Toral and exponential stabilization for homotopy spherical spaceforms

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Abstract

The Atiyah–Singer equivariant signature formula implies that the products of isometrically inequivalent classical spherical space forms with the circle are not homeomorphic, and in fact the same conclusion holds if the circle is replaced by a torus of arbitrary dimension. These results are important in the study of group actions on manifolds. Algebraic K-theory yields standard classes of counterexamples for topological and smooth analogs of spherical spaceforms. The results of this paper characterize pairs of nonhomeomorphic topological spherical space forms whose products with a given torus of arbitrary dimension are homeomorphic, and the main result is that the known counterexamples are the only ones that exist. In particular, this and basic results in lower algebraic K-theory show that if such products are homeomorphic, then the products are already homeomorphic if one uses a 3-dimensional torus. Sharper results are established for important special cases such as fake lens spaces. The methods are basically surgery-theoretic with some input from homotopy theory. One consequence is the existence of new infinite families of manifolds in all dimensions greater than three such that the squares of the manifolds are homeomorphic although the manifolds themselves are not. Analogous results are obtained in the smooth category.

1. *Introduction*

Given a topological space B and two topological spaces X and Y such that $X \times B$ and $Y \times B$ are homeomorphic, it is well known that X and Y need not be homeomorphic even if one takes B to be the real line $\mathbb{R}$ or the unit circle S^1. For example, if $B = \mathbb{R}$ then there is a space Y such that $Y \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4 \approx \mathbb{R}^3 \times \mathbb{R}$ but Y is not a

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topological manifold (compare Bing [**6**], and if $B = S^1$ one can even find spaces X and Y with nonisomorphic fundamental groups such that $X \times S^1$ and $Y \times S^1$ are homeomorphic (compare Conner and Raymond [**7**] and [**22**]). Similar conclusions hold in the categories of piecewise linear spaces and smooth manifolds. For these and other choices of B, results of this sort have numerous implications for geometric topology; a few examples involving transformation groups are considered in the authors' survey paper [**26**].

In several contexts it is interesting and useful to know when two objects X and Y (in the appropriate category) become isomorphic after taking products with the torus T^k or Euclidean space $\mathbb{R}^k$. Products with tori have figured importantly in geometric topology for nearly four decades. Examples include S. P. Novikov's work on the topological invariance of rational Pontrjagin classes (*e.g.*, see [**35**] or [**36**]), the Kirby–Siebenmann results on topological manifolds [**20**] (see also Rudyak [**39**]), and numerous formulas in algebraic K- and L-theory like the Bass–Heller–Swan decomposition [**5**] or Shaneson's formula (compare Anderson–Hsiang [**1**], Hsiang [**17**], Shaneson [**43**] or Ranicki [**38**]), and products of lens spaces with tori have been known to figure crucially in the topological classification of linear and smooth group actions for at least a quarter century; references in this direction include papers of R. Lashof and M. Rothenberg [**27**], I. Madsen and M. Rothenberg ([**28–31**]), M. Rothenberg and S. Weinberger [**40**] and I. Hambleton and E. Pedersen [**16**]. In most of the latter papers it is important to have a version of the following **toral stability property** for lens spaces: *Let L and L' be lens spaces such that $L \times T^k$ and $L' \times T^k$ are homeomorphic. Then L and L' are diffeomorphic.*

If $k = 1$ this is an important consequence of the Atiyah–Singer G-Signature Formula (compare [**2**, p. 590]). Strong partial results on this question for arbitrary values of k have been known for some time, and they have sufficed for the purposes mentioned above. Using S. Weinberger's theory of higher Atiyah–Singer ρ-invariants [**47**], one can prove more general results of this sort where T^k can be replaced by certain other aspherical manifolds. More precisely, this holds if one has a strong form of the Novikov Conjecture for the associated fundamental group, and in particular it holds for, say, products with symmetric spaces of noncompact type. The methods immediately yield analogous results if L and L' are arbitrary spherical spaceforms.

1·1. *Main results for homotopy spherical spaceforms*

In this paper we are interested in quotient spaces of arbitary free actions of finite groups on homotopy spheres. To streamline the discussion we shall often use the term *homotopy spherical spaceforms* to denote such manifolds; these include both fake and genuine spherical spaceforms.

It is well known that the toral stability property fails for homotopy spherical spaceforms. This is true because (i) in general there are fake lens spaces that are h-cobordant to their linear models, (ii) the products of h-cobordant manifolds with S^1 are isomorphic in the appropriate category. However, homotopy spherical spaceforms turn out to have modified toral stability properties that are reasonably good.

THEOREM 1·1. *Let M and M' be homotopy spherical spaceforms such that $M \times T^k$ is isomorphic to $M' \times T^k$. Then $M \times T^3$ is isomorphic to $M' \times T^3$.*

Stronger results hold if the fundamental groups of M and M' are cyclic or quaternionic, in which cases one can replace T^3 with T^2 or S^1, respectively. These results are sharp; one cannot replace T^2 by S^1 for cyclic fundamental groups and one cannot replace S^1 by $\{1\}$

for quaternionic groups. In fact, in the quaternionic case the corresponding spaceforms are completely classified.

1·2. *Related results*

If L and L' are nonhomeomorphic 3-dimensional lens spaces with isomorphic fundamental groups, then [**25**] shows that $L \times L$ and $L' \times L'$ are diffeomorphic. One motivation for our interest in toral stability was to find parallel systematic families of n-manifold pairs (X^n, Y^n) in dimensions $n > 3$ such that X^n and Y^n are not isomorphic but $X^n \times X^n$ and $Y^n \times Y^n$ are. If L and L' are as above, then the Toral Stability Theorem and [**25**] imply that $L \times T^{k-2}$ and $L' \times T^{k-2}$ are not homeomorphic but their cartesian squares are diffeomorphic. One also has results for higher cartesian powers analogous to those of [**25**], and in fact one can combine the methods of [**25**] with the methods of this paper and [**47**] to find infinite families of examples.

THEOREM 1·2. *Let L be a 3-dimensional lens space and let $n > 4$. Then there are infinitely many smooth manifolds X_k that are simple homotopy equivalent to $L \times T^{n-3}$ such that* (a) *the manifolds X_k are pairwise nonhomeomorphic and have distinct higher ρ-invariants,* (b) *the cartesian nth powers $\prod^m X_k$ and $\prod^m (L \times T^{n-3})$ are diffeomorphic for all $m > 1$. Similar results hold in the topological category if $n = 4$.*

If the Thurston Geometrization Conjecture for 3-manifolds is true, then analogous infinite families cannot exist when $n = 3$.

1·2·1. *Stabilization by Euclidean spaces*

A second question related to our main result is the stability of homotopy spherical spaceforms under taking products with Euclidean spaces. In the classical 3-dimensional case of lens spaces, we have the following result due to J. Milnor (compare [**34**, corollary 12·12] and [**33**, section 2]).

THEOREM 1·3. *Let L and L' be 3-dimensional lens spaces. Then $L \times \mathbb{R}^n \approx L' \times \mathbb{R}^n$ if and only if $L \approx L'$ for $n \leqslant 1$, and $L \times \mathbb{R}^n \approx L' \times \mathbb{R}^n$ if and only if $L \simeq L'$* (*homotopy equivalent*) *for $n \geqslant 3$.*

These results lead naturally to two further questions:

Question 1. *What happens when $n = 2$?*
Question 2. *What happens for higher dimensional lens spaces, or more generally for homotopy spherical spaceforms in higher dimensions?*

The techniques used by Milnor cannot be extended, or adapted, to cover the missing case of stabilization by $\mathbb{R}^2$. In fact, one even encounters rather serious difficulties when applying the full strength of surgery theory to this problem. This reflects the highly exceptional nature of codimension two embeddings (compare Cappell–Shaneson [**9**]). However, it turns out that in this case one can wrap (or furl) $\mathbb{R}^2$ around the torus, and thus obtain the following results, which answer Question 1 as well as its analogs in higher dimensions.

