\kappa(\pi^G(M \times \mathbf{B}), -)$ are isomorphic (see [DoS], Thms. III.2.5 and III.2.7–9). It is possible to prove extensions of Theorem 2.2 without the restriction on orbit structure, but some additional steps and constructions are needed. On the other hand, if $G = \mathbb{Z}_2$ the situation is considerably more complicated. Results of Dovermann [Do] imply that similar procedures do not work as well for $\mathbb{Z}_2$ as for odd order groups but results of M. Yan ([Y1], [Y2]) imply that there are periodicity results for all finite groups if one works in the isovariant stratified surgery theory developed by S. Weinberger [Wb] (see also Weinberger-Yan [WY]).

*Final Remarks*

It is natural to expect that a great deal of information is lost under the periodic stabilization map

$$\mathcal{S}_G^T(D^q \times M, \partial) \to \widehat{\mathcal{S}}_G^T(D^q \times M, \partial)$$

if the Gap Hypothesis does not hold for $D^q \times M$. Explicit examples of this sort follow from [DuS], Section 8, with $G = \mathbb{Z}_{pr}$, where $p$ and $r$ are distinct odd primes, and $M$ is the unit disk $D(W)$ in a suitably chosen $G$-representation $W$; strictly speaking this requires a relative version of the preceding for bounded manifolds such that the restriction of everything over $\partial M$ is a diffeomorphism, but the surgery sequences of [LM2] are formulated in these cases and everything we have done can be generalized directly to cover structures relative to $\partial M$. If we take the restriction type T to be ordinary or simple equivariant homotopy equivalence then the results of [DuS], Section 8, yield an increasing sequence of positive integers $q(k)$ for which the rational vector spaces

$$\mathcal{S}_G^T(D^{q(k)} \times D(W), \partial) \otimes \mathbb{Q}$$

contain finite-dimensional subspaces $\Omega_k$ such that $\dim_{\mathbb{Q}} \Omega_k$ grows exponentially. On the other hand, the dimensions of the stabilized groups

$$\widehat{\mathcal{S}}_G^T(D^q \times D(W), \partial) \otimes \mathbb{Q}$$

are uniformly bounded.

For over a quarter of a century the Gap Hypothesis has been formidable obstacle in the application of surgery theory to group actions. Usually progress outside the range of the Gap Hypothesis has involved isovariance assumptions as in the transverse linear, isovariant surgery theory of Browder and Quinn [BQ]. The relation between the Gap Hypothesis and isovariance is strengthened by the previously mentioned result that an equivariant homotopy equivalence can be deformed to an isovariant one if the Gap Hypothesis holds (see the discussion before Theorem 4). The obstruction theoretic approach to isovariant homotopy theory in [DuS] should allow one to apply homotopy theory more effectively to equivariant surgery outside the Gap Hypothesis range. Certainly some problems are likely to translate into very hard questions in unstable homotopy theory, but it also seems likely that a great deal of interesting and useful information can be extracted.

## 3. Controlled topology, stabilization and 4-manifolds

In Section 1 we noted that multiplicative stabilization yields examples of exotic, nonsmoothable free circle actions on 4-manifolds. However, in some respects a circle or finite group action on a 4-manifold is still relatively well behaved. For example, many basic restrictions on the fundamental groups of closed 4-manifolds with smooth circle actions have strong analogs for arbitrary topological circle actions [KwS4]. Furthermore, in sharp contrast to higher dimensions, for orientation preserving actions the fixed-point sets are all manifolds (which may be wildly embedded), and there is a Lefschetz fixed point formula for finite cyclic group actions on closed 4-manifolds that is identical to the corresponding, well known formula for smooth actions [KwS2]. A quite different local regularity result is obtained in [KwS3].

There are (at least) two reasons why controlled topology is particularly useful for analyzing group actions on 4-manifolds. One is given by the following result:

**Proposition 8.** *If $M^*$ is the orbit space of a nontrivial topological circle action on a 4-manifold $M$, then $M^* \times \mathbb{R}^2$ is a topological manifold.*