THEOREM 1·4. *Let M and N be linear spherical spaceforms* (*of any dimension*). *Then $M \times \mathbb{R}^n \approx N \times \mathbb{R}^n$ if and only if $M \approx N$ for $n \leqslant 2$.*

In view of this result, the second question reduces to determining when $M \times \mathbb{R}^n \approx N \times \mathbb{R}^n$ for some $n \geqslant 3$ if M and N are spherical spaceforms. Significant partial results on this question follow from the work of J. Folkman [**11**] and B. Mazur [**32**]. In particular, Mazur's techniques and elementary considerations imply the following:

THEOREM 1·5. *Let M and N be linear spherical spaceforms (of any dimension). Then the following hold::*

(i) *If $M \times \mathbb{R}^n \approx N \times \mathbb{R}^n$ for some $n \geqslant 1$ then M and N are tangentially homotopy equivalent in the sense of Mazur (compare* [**32**], [**44**]*).*
(ii) *If M and N are tangentially homotopy equivalent in this sense, then $M \times \mathbb{R}^n \approx N \times \mathbb{R}^n$ for all $n > \dim M = \dim N$.*

Since Folkman's results provide a complete number-theoretic characterization of the tangential homotopy equivalence relation for lens spaces in terms of the algebra of the group ring $\mathbb{Z}[\pi]$ (see [**11**, Theorem 7·1]), this yields essentially complete information on $\mathbb{R}^n$ stabilization for spherical spaceforms (and in particular for lens spaces) when n is sufficiently large. Milnor's result fits in with the general result because all orientable 3-manifolds are parallelizable and hence all homotopy equivalences between them are tangential.

In the remaining cases where $3 \leqslant n \leqslant \dim M = \dim N$, the methods of [**32**] reduce the question of equivalence under $\mathbb{R}^n$ stabilization to problems about smooth embeddings up to homotopy type (see [**21**] for information about currently available techniques for studying such problems), but quantitatively the results are less illuminating much of the time. However, in order to illustrate the differences between $\mathbb{R}^2$ and $\mathbb{R}^3$ stabilization, at the end of this paper we shall describe infinite families of pairs of lens spaces (M, N) such that $M \times \mathbb{R}^n \approx N \times \mathbb{R}^n$ for all $n \geqslant 3$ but not for $n \leqslant 2$ (see Section 3).

1·3. *Methods and organization of this paper*

Since homotopy spherical spaceforms with a given fundamental group are classified up to finite ambiguity by their Atiyah–Singer ρ-invariants, the results of Weinberger [**47**] or other considerations (for example, those of Hambleton–Pedersen [**16**]) immediately imply that the products of spherical spaceforms with tori are classified up to finite ambiguity by their higher ρ-invariants.

Our objective in this paper is not simply to reprove these results but rather to go beyond finite ambiguity and study the actual classification of such products up to homeomorphism. This requires a direct analysis of the classical surgery structure sets and the actions of groups of homotopy self-equivalences on such sets. More specialized consequences as in Theorems 3·9 and 3·10 also use the work of D. Carter on the vanishing of the lower algebraic K-theory for group rings of finite groups [**8**] and the computations of Wall groups for quaternionic groups in Hambleton–Milgram [**14**]. The subsequent material in this paper is divided into four sections. Section 2 contains various observations related to surgery and homotopy theory that are needed for the proofs. Proofs of the toral stability theorems for linear and fake spherical spaceforms appear in Section 3. In Section 4 we prove the result on Euclidean space stabilizations and describe systematic examples illustrating the difference between stabilization by $\mathbb{R}^2$ and $\mathbb{R}^3$. Finally, in Section 5 we prove the results on exponential stabilization.

2. *Surgery and homotopy*

In this section we shall collect several observations that will be needed in the proofs of Theorems 3·1 and 1·1. Some of these represent restatements of well-known facts, while others are fairly straightforward but do not seem to be in the literature.

2·1. *Homotopy self-equivalences in the surgery sequence*

Let X, Y be topological spaces, and let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. If X is compact Hausdorff, then the monoid $E(X)$ self-homotopy equivalences of X becomes a topological monoid in the compact open topology; as usual, we shall let the identity be its default basepoint.

Let M^n be a closed, oriented n-dimensional manifold with $\pi_1(M^n) \cong \pi$. If $n \geqslant 5$, we have the following associated Sullivan–Wall exact surgery sequence (compare [**45**]):

$$\cdots \longrightarrow L^s_{n+1}(\pi) \xrightarrow{\gamma} \mathcal{S}^s_{\mathrm{Top}}(M^n) \xrightarrow{\eta} [M^n; G/\mathrm{Top}] \xrightarrow{\theta} L^s_n(\pi)\ .$$

Consider now the action of $\pi_0(E(M))$ on $\mathcal{S}^s_{\mathrm{Top}}(M^n)$ given by choosing a self-equivalence h and a homotopy structure $(X,\ f\colon X \to M)$ for classes in the respective sets and sending the pair to the class represented by $(X,\ h \circ f\colon X \to M)$. It is elementary to verify that this construction does not depend upon the choices of representatives.

CLAIM 2·1. *The inverse image of* 0 *under the composite* $\pi_0(E(M)) \to \mathcal{S}^s_{\mathrm{Top}}(M^n) \to [M^n, G/\mathrm{Top}]$ *is a subgroup.*

Proof. The normal invariant formula

$$\eta(f \circ g) = \eta(f) + (f^*)^{-1}\eta(g)$$

shows that the inverse image is closed under multiplication. To see the closure under inverses, let $f \circ g \simeq \mathrm{id}$. Then

$$0 = \eta(f) + (f^*)^{-1}\eta(g)$$

shows that $\eta(f) = 0$ if and only if $\eta(g) = 0$, and thus the set in question is closed under taking inverses. ☐

We next consider homotopy self-equivalences that are normally cobordant to the identity, so assume that we are given $[h_1], [h_2] \in E(M^n)$ having the form $[h_i] = \gamma(\alpha_i)$ where $\alpha_i \in L^s_{n+1}(\pi)$ for $i = 1, 2$.

CLAIM 2·2. *In the structure set for* M^n*, the class of the composite* $[h_2 h_1]$ *represents* $\gamma(\alpha_2 + (h_2)_*\alpha_1)$ *where* $(h_2)_*\colon L^s_{n+1}(\pi) \to L^s_{n+1}(\pi)$ *is the automorphism of Wall groups associated to* $(h_2)_*$ *on* $\pi = \pi_1(M)$.

Proof. For $i = 1, 2$ let $h_i : M^n \to M^n$ be a homotopy self-equivalence that is normally cobordant to the identity with $[h_i] = \gamma(\alpha_i)$ for $\alpha_i \in L^s_{n+1}(\pi_1(M))$. Let $(F_1, B_1)\colon (W_i; \partial_0, \partial_1) \to M^n \times I$ be a normal cobordism between the identity and h_1 (i.e., the restrictions to the boundary components satisfy $F_1|\partial_0 = \mathrm{id} : M^n \to M^n$ and $F_1|\partial_1 = h_1 : M^n \to M^n$).

Analogously, let $(F_2, B_2)\colon (W_2; \partial_0, \partial_1) \to M^n \times I$ be a normal cobordism between the identity and h_2. By stacking one cobordism on the top of the other we can form a normal cobordism $(F, B) : W_1 \cup W_2 \to M \times [0, 2]$ with $F|W_2 = F_2$ and $F|W_1 = (h_2 \times T_1) \circ F_1$, where T_1 is the translation $[0,1] \to [1,2]$. Here the linear bundle map B is obtained by putting together B_1 and B_2. Note that $F\big|\partial_0 = \mathrm{id}$ and $F\big|\partial_1 = h_2 \circ h_1 : M^n \to M^n$.

If $\theta(-)$ is the surgery obstruction in $L^s_{n+1}(\pi_1(M^n))$, then $\theta(F_1) = \alpha_1$, $\theta(F_2) = \alpha_2$ and it turns out that $\theta(F) = (\alpha_2 + h_{2*}\alpha_1)$. To be more specific, $\theta(F,B) = \theta(F_2, B_2) + \theta\left((h_2 \times \mathrm{id}) \circ F_1, (h_2 \times \mathrm{id})^{*-1} B_1\right)$ and by the bordism definition of Wall groups the second term on the right-hand side is merely $h_{2*}\theta(F_1, B_1)$. It follows then (by construction) that $[h_2 h_1] = \gamma(\theta(F)) = \gamma(\alpha_2 + h_{2*}\alpha_1)$ as claimed. $\square$

2·2. *Decompositions for certain groups of homotopy classes*

Let Y be an arbitrary topological monoid with the identity as the basepoint. Assume that A and B are connected finite CW complexes. Then one has the following split group extension:

$$1 \longrightarrow [A \wedge B, Y] \longrightarrow [A \times B, Y] \longrightarrow [A, Y] \times [B, Y] \longrightarrow 1$$

This sequence and the cellular filtration of the torus T^k can be used to show that $[T^k, Y]$ is an iterated split extension such that the factors are all products of homotopy groups of Y. Here is a very brief sketch of the argument.

For each subset $\alpha \subset \{1, 2, 3, \ldots, k\}$, let $|\alpha|$ denote the number of elements in α, and let T^α be the $|\alpha|$-dimensional torus corresponding to the elements of α (e.g., if $\alpha = \{1,3,5\} \subset \{1,2,3,4,5,6\}$, then $T^\alpha = S^1 \times \{pt.\} \times S^1 \times \{pt.\} \times S^1 \times \{pt.\}$). Take $P_\alpha : T^k \to T^\alpha$ to be the associated projection and let $J_\alpha : T^\alpha \to T^k$ be the associated injection. Then one has an iterated split extension corresponding to the skeletal filtration $F_j[T^k, Y]$ such that F_j consists of all classes whose restriction to the j-skeleton

$$\bigcup_{|\alpha| \leqslant j} T^\alpha$$

is null homotopic, and

$$F_j \Big/ F_{j+1} \cong \prod_{|\alpha| = j} \pi_{|\alpha|}(Y) \ .$$

Notice that F_j is normal in $[T^k, Y]$. One then gets various factorizations of an element in $[T^k, Y]$. Let $\varphi_j : F_j \Big/ F_{j+1} \to [T^k, Y]$ be the map (not necessarily a homomorphism!) corresponding to the following composite:

$$\prod\nolimits_{|\alpha|=j} \pi_{|\alpha|}(Y) \xrightarrow{(\text{collapses})^*} \prod\nolimits_{|\alpha|=j} [T^\alpha, Y] \xrightarrow{(\text{proj})^*} \prod [T^k, Y] \xrightarrow{\text{multiplication}} [T^k, Y] \ .$$

Then we can write an element $u \in [T^k, Y]$ as a product

$$\varphi_i(u_i)\varphi_{i+1}(u_{i+1}) \ \ldots \ \varphi_k(u_k) \cdot \varphi_1(u_1) \tag{2·1}$$

for suitable $u_\ell \in F_\ell / F_{\ell+1}$.

For our purposes it is important to understand the effect of certain finite torus unfurlings (*i.e.*, finite covering maps from the torus to itself) on the factorization in Equation 2·1.

PROPOSITION 2·3. *Let $r > 1$ be an integer, let $\psi^r : T^k \to T^k$ be the rth power map, and let $u \in [T^k, Y]$ be given with the factorization described above. Then the composite $u \circ [\psi^r]$ has a factorization of the following form:*

$$\varphi_i(r^i u_i)\varphi_{i+1}(r^{i+1} u_{i+1}) \ \ldots \ \varphi_k(r^k u_k) \cdot \varphi_1(r u_1) \ \ldots \ \varphi_{i-1}(r^{i-1} u_{i-1}) \ .$$

The proof of the preceding result is an elementary exercise.

We are primarily interested in topological monoids that are spaces of homotopy self-equivalences of some finite CW complex X. If $E(X)$ is such a monoid and W is another finite CW complex, then the standard adjoint functor relationship on spaces of continuous functions

$$\mathbf{F}\left(A, \mathbf{F}(B, C)\right) \approx \mathbf{F}(A \times B, C)$$

defines a continuous mapping

$$Ad' : \mathbf{F}(W, E(X)) \longrightarrow E(W \times X)$$

sending a function $g : W \to E(X)$ to a map $g_{\#}$ satisfying

$$g_{\#}(w, x) = (w, [g(w)](x)) \, .$$

It is an elementary exercise to verify that the map Ad' is a monoid homomorphism if the operation on the domain is given by pointwise composition.

2·3. *Product decompositions of Wall groups*

Since we shall be working extensively with the Wall groups of products having the form $\pi \times \mathbb{Z}^k$ it will be useful to summarize some facts about them in order to avoid digressions later.

If π is a finitely presented group, then the Wall group $L^s_{n+k+1}(\pi \times \mathbb{Z}^k)$ can be expressed in terms of lower quadratic L-groups of π, $L^{\langle 2-i \rangle}_{*}(\pi)$, as follows (see Ranicki [**38**, proposition 17·3]):

$$L^s_{n+k+1}\left(\pi \times \mathbb{Z}^k\right) \;\cong\; \sum_{i=0}^{k} \binom{k}{i} L^{\langle 2-i \rangle}_{n+k+1-i}(\pi) \, .$$

It will be convenient to rewrite this in the form

$$L^s_{n+k+1}(\pi \times \mathbb{Z}^k) \approx \sum_{\alpha} L^{\langle 2-|\alpha| \rangle}_{n+k+1-|\alpha|}(\pi) \, .$$

Here α ranges over the subsets of $\{1, ..., k\}$ and as usual $|\alpha|$ denotes the number of elements in α; if $k = 1$ this is the Shaneson splitting [**43**], so we shall call this the *generalized Shaneson splitting.* This decomposition is useful because the splitting injection for the factor indexed by α corresponds to crossing with the torus T^{β} geometrically, where β is the complement of α in $\{1, 2, \cdots k\}$ (we are using the same notation as in the discussion of $[T^k, Y]$), or by tensoring algebraically with the group ring

$$\mathbb{Z}\left[t_j, t_j^{-1} | j \in \beta\right] .$$

In particular the preceding yields a split surjection

$$H : L^s_{n+k+1}(\pi \times \mathbb{Z}^k) \longrightarrow L^{\langle 2-k \rangle}_{n+1}(\pi) \, .$$

If $P : L^h_*(\pi) \to L^s_{*+k}(\pi \times \mathbb{Z}^k)$ denotes the homomorphism induced geometrically by crossing with T^k, then by construction the composite $H \circ P$ is the forgetful map.

In discussing the implications of the main results we shall need the following well understood point:

PROPOSITION 2·4. *The kernel of the product map P is a finite abelian 2-group.*

Proof. It suffices to show this for the forgetful map because the latter is just $H \circ P$. But an appropriate version of the Rothenberg exact sequence (Wall [**46**, p. 4]; see also Ranicki [**37**, p. 146]), implies that the kernels and cokernels of the forgetful maps $L^h_{n+1}(\pi) \to L^{\langle 2-k\rangle}_{n+1}(\pi)$ are finite abelian 2-groups. □

3. *Stabilization by tori*

The main results of this paper may be viewed as specializations of the following abstract result.

THEOREM 3·1. *Let M^n and N^n be homotopy spherical spaceforms such that $M \times T^k$ is homeomorphic to $N \times T^k$ (i.e., $M \times T^k \approx N \times T^k$) for some $k \geqslant 1$. Then there is a homotopy equivalence $h\colon M^n \to N^n$ such that h is normally cobordant to the identity, and if $\mathbf{x} \in L^h_{n+1}(\pi_1(N^n))$ is the surgery obstruction associated to h, then there is a relative degree one normal map $\mathbf{G}$ into $N \times I$ (i.e., it is a homeomorphism on the boundary) such that the images of $\mathbf{x}$ and the surgery obstruction $\sigma(\mathbf{G}) \in L^s_{n+1}(\pi_1(N^n))$ are equal in the lower L-group $L^{\langle 2-k\rangle}_{n+1}(\pi_1(N^n)))$ described above.*

The toral stability property of lens spaces follows immediately from Theorem 3·1 and Proposition 2·4 because nonhomeomorphic (genuine) spherical spaceforms have distinct ρ-invariants. We have already noted that this also follows from [**47**]; results of Hambleton and Pedersen involving bounded surgery (see [**16**], especially Section 5) also provide an illuminating conceptual approach. As noted, the key difference between the linear and nonlinear cases is that the latter involve invariants arising from 2-primary torsion in Wall groups. Although it is immediately clear, from Whitehead torsion considerations, that one cannot have similarly strong toral stability for homotopy spherical spaceforms, Theorem 3·1 leads to reasonably good general results on toral stability.

In the piecewise linear and smooth categories one can say slightly more.

THEOREM 3·2. *If the manifolds M and N in Theorem 3·1 are smooth or piecewise linear and $n + k \geqslant 5$, then M and N are respectively diffeomorphic or piecewise linearly homeomorphic.*

Proof. This follows because Theorem 3·1 implies that the product of the homotopy equivalence h with the identity is s-cobordant to the identity in whichever category of manifolds one considers, at least if $n + k \geqslant 5$. □

3·1. *Proof of Theorem 3·1*

Let L and L' be homotopy spherical spaceforms of the same dimension; in view of the classification results for 1- and 2-manifolds, we may as well assume the dimension is at least 3. Assume $L \times T^k$ is homeomorphic to $L' \times T^k$ for some $k \geqslant 1$, and let $\Phi : L \times T^k \to L' \times T^k$ be a specific homeomorphism. If $k = 1$, then Theorem 3·1 follows from Wall's extensions of the G-signature theorem to topological actions (compare Wall [**45**], p. 189). More precisely, a homeomorphism $L \times S^1 \approx L' \times S^1$ leads to a homeomorphism between $L \times \mathbb{R}$ and $L' \times \mathbb{R}$, and hence a topological h-cobordism between L and L'. Because of this, from now on we shall assume $k \geqslant 2$.