**Proof.** Without loss of generality we may assume the action is effective (by hypothesis the ineffective kernel $K$ is finite, so that one obtains an effective action of $S^1/K \approx S^1$). Let $N \subseteq M$ be the set of points where the action is free and let $N^*$ be its orbit space. Then every point of $N^*$ has a neighborhood $U^*$ such that the inverse image of $U^*$ in $N$ is equivariantly homeomorphic to $S^1 \times U^*$, so it follows that $N^* \times \mathbb{R}$ is a manifold. This implies that the Quinn invariant of $N^*$ from [Q2] is 1; since $M^*$ is a generalized manifold containing $N^*$ as an open subset, it follows that the Quinn invariant of $M^*$ is also 1. But now the results of [Q2] imply that $M^* \times \mathbb{R}$ has a resolution (*i.e.,* there is a map from some topological manifold onto $M^* \times \mathbb{R}$ with acyclic fibers), and if one takes products with $\mathbb{R}$ again then the results of [Q2] imply that $M^* \times \mathbb{R}^2$ is a genuine topological manifold.

A somewhat different connection between controlled topology and symmetries of 4-manifolds comes from the fact that the nested fixed point sets of orientation preserving actions are submanifolds. In other words, the orbit space is somehow built out of open strata that are manifolds. These are generally not homotopically stratified sets in the sense of Quinn [Q3] or Weinberger [Wb], but they often become homotopically stratified if one takes products with suitable linear actions on Euclidean space. Thus one can use the methods of stratified surgery theory [Hu, HW, Wb] to study the stabilized actions. In this paper we shall focus on one application of the first type and simply mention some applications involving stratified surgery and stabilization.

During the past two decades our knowledge of finite group actions on 4-manifolds has increased dramatically (*e.g.,* see papers of Edmonds or Wilczyński such as [E2], [E3] or [Wi]). Much of this progress involves applications of M. Freedman's extension of topological surgery theory to certain 4-manifolds [FQ]. One particularly striking consequence of Freedman's work is the existence of a topological 4-manifold **Ch** (the *Chern manifold*) that is homotopy equivalent but not homeomorphic to $\mathbb{CP}^2$. One criterion for distinguishing these is that $\mathbb{CP}^2 \times \mathbb{R}^k$ has a canonical smooth structure for each $k \geq 0$ but $\mathbf{Ch} \times \mathbb{R}^k$ has no smooth structures whatsoever.

Finite group actions on **Ch** have been studied extensively (for example, by the first author [Ksk2] and D. Wilczyński [Wi]) and the results show that **Ch** admits topological $\mathbb{Z}_k$ actions for every odd positive integer $k$. On the other hand, the results of the first author and P. Vogel [KV] imply that **Ch** does not admit a locally linear involution, and the question of whether nontrivial circle actions exist on **Ch** has been discussed for some time (see Adams [A2] or Kirby [Ki]). In this section we shall show that **Ch** has no semifree circle actions; we shall study further problems along this line elsewhere.

Our result for **Ch** depends upon a taming principle for semifree circle actions on 4-manifolds; this is weakly analogous to a result of A. Edmonds for free circle actions on higher dimensional manifolds [E1].

**Theorem 9.** *Let $M$ be a closed 4-manifold with a semifree circle action $\Phi$. Then the product manifold $(M, \Phi) \times (T^2$, trivial action$)$ is equivariantly $h$-cobordant to a semifree $S^1$ manifold $N$ such that*
*$(i)$ the $h$-cobordism is a product on the fixed point set,*
*$(ii)$ all 4-dimensional components of the fixed point set of $N$ have equivariant linear tubular neighborhoods.*

**Proof.** Let $M^*$ be the orbit space, let $F$ be the fixed point set, and let $F^*$ denote the image of $F$ in $M^*$. By the results of [KwS4], Section 1, the set $F$ is a disjoint union of finitely many points and compact surfaces, and the surface components map bijectively to $\partial M^*$. Let $F_0 \subseteq F$ be the set of isolated fixed points and let $F_0^*$ be its image in $M^*$.

Since $S^1$ acts freely on $M - F$ it follows that the orbit space projection defines a principal $S^1$ bundle

$$\omega : M - F \to M^* - F^*.$$

Duality considerations imply that $\omega$ extends to a principal bundle

$$\omega' : M' \to M^* - F_0^*.$$

Let $N^* = M^* \cup_\partial (\partial M^* \times [0,1])$, where $\partial M^*$ is identified with $\partial M^* \times \{0\}$, and form the $S^1$ space

$$N = M' \cup_\partial D(\omega' | \partial M^*),$$

where $D$ denotes the associated $\partial$-disk bundle. By construction $N^* \cong N/S^1$. Results of [Bo] and [CF] show that $N$ and $N^*$ are compact $ANRs$, and results of [Ra] show that both are also generalized 4-manifolds, with $\partial N = \emptyset$. Let $J$ be the 2-simplex