If we pass to covering spaces associated to torsion subgroups, we obtain obtain a homeomorphism ϕ from $L \times \mathbb{R}^k$ to $L' \times \mathbb{R}^k$, and the composite

$$L \approx L \times \{0\} \subset L \times \mathbb{R}^k \approx L' \times \mathbb{R}^k \to L'$$

is a homotopy equivalence. Let $\widehat{\Phi}\colon L \to L'$ be the homotopy equivalence determined by this composite.

CLAIM 3·3. *Without loss of generality, one can assume that the diagram*

$$\begin{array}{ccc} L \times T^k & \xrightarrow{\Phi} & L' \times T^k \\ {\scriptstyle\text{proj}}\big\downarrow & & {\scriptstyle\text{proj}}\big\downarrow \\ T^k & \xrightarrow{=} & T^k \end{array}$$

is homotopy commutative.

Proof. Let $A \in GL(k,\mathbb{Z})$ be the matrix for the isomorphism $\pi_1(L' \times T^k)/\text{TORSION} \approx \pi_1(L \times T^k)/\text{TORSION}$ associated to Φ with respect to the standard free generators. Set B equal to A^{-1} and let $\overline{B}_0$ be the induced endomorphism of T^k. Then $(\mathrm{id} \times \overline{B}_0) \circ \Phi$ is the homeomorphism with the desired property.

Let $\widehat{\Phi}^{-1}$ be a homotopy inverse to $\widehat{\Phi}$. Then

$$g : L \times T^k \xrightarrow{\Phi} L' \times T^k \xrightarrow{\widehat{\Phi}^{-1} \times \mathrm{id}} L \times T^k$$

is a homotopy self-equivalence with the following properties:

(i) g homotopy commutes with the projection on T^k;

(ii) the composite $L \to L$ is homotopic to the identity;

(iii) g and $\widehat{\Phi}^{-1} \times \mathrm{id}$ determine the same element of $\mathcal{S}^s_{\mathrm{Top}}(L \times T^k)$.

The first two properties follow immediately from the construction. The equality $[g] = [\widehat{\Phi}^{-1} \times \mathrm{id}]$ in $\mathcal{S}^s_{\mathrm{Top}}(L' \times T^k)$ follows from the fact that $\varphi : L \times T^k \to L' \times T^k$ is a homeomorphism and $g = (\widehat{\Phi}^{-1} \times \mathrm{id}) \circ \Phi$. $\square$

Although the claim 3·3 is quite elementary, it has far reaching consequences. We begin with a property of $\widehat{\Phi}$ and its homotopy inverse that is stated explicitly below.

CLAIM 3·4. *The homotopy equivalences $\widehat{\Phi}$ and its homotopy inverse(s) are normally cobordant to the identity.*

Proof. It suffices to show that $\widehat{\Phi} \times \mathrm{id}\,(T^k)$ is normally cobordant to the identity and, by property (iii) above, this is equivalent to verifying the same thing for g. The projection $p_L \circ g$ of g onto L is coadjoint to a continuous map $g_\flat : T^k \to E(L)$, and if we choose another map φ that is homotopic to $g_\flat$ with adjoint $\varphi' : L \times T^k \to L$, then g and the map $\varphi_\#$ are homotopic and therefore determine the same class in the structure set. Choose φ homotopic to $g_\flat$ so that φ sends all points on some closed coordinate disk $D \subset T^k$ to the identity. It follows that $\varphi_\#$ is a homotopy self-equivalence of manifold triads on $L \times \left(T^k; D, T^k - \mathrm{Int}\, D\right)$ that is the identity on $L \times D$. A diagram chase then shows that the restriction of the normal invariant to L is trivial. Since the relation $[g] = [\widehat{\Phi}^{-1} \times \mathrm{id}]$ implies that the normal invariant of g is detected by its restriction to L, this proves the claim. $\square$

The preceding discussion also implies that the simple homotopy self-equivalence g is normally cobordant to the identity and therefore is given by the action of the Wall group

$$L^s_{n+k+1}(\pi \times \mathbb{Z}^k)$$

on the structure set, where $\pi = \pi_1(L)$. Problems of this sort are often quite difficult, but torus unfurling will yield the following means for bypassing such questions here.

CLAIM 3·5. (Main Claim) *If there is a simple homotopy self-equivalence $g\colon L\times T^k \to L\times T^k$ with properties* (i)–(iii) *as above, then there is also one that has the same properties and represents the trivial element of the structure set.*

Assuming this, we may complete the proof of Theorem 3·1 as follows: suppose we have a homeomorphism $\Phi : L \times T^k \to L' \times T^k$ (in the smooth categories assume it is a PL homeomorphism or diffeomorphism respectively). Let $\widehat{\Phi} : L' \to L$ be defined as above. Then there exists a homotopy self-equivalence g of $L \times T^k$ satisfying properties (i)–(iii) above. By the Main Claim we can choose g so that it represents the trivial element of the structure set $\mathcal{S}^s_{\mathrm{CAT}}(L \times T^k)$, where CAT is Top, PL or Diff depending upon the category in which we are working. Since g and $\widehat{\Phi}^{-1} \times \mathrm{id}$ determine the same element of the structure set it follows that $\widehat{\Phi}^{-1} \times \mathrm{id}$ also represents the trivial element of the structure set. If we choose $\mathbf{x} \in L^h_{n+1}(\pi)$ so that $[\widehat{\Phi}^{-1}] \in \mathcal{S}^h_{\mathrm{CAT}}(L)$ comes from the action of $\mathbf{x}$ on the "identity element" of the structure set $\mathcal{S}^h_{\mathrm{CAT}}(L)$, it follows that the image $P_k(\mathbf{x})$ of $\mathbf{x}$ in $L^s_{n+1+k}(\pi \times \mathbb{Z}^k)$ acts trivially on the "identity element" in the structure set $\mathcal{S}^s_{\mathrm{CAT}}(L \times T^k)$. By the surgery exact sequence this in turn implies that $P_k(\mathbf{x})$ lies in the image of the surgery obstruction homomorphism

$$\Theta_k : [\Sigma(L \times T^k), G/\,\mathrm{Top}\,] \longrightarrow L^s_{n+1+k}(\pi \times \mathbb{Z}^k)\ .$$

We need to show that $P_k(\mathbf{x}) = 0$.

In the previous section we described a generalized Shaneson splitting of the codomain of Θ_k:

$$L^s_{n+k+1}(\pi \times \mathbb{Z}^k) \approx \sum_{\alpha} L^{\langle 2-|\alpha|\rangle}_{n+k+1-|\alpha|}(\pi)\ .$$

Furthermore, the observations of the preceding section, the commutativity of the group operation on $[\Sigma(L \times T^k), G/\,\mathrm{Top}\,]$, the homotopy splitting of $\Sigma(L \times T^k)$ as a wedge

$$\Sigma(L \wedge T^k) \vee \Sigma(L) \vee \Sigma(T^k)$$

and standard adjoint functor relationships also yield a parallel splitting of the domain of

$$\Theta_k : [\Sigma(L \times T^k), G/\,\mathrm{Top}\,] \approx$$

$$\sum_{\beta} [\Sigma(L \wedge S^{|\beta|+1}), G/\,\mathrm{Top}\,] \approx \sum_{\alpha} [\Sigma(L \wedge S^{k+1-|\alpha|}), G/\,\mathrm{Top}\,]$$

where the last isomorphism represents the change of variable $\beta = \{1, ..., k\} - \alpha$.

We claim that *for each α, the surgery obstruction homomorphism Θ_k maps the summand $[\Sigma(L \wedge S^{k+1-|\alpha|}), G/\,\mathrm{Top}\,]$ in the domain to the summand $L^{\langle 2-|\alpha|\rangle}_{n+k+1-|\alpha|}(\pi)$ in the codomain.*

To prove this, consider the surgery obstruction map on the summand

$$[\Sigma(L \wedge S^{k+1-|\alpha|}), G/\,\mathrm{Top}\,]\ .$$

An element of this group corresponds to a degree 1 normal map into $S^1 \times N \times T^\beta$ whose surgery obstruction lies in $L^s_{n+k+1-|\alpha|}(\pi)$, where as before α and β are complementary subsets of $\{1, ..., k\}$, and the latter surgery obstruction group maps into $L^s_{n+k+1}(\pi \times \mathbb{Z}^k)$

geometrically by taking products with T^α. On the other hand, the map of homotopy groups from $[\Sigma(L\wedge S^{k+1-|\alpha|}), G/\operatorname{Top}]$ to $[\Sigma(L\times T^k), G/\operatorname{Top}]$ is induced by the projection from T^k to $T^{|\beta|}$, and geometrically this corresponds to taking the product of the original normal map into $S^1\times N\times T^{|\beta|}$ with the identity on $T^{|\alpha|}$. In particular, this implies that the map Θ_k is consistent with the splittings of the domain and codomain.

The assertion implies that $P_k(\mathbf{x})$ is the image of the surgery obstruction for some normal map $\mathbf{G}$ under the composite

$$[\Sigma L, G/\operatorname{Top}] \longrightarrow L^s_{n+1}(\pi) \longrightarrow L^{\langle 2-k\rangle}_{n+1}(\pi_1(N^n)))$$

and this is precisely the conclusion of Theorem 3·1.

3·2. *Proof of the Main Claim*

The crucial points are that finite toral unfurlings of g also satisfy properties (i)–(ii) (and also property (iii) if applicable), and one can simplify both g and the Wall group element α by taking such unfurlings.

Let $F_\pi(\widetilde{L})$ be the topological monoid of all homotopy self-equivalences of $\widetilde{L}\sim S^n$ that are π-equivariant. Then there is a principal fibration

$$F_\pi(\widetilde{L}) \to E(L)$$

whose fiber is equivalent to the space Γ or all π-equivariant maps from $\widetilde{L}$ to π^{ad}, where π acts on the latter by conjugation (compare [**19**] or [**41**]). Since π is discrete and $\widetilde{L}$ is connected, this reduces to the space of all continuous maps from L to the center of π, so that Γ is in fact equivalent to the center $\mathbf{C}(\pi)$ of π. This means that

$$\pi_\ell(F_\pi(\widetilde{L})) \;\cong\; \pi_\ell(E(L))\ , \quad \text{for } \ell \geqslant 2$$

and for $\ell = 1$, we have a short exact sequence

$$0 \longrightarrow \pi_1(F_\pi(\widetilde{L})) \longrightarrow \pi_1(E(L)) \longrightarrow \mathbf{C}(\pi) \longrightarrow 0\ .$$

If the dimension n of L and L' is odd, then π acts orientation preservingly and there is a spectral sequence with $E^2_{pq} = H_p(L;\pi_{n+q}(\widetilde{L}))$ converging to $\pi_{p+q}(F_\pi(\widetilde{L}))$ (compare Schultz [**41**]); the spectral sequence implies that $\pi_\ell(E(L))$ is finite for $\ell\neq n$ and $\pi_n(E(L))$ is the direct sum of $\mathbb{Z}$ and a finite group.

On the other hand, if n is even, then π must be $\mathbb{Z}_2$ and act orientation reversingly. In these cases one has a similar spectral sequence for cohomology with twisted coefficients, and in this case the rational calculation is given as follows.

Proposition 3·6. *In the orientation reversing case the homotopy groups $\pi_\ell(E(L))$ are finite for $\ell\neq 4n-1$, and $\pi_{4n-1}(E(L))$ is the direct sum of $\mathbb{Z}$ and a finite group.*

Proof. In this case the E^2 term of the spectral sequence has the form

$$E^2_{s,t} \approx \mathcal{H}_s\left(\mathbb{RP}^{2n};\pi_{2n+t}(S^{2n})\right)$$

where the twisting of the coefficients by $\mathbb{Z}_2$ is given by the involution $-(-1)_*$ on $\pi_\ell(S^{2n})$; this involution is the identity in the stable range but not in general because right composition in homotopy groups is not necessarily bilinear. The only homotopy groups of the sphere S^{2n} with infinite order are the groups $\pi_{2n}(S^{2n})$ and $\pi_{2n}(S^{4n-1})$, each of which is finitely generated of rank one. The torsion free parts of these groups are generated by the identity ι and its self Whitehead product $[\iota,\iota]$ respectively. The automorphisms $(-1)_*$ on these elements send them to $-\iota$ and $[\iota,\iota]$ respectively (because the Whitehead product is bilinear). Therefore the coefficient twisting involution $-(-1)_*$ sends the first class to itself and the second to minus itself. Consider now the corresponding chain complexes for computing cohomology that come from the standard CW decomposition of a real projective space with one cell in each dimension:

$$E^1_{*,t} \approx \mathcal{C}_* \left(\mathbb{RP}^{2n}; \pi_{2n+t}(S^{2n})\right) \ .$$

These complexes consist entirely of finite groups unless $t = 0$ or $t = 2n-1$, and therefore $E^2_{s,t}$ is finite except perhaps for these values of t. If $t = 0$ then the action of $\mathbb{Z}_2$ on $\pi_{2n}(S^{2n})$ is trivial, and the finiteness of $E^2_{s,0}$ for $s > 0$ follows from the finiteness of the reduced homology of $\mathbb{RP}^{2n}$. On the other hand, if $t = 2n-1$ then the action of $\mathbb{Z}_2$ on $\pi_{4n-1}(S^{2n})$ modulo torsion is nontrivial, and accordingly the groups $E^2_{s,2n-1}$ are isomorphic modulo torsion to the twisted homology groups $\mathcal{H}_s(\mathbb{RP}^{2n}; \mathbb{Z}^-)$, which by duality are the untwisted cohomology groups $H^{2n-s}(\mathbb{RP}^{2n}; \mathbb{Z})$. The latter are finite unless $s = 2n$, in which case they are infinite cyclic. From the structure of the spectral sequence it then follows that $E^\infty_{s,t}$ is finite in positive total degrees unless $(s,t) = (2n-1, 2n)$, in which case the E^∞ term is finitely generated of rank one. □

Remark. It is well known that the stabilization map from $\pi_{4n-1}(SO_m)$ to $\pi_{4n-1}(SO) \approx \mathbb{Z}$ is nontrivial if $m \geqslant 2n+1$ and trivial if $m \leqslant 2n$ (in fact, by the Barratt–Mahowald Theorem the map is onto for $m \geqslant 2n+1$ and all but finitely many values of n [**4**]). The results of [**41**] show that a class of infinite order in $\pi_{4n-1}(SO_{2n+1})$ maps to a class of infinite order in $\pi_{4n-1}(F_{\mathbb{Z}_2}(S^{2n}))$.

The next result goes a long way towards establishing the main result.

PROPOSITION 3·7. *If there is a simple homotopy self-equivalence $g : L \times T^k \to L \times T^k$ with properties* (i)–(iii) *as above, and a factorization of g as in Equation* 2·1 *then there is also one that has the same properties but has the form $\varphi_j(v_j)$ for some $v_j \in F_j/F_{j+1}$. If n is odd then we can take $j = n$, and if n is even then we can take $j = 2n-1$.*

In particular, we can choose g to be homotopic to the identity if $k < n$ when n is odd or $n < 2n-1$ when n is even; in these cases Claim 3·5 follows immediately.

Proof. We start with the factorization of u as in Claim 2·1:

$$\varphi_i(u_i)\varphi_{i+1}(u_{i+1}) \ \ldots \ \varphi_k(u_k)\cdot\varphi_1(u_1)$$

We have already noted that $\pi_j(E(L))$ is finite for all but one value j_0 of j, which is given in the statement of the proposition, and accordingly we choose $i = j_0$ in the factorization above. Take $r > 0$ to be the products of the orders of the groups $\pi_j(E(L))$ for all $j \neq j_0$ between 1 and k, and let ψ^r be the rth power map on T^k.

If g satisfies properties (i)–(iii), then so do its finite unfurlings. Consider the particular case $g^{\langle r\rangle}$ where one unfurls using the rth power map ψ^r on T^k. If the coadjoint of g corresponds to $u \in [T^k, E(L)]$, then the coadjoint of $g^{\langle r\rangle}$ corresponds to $v = u \circ [\psi^r]$. By the choice of r and Proposition 2·3 we know that the factors $\varphi_j(r^j u_j)$ are trivial for $j \neq j_0$ and therefore v can be written simply in the form $\varphi_i(v_i)$ for $i = j_0$, as asserted in the proposition. □

To conclude the proof of the Main Claim, it suffices to show that some finite unfurling of an adjoint to v determines the trivial element in the structure set. The class v_i is expressible as a product $\prod v_\alpha$ under the isomorphism

$$F_i/F_{i+1} \approx \prod_{|\alpha|=i} \pi_{|\alpha|}(E(L))$$

and it suffices to show that every one of the factors v_α has an adjoint for which some finite unfurling determines the trivial element in the structure set.

Given α as above, let $\mathbf{CO}(\alpha)$ denote its complement in $\{1, ..., k\}$. By construction, the homotopy group $\pi_{|\alpha|}(E(L))$ maps to $\mathcal{S}^s_{\mathrm{Top}}(T^k \times L)$ as follows: first of all, one sends $\pi_{|\alpha|}(E(L))$ to the relative structure group

$$\mathcal{S}^s_{\mathrm{Top}}(D^{|\alpha|} \times L, \partial)$$

by taking the associated homotopy self-equivalences of $D^{|\alpha|} \times L$ that are the identity on the boundary, then one maps to $\mathcal{S}^s_{\mathrm{Top}}(T^\alpha \times L)$ by gluing on a copy of $T^\alpha - \mathrm{Int} D^{|\alpha|}$ along the boundary, and finally one maps to $\mathcal{S}^s_{\mathrm{Top}}(T^k \times L)$ by taking products with $T^{\mathbf{CO}(\alpha)}$ and reshuffling the circle factors. In view of this it suffices to show that the intial homomorphism

$$\sigma : \pi_{|\alpha|}(E(L)) \longrightarrow \mathcal{S}^s_{\mathrm{Top}}(D^{|\alpha|} \times L, \partial)$$

has a finite image. Since $|\alpha| \geqslant 2$ all groups in sight are abelian

and thus we can tensor with the rationals if necessary.

Suppose first that π acts orientation preservingly, so that the dimension n is even and $i = n$. We claim that the composite of σ with the normal invariant map is rationally trivial. But

$$[D^{|\alpha|} \times L/\partial, G/\mathrm{Top}] \otimes \mathbb{Q} \approx 0$$

because $i + n = 2n$ is congruent to 2 mod 4. On the other hand, the Wall group $L^s_{2n+1}(\pi)$ is also finite, so this forces the image of σ to be finite. In particular this means that we can again use a finite torus unfurling to kill the image of the α component in the structure set for $T^k \times L$. Since α was an arbitrary subset satisfying $|\alpha| = i$, it follows that we can kill the image of the entire homotopy class in this fashion. This completes the proof of the Main Claim if π acts orientation preservingly.

Assume now that $\pi \approx \mathbb{Z}_2$ acts orientation reversingly. In this case n is even and $i = 2n - 1$; it follows that the group of normal invariants

$$[D^{2n-1} \times \mathbb{RP}^n/\partial, G/\mathrm{Top}]$$

is again finite. On the other hand, all of the Wall groups $L^x_*(\mathbb{Z}_2^-)$ for $\mathbb{Z}_2$ with the nontrivial orientation map are finite, so the finiteness of the image of σ follows in this case too. This completes the proof of the Main Claim.

3·3. *Refinements of the main results*

We shall now state some sharpened forms of Theorem 1·1.

THEOREM 3·8. *Let M, N be homotopy spherical spaceforms such that $M \times T^k \approx N \times T^k$, for some $k \geqslant 1$. Then $M \times T^3 \approx N \times T^3$.*

THEOREM 3·9. *Let L, L' be homotopy lens spaces with $L \times T^k \approx L' \times T^k$, for some $k \geqslant 1$. Then $L \times S^1$ and $L' \times S^1$ are h-cobordant; in particular $L \times T^2 \approx L' \times T^2$.*

THEOREM 3·10. *Let M, N be homotopy quaternionic spaceforms with fundamental groups isomorphic to $Q[2^n]$ with $n \geqslant 3$. where $Q[2^n]$ denotes the quaternionic group of order 2^n. Then $M \times T^k \approx N \times T^k$ for some $k \geqslant 1$ is equivalent to M and N being h-cobordant; in particular $M \times S^1 \approx N \times S^1$.*

Proof of Theorem 3·8. Let M, N be n-dimensional homotopy spherical spaceforms with $M \times T^k \approx N \times T^k$ for some $k \geqslant 4$ (the cases with $k \leqslant 3$ are trivial). This implies the existence of a homotopy equivalence $f\colon M \to N$ which is normally cobordant to the identity such that if $\mathbf{x}$ is the associated surgery obstruction for the normal cobordism in $L^h_{n+1}(\pi)$, then there is a relative degree 1 normal map $\mathbf{G}$ into $N \times I$ whose surgery obstruction $\sigma(\mathbf{G})$ maps to the same element in $L^{(2-k)}_{n+1}(\pi)$ as $\mathbf{x}$. In other words, the image of $\mathbf{x} - \sigma(\mathbf{G})$ in the lower L-group is trivial. The vanishing of the lower algebraic K-groups $K_{-i}(\pi)$ for $i \geqslant 2$ (compare Carter [**8**]) and the Rothenberg exact sequence for lower L-groups ([**38**, theorem 17·2]) then imply the vanishing of $\mathbf{x} - \sigma(\mathbf{G})$ in $L^{\langle 2-r\rangle}_{n+1}(\pi)$ for all $r \geqslant 3$.

Geometrically this vanishing condition means that $f \times \mathrm{id}_{T^3}$ represents the trivial element in the structure set $\mathcal{S}^s_{\mathrm{Top}}(N \times T^3)$ and hence $f \times \mathrm{id}_{T^3} : M \times T^3 \to N \times T^3$ must be homotopic to a homeomorphism.

Proof of Theorems 3·9 *and* 3·10. Let L, L' be homotopy lens spaces of dimension $2n-1 \geqslant 3$. Assume $L \times T^k \approx L' \times T^k$ for some $k \geqslant 1$. It follows that there is a homotopy equivalence $f : L' \to L$ with trivial normal invariant in $[L, G/\mathrm{Top}]$. This implies that $[f] \in \mathcal{S}_{\mathrm{Top}}(L)$ is obtained from an action of $L^h_{2n}(\mathbb{Z}_t)$ on $\mathcal{S}_{\mathrm{Top}}(L)$; i.e., $[f] = \gamma(\alpha)$ for some $\alpha \in L^h_{2n}(\mathbb{Z}_t)$. The proof of 3·1 has shown that *$\alpha \in L^h_{2n}(\mathbb{Z}_t)$ cannot be an element of infinite order*. Let $F\colon (W; \partial_0 W, \partial_1 W) \to (L \times I; L \times \{0\}, L \times \{1\})$ be the normal cobordism realizing $\alpha \in L^h_{2n}(\mathbb{Z}_t)$. Since $\alpha \in \mathrm{T{\scriptsize ORSION}} L^h_{2n}(\mathbb{Z}_t)$, then it follows that $\rho(L) = \rho(L')$, because the free part of $L^h_{2n}(\mathbb{Z}_t)$ is given by the multisignature (compare Bak [**3**, theorem 1] and Hambleton-Milgram [**14**]). Consider the normal cobordism

$$F \times \mathrm{id}_{S^1} : W \times S^1 \to (L \times I) \times S^1$$

and its surgery obstruction $\Theta(F \times \mathrm{id}_{S^1}) \in L^h_{2n+1}(\mathbb{Z}_t \times \mathbb{Z})$. The latter group is isomorphic to $0 \oplus L^p_{2n}(\mathbb{Z}_t)$ if n is even and $\mathbb{Z}_2 \oplus L^p_{2n}(\mathbb{Z}_t)$ if n is odd. The group $L^p_{2n}(\mathbb{Z}_t)$ is finitely generated free abelian if n is even and is the direct sum of a finitely generated free abelian group and $\mathbb{Z}_2$ if n is odd (compare Bak [**3**, theorem 1]).

Elementary considerations imply that $\Theta(F \times \mathrm{id}_{S^1})$ is an element of infinite order. In particular, $\Theta(F \times \mathrm{id}_{S^1})$ is normally cobordant to a homotopy equivalence; i.e., there is an h-cobordism $\overline{W}$ between $L \times S^1$ and $L' \times S^1$. This in turn implies that $\overline{W} \times S^1$ is a product and hence $L \times T^2 \approx L' \times T^2$. This completes the proof of Theorem 3·9.

For Theorem 3·10 we observe that $L^h_0(Q[2^k])$ is torsion free detected by the multisignature (compare Hambleton-Milgram [**14**]). This implies that the corresponding homotopy equivalence $f : M \to N$ is in fact h-cobordant to the identity and hence $M \times S^1 \approx N \times S^1$.

Remark. It turns out that the stabilization by T^2 in Theorem 3·9 is the best possible; i.e., stabilization by S^1 will not be enough in general. To see this, one can appeal to a classification of fake lens spaces with odd order fundamental groups due to Browder, Petrie and Wall (compare [**45**]). More precisely, by using [**45**, theorem 14E.7], one can show the existence of fake lens spaces L, L' with $\rho(L) = \rho(L')$ but with Reidemeister torsions $\Delta(L)$, $\Delta(L')$ not related by an h-cobordism (i.e., $\Delta(L) \not\sim u^2 \Delta(L')$ where u is a unit in $\mathbb{Z}[\mathbb{Z}_t]$). This choice of $\Delta(-)$ will guarantee $L \times S^1 \not\approx L' \times S^1$ (otherwise L and L' would be h-cobordant). On the other hand, the equality $\rho(L) = \rho(L')$ will give $L \times T^2 \approx L' \times T^2$ analogously as in the proof of Theorem 3·9. Also, the realization of Whitehead torsion by h-cobordisms implies the stabilization by S^1 in Theorem 3·10 is the best possible as well. The comparison of results in [**24**] with Theorem 3·10 leads to a complete classification of quaternionic 2-group spaceforms with respect to toral stabilization:

PROPOSITION 3·11. *There are precisely*

$$2^{2^{n-2}-n+1}$$

different quaternionic homotopy spherical spaceforms M_i with $\pi_1(M_i) \approx Q[2^n]$, $n \geqslant 3$ and $M_i \times T^k \approx M_j \times T^k$, for each i, j, $k \geqslant 1$.

4. *Stabilization by Euclidean spaces*

In this section we consider stabilization by Euclidean spaces $\mathbb{R}^n$, beginning with proofs of Theorems 1·4 and 1·5. The second of these follows from the stable equivalence theorem of Mazur (see [**32**]; compare also Siebenmann [**44**, theorem 2·3]), and the first is an immediate consequence of the toral stability property for lens spaces and the following result:

PROPOSITION 4·1. *Let M and N be closed (connected) topological manifolds of dimension $\geqslant 3$ such that $M \times \mathbb{R}^2$ and $N \times \mathbb{R}^2$ are homeomorphic. Then $M \times T^2$ and $N \times T^2$ are homeomorphic. Similar results hold in the piecewise linear and smooth categories.*

Proof. The argument is similar to that of [**32**], so we shall concentrate on the differences. If $r > 0$ is a real number it will be convenient to let $D^2(r)$ and $S^1(r)$ denote the disk and circle of radius r in $\mathbb{R}^2$. It will also be convenient to identify $\mathbb{R}^2 - \text{Int}\, D^2(r)$ with $S^1 \times [r, \infty)$ by the standard homeomorphism sending z to $\left(|z|^{-1}z, |z|\right)$.

Suppose that $f : M \times \mathbb{R}^2 \to N \times \mathbb{R}^2$ is a homeomorphism. Then one can find real numbers a and b such that $1 < a < b$ and

$$N \times D^2 \subset \text{Int}\, f\left(M \times D^2(a)\right) \subset f\left(M \times D^2(a)\right) \subset \text{Int}\, N \times D^2(b)\ .$$

Define W and V to be the cobordisms

$$W = f\left(M \times D^2(a)\right) - \text{Int}\, N \times D^2$$

$$V = N \times D^2(b) - \text{Int}\, f\left(M \times D^2(a)\right)$$

so that $\partial W = f\left(M \times S^1(a)\right) \sqcup N \times S^1$ and $\partial V = N \times S^1(b) \sqcup f\left(M \times S^1(a)\right)$; in each case we shall use ∂_+ and ∂_- to denote the first and second listed boundary components (if M and N are orientable this corresponds to the orientations inherited by the respective boundary components).

As in [**32**], by excision and homotopy invariance we know that the pairs $(W, \partial_- W)$ and $(V, \partial_- V)$ are homologically acyclic, and in fact this holds for arbitrary local coefficients in the group ring $\mathbb{Z}[\pi]$, where $\pi = \pi_1(N)$. The local coefficients arise from the free action of π on the pullback of the universal covering $\widetilde{N} \times \mathbb{R}^2 \to N \times \mathbb{R}^2$ to a subset of $N \times \mathbb{R}^2$. By duality it follows that $(W, \partial_+ W)$ and $(V, \partial_+ V)$ are also homologically acyclic for arbitrary local coefficients in the group ring $\mathbb{Z}[\pi]$.

In complete analogy with [**32**], the goal is to show that W is an h-cobordism; if we know this then we can use the identity $W \times S^1 \approx \partial_- W \times I \times S^1$ to conclude that the ends of $W \times S^1$ — which are homeomorphic to $M \times T^2$ and $N \times T^2$ respectively — are themselves homeomorphic. This is the point at which one must look more carefully at the construction in [**32**].

Let $X = N \times S^1 \times [b, \infty) \subset N \times \mathbb{R}^2$. It follows that $V \cup X \approx \partial_+ W \times [a, \infty)$ so that the inclusion of W in $W \cup V \cup X$ is a homotopy equivalence. But we also know that $W \cup V \cup X$ is equal to $N \times S^1 \times [1, \infty)$, so that the inclusion of $\partial_- W$ in $W \cup V \cup X$ is also a homotopy equivalence, and a diagram chase then shows that the inclusion $\partial_- W \subset W$ is a homotopy equivalence. Therefore the proof reduces to verifying that the inclusion $\partial_+ W \subset W$ is also a homotopy equivalence.

Let f_0 be the homeomorphism from $M \times S^1 \approx M \times S^1(a)$ to $\partial_+ W$ determined by f, and let J_+ denote the inclusion $\partial_+ W \subset W$. Then we have the following commutative diagram:

$$\begin{tikzcd}
M \times S^1 \arrow[r, "J_+ \circ f_0"] \arrow[d, "f_0"'] & W \arrow[d, "\text{inclusion}"] \\
\partial_+ W \arrow[r, "\text{inclusion}"] \arrow[d, "\text{inclusion}"] & N \times \left(\mathbb{R}^2 - \{0\}\right) \arrow[d, "\text{inclusion}"] \\
M \times \mathbb{R}^2 \arrow[r, "f"] & N \times \mathbb{R}^2
\end{tikzcd}$$

It follows that all the maps in the diagram induce isomorphisms of homotopy groups in dimensions $\geqslant 2$ and in dimension 1 there is the following commutative diagram:

$$\begin{tikzcd}
\pi_1(M) \times \mathbb{Z} \arrow[r, "(J_+ \circ f_0)_*"] \arrow[d, "(f_0)_*"'] & \pi_1(W) \arrow[d, "(J_2)_*"] \\
\pi_1(\partial_+ W) \arrow[r, "(J_1)_*"] \arrow[d, "(J_3)_*"] & \pi_1(N) \times \mathbb{Z} \arrow[d, "(J_4)_*"] \\
\pi_1(M) \arrow[r, "f_*"] & \pi_1(N)
\end{tikzcd}$$

Many of the maps in this diagram are already known to be isomorphisms; specifically, f_* and $(f_0)_*$ are isomorphisms because they are homeomorphisms, while $(J_2)_*$ is an isomorphism because J_2 is a homotopy equivalence (this inclusion map merely attaches an open collar to each boundary component). Furthermore, the composites $(J_3)_* \circ (f_0)_*$ and $(J_4)_*$ are simply projection onto the respective first factors. In order to complete the verification that J_+ is a homotopy equivalence, we need to show that the composite $\eta = (J_2)_* \circ (J_+ \circ f_0)_*$ is an isomorphism. Algebraically we have the situation summarized by the following commutative diagram:

$$\begin{array}{ccc} \pi_1(M) \times \mathbb{Z} & \xrightarrow{\eta} & \pi_1(N) \times \mathbb{Z} \\ \downarrow \text{proj} & & \downarrow \text{proj} \\ \pi_1(M) & \xrightarrow[\approx]{f_*} & \pi_1(N) \end{array}$$

This means that η sends $\{1\} \times \mathbb{Z} \subset \pi_1(M) \times \mathbb{Z}$ into $\{1\} \times \mathbb{Z} \subset \pi_1(N) \times \mathbb{Z}$, and the proof of the result amounts to showing that this map of infinite cyclic groups is an isomorphism. The latter in turn can be studied by looking at the induced homomorphism of abelianizations for the domain and codomain of η. However, this map of abelianizations is equivalent to the map on one-dimensional integral homology induced by J_+, and we know that this homology map is an isomorphism. This completes the proof that J_+ is a homotopy equivalence and W is an h-cobordism. □

Although the results of Folkman [**11**] give explicit conditions for two lens spaces to be tangentially homotopy equivalent, one cannot write down a general result as neatly as in the classification up to ordinary or simple homotopy equivalence, and one quickly encounters intricate number-theoretic considerations (for example, consider the results of Ewing, Moolgavkar, Smith and Stong [**10**] on which lens spaces have stably trivial tangent bundles). However, it is possible to construct systematic examples of lens space pairs that are tangentially homotopy equivalent but not homeomorphic, and of course the second part of Theorem C applies to such pairs.

PROPOSITION 4·2. *Let $\pi \approx \mathbb{Z}_{4q}$, where $q \geqslant 3$ is odd. Let a be an integer such that $a \equiv 1 (\mathrm{mod}\ 4)$ and $a \equiv -1 (\mathrm{mod}\ q)$. Consider two lens spaces of the same dimension $n \geqslant 5$ given by:*

$$L = L(4q; a, a, 1, \ldots 1), \ L' = L(4q; 1, \ldots, 1) \ .$$

Then L and L' are not diffeomorphic but $L \times \mathbb{R}^k$ is diffeomorphic to $L' \times \mathbb{R}^k$ for $k > n$.

Proof. It follows from [**11**, p. 27] that $L \not\approx L'$ but L and L' are tangentially homotopy equivalent. □

4·1. *Desuspending stable equivalences*

The preceding examples show that one can have pairs of lens spaces L, L' such that $L \times \mathbb{R}^m \approx L' \times \mathbb{R}^m$ for all sufficiently large m but not for $m \leqslant 2$. It is natural to ask for the smallest value of m such that $L \times \mathbb{R}^m \approx L' \times \mathbb{R}^m$. In general this is a difficult problem, but for the examples of Proposition 4·2 one already has diffeomorphisms when $m = 3$.

PROPOSITION 4·3. *In every dimension $\geqslant 5$, the nonhomeomorphic, tangentially homotopy equivalent lens spaces L and L' in 4·2 satisfy $L \times \mathbb{R}^m \approx L' \times \mathbb{R}^m$ for each $m \geqslant 3$.*

By the previously mentioned results of Milnor, the minimum value of m for 3-dimensional lens spaces is also 3.

In order to prove Proposition 4·3 we shall need a little more information on the tangential homotopy equivalence between the lens spaces in the examples. The first step is to choose polarizations of the lens spaces (see [**25**] for background); these are 2-connected maps into $K(\pi,1)$ that we may as well assume are basepoint preserving. Canonical choices for these polarizations are given by viewing the lens spaces as orbit spaces of the free orthogonal actions on spheres described in Proposition 4·2.

There is a corresponding notion of a polarization preserving map that will be useful for our purposes. Given two pointed spaces X and Y with the same fundamental group π (not necessarily isomorphic to $\mathbb{Z}_{4q}$ as above) and preferred polarizations c_X and c_Y, we say that a basepoint preserving homotopy equivalence $f : X \to Y$ is polarization preserving if $f \circ c_Y$ is homotopic to c_X. If X and Y are both presented as orbit spaces of free π-actions on, say, S and T respectively and one chooses the canonical polarizations associated to the free π-actions, then f is polarization preserving if and only if its lifting $\widetilde{f}: S \to T$ is π-equivariant.

It will also be helpful to adopt some standard notation for linear representations of cyclic groups. If we express the common dimension of the lens spaces as $n = 2k+1$, then the characters of these representations are $(k+1)t$ and $2t^a + (k-1)t$; we shall use these characters to denote the representations when necessary.

PROPOSITION 4·4. *Let L and L' be as in Proposition 4·2 and for each prime p dividing $4q$ let L_p and L'_p be the coverings associated to the Sylow p-subgroup π_p of π. Then there is a polarization preserving tangential homotopy equivalence $h : L' \to L$ such that for each p dividing $|\pi| = 4q$ the lifted homotopy equivalence $h_p : L'_p \to L_p$ is homotopic to a diffeomorphism.*

Proof. It suffices to find a polarization preserving homotopy equivalence with the desired lifting property; in other words, such a map will automatically be tangential. To see this, note that the reduced real K-groups of L and L' are both finite and thus detected by passage to all the Sylow subgroup coverings. This means that h is tangential if and only if each h_p is tangential. But diffeomorphisms, and maps homotopic to diffeomorphisms, are always tangential, so h will be a tangential homotopy equivalence if each h_p is homotopic to a diffeomorphism.

Since a^2 is congruent to 1 mod $4q = |\pi|$, it follows that there is an equivariant degree one map $\widetilde{h}$ from the unit sphere $S\left((k+1)t\right)$ of $(k+1)t$ to the corresponding unit sphere $S\left(2t^a + (k-1)t\right)$. The induced map h of orbit spaces is a degree one polarization preserving homotopy equivalence from L' to L. We need to verify that each lifting h_p is homotopic to a diffeomorphism. It will suffice to show that h_2 and the lifting h_q to the characteristic subgroup $\mathbb{Z}_q \subset \mathbb{Z}_{4q} = \pi$ are both homotopic to diffeomorphisms.

By construction $a \equiv 1(\mathrm{mod}\ 4)$, and therefore h_2 is a polarization preserving homotopy equivalence from L_2 to itself of degree one; it follows that h_2 is homotopic to the identity and thus is homotopic to a diffeomorphism. However, since $a \equiv -1(\mathrm{mod}\ q)$ it follows that h_q is a polarization preserving homotopy equivalence from the lens space $L\left((k+1)t\right)$ to the corresponding lens space $L\left(2t^a + (k-1)t\right)$ whose degree is equal to one, and h_q will be homotopic to a diffeomorphism if the lifting $\widetilde{h}$ to the universal covering spheres is $\mathbb{Z}_q$-equivariantly homotopic to a diffeomorphism. Since equivariant maps between spheres with free actions of a finite group π are classified up to equivariant homotopy by their degrees, it will suffice to find an orientation preserving equivariant diffeomorphism from the $\mathbb{Z}_q$-linear sphere $S\left((k+1)t\right)$ to the corresponding $\mathbb{Z}_q$-linear sphere $S\left(2t^{-1} + (k-1)t\right)$; we can replace a by -1 because we are restricting the group actions to $\mathbb{Z}_q$. An explicit choice of such an equivariant diffeomorphism is given by the map f

sending $(z_1, z_2, z_3, ...) \in S^{2k+1} \subset \mathbf{C}^k$ to $(\overline{z_1}, \overline{z_2}, z_3, ...)$. It follows that h_q is homotopic to a diffeomorphism and that the same also holds for h. $\square$

COROLLARY 4·5. *If h is the map in Proposition* 4·4*, then h is normally cobordant to the identity.*

COROLLARY 4·6. *If h is the map in Proposition* 4·4*, then $h \times \mathrm{id}\,(\mathbb{R}^3)$ is homotopic to a diffeomorphism.*

Proof. The second corollary follows from the first corollary and the $\pi - \pi$ theorem in nonsimply connected surgery. To prove the first corollary, observe that group of normal invariants $[L, G/O]$ is detected by passage to covering spaces corresponding to Sylow p-subgroup coverings (for example, this holds because G/O is an infinite loop space; compare [**23**]). Therefore h will be normally cobordant to the identity if the same is true for every lifting h_p. But the latter are homotopic to diffeomorphisms and thus automatically normally cobordant to the identity (in fact, they represent the neutral elements in the corresponding structure sets). It follows that h itself is normally cobordant to the identity as desired. $\square$

4·2. *Minimal stabilizations in general*

Given any two tangentially homotopy equivalent lens spaces L and L' it is natural to ask for the smallest value of m such that $L \times \mathbb{R}^m \approx L' \times \mathbb{R}^m$. There are no known examples where m is greater than 3, and the argument proving Corollary 4·6 shows that the least value is always 3 if the answer to the following question is affirmative.

Question. *If $f : L' \to L$ is a tangential homotopy equivalence, is f normally cobordant to the identity?*

It should be possible to answer such questions by methods resembling those of [**23**], but the necessary calculations could turn out to be quite formidable.

Of course, the same question about finding the smallest value of m can be also be formulated for homotopy lens spaces in the smooth, piecewise linear and topological categories. We shall merely state one result that reflects