$$\{(s,t) \in I \times I | s \leq t\}$$

($i.e.$, the isosceles right triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$ ) and let $W^*$ be the union of $M^* \times I$ with $\partial M^* \times J$ where $\partial M^* \times I$ in $M^* \times I$ is identified with $\partial M^* \times \{0\} \times I$ in $\partial M^* \times J$. Then $W^*$ is an $h$-cobordism of generalized manifolds from $M^*$ to $N^*$, and it lifts to an $S^1$ equivariant $h$-cobordism of generalized manifolds $W$ from $M$ to $N$ that is a product on the fixed point set. If we take the product of $W$ with $T^2$, then it follows that $W \times T^2$ is an $S^1$ equivariant $h$-cobordism that is a genuine topological manifold, and the desired properties follow from the construction.

We can now prove the result for **Ch**.

**Theorem 10.** *If $M$ is a 4-manifold that admits a nontrivial semifree circle action and is homotopy equivalent to $\mathbb{CP}^2$, then $M$ is homeomorphic to $\mathbb{CP}^2$.*

**Proof.** Suppose $M$ satisfies the given hypotheses, and let $M^*$ be its orbit space. Then $M^*$ is a contractible generalized 3-manifold with $\partial M^* = S^2$, and the fixed point set is homeomorphic to a disjoint union $\partial M^* \amalg \{pt.\}$. There is an $S^1$ equivariant homotopy equivalence $\varphi$ from $M$ to $\mathbb{CP}^2$ with a linear semifree circle action (which is unique up to equivalence), and if $W$ is the equivariant $h$-cobordism constructed in Theorem 9, then the map $\varphi$ extends to an equivariant homotopy equivalence $W \to \mathbb{CP}^2$ whose restriction to $N$ is transverse to the standardly embedded $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$ that is fixed under the circle action and maps $\partial M \subseteq N$ homeomorphically to $\mathbb{CP}^1$.

If we take products with $T^2$ we obtain a manifold $h$-cobordism from $f \times id : M \times T^2 \to \mathbb{CP}^2 \times T^2$ to $g \times id : N \times T^2 \to \mathbb{CP}^2 \times T^2$, and by construction the restriction of the normal invariant of $g \times id$ to $\mathbb{CP}^1 \times \{e\} \subseteq \mathbb{CP}^2 \times T^2$ is trivial. On the other hand, if $M$ is homeomorphic to **Ch** then the restriction of the normal invariant of $f \times id$ to $\mathbb{CP}^1 \times \{e\}$ is nontrivial; in fact, this normal invariant is the means for distinguishing $\mathbb{CP}^2$ and **Ch**. It follows that $M$ must be homeomorphic to $\mathbb{CP}^2$.

*Some further results*

Theorem 9 is just one example of how group actions on 4-manifolds simplify after stabilization. Here are two results that follow from work of B. Hughes, L. Taylor, S. Weinberger and B. Williams [HTW].

**Theorem 11.** *If $S^1$ acts semifreely on a 4-manifold $M$ and $D(\mathbb{C})$ is the unit disk in the complex plane with the $S^1$ action given by scalar multiplication then the product action on $M \times D(\mathbb{C})$ is locally linear.*

**Theorem 12.** *If $\mathbb{Z}_2$ acts nontrivially on a 4-manifold $M$ and $x \in M$ lies in a 2-dimensional component of the fixed point set, then $(x,0) \in M \times D(\mathbb{C})$ has an invariant neighborhood that is equivariantly a product $\mathbb{R}^2 \times U$, where $U$ is a neighborhood of an isolated fixed point of an involution on $\mathbb{R}^4$.*

The possibilities for $U$ up to germ equivalence can be described fairly well by means of surgery theory (compare [KwS1]). Specifically, they are given by taking the universal coverings of topological manifolds $X$ homotopy equivalent to $S^1 \times \mathbb{RP}^3$ and adding a point at infinity to one end. The results of [FQ] completely determine the possibilities for $X$. Theorem 12 will figure in a subsequent paper by the authors on involutions of **Ch** (see [E2] for some background information).

The methods of controlled topology have yielded a great deal of new information about group actions during the past two decades. An excellent overview of this work is given in an article of Cappell and Weinberger [CW].

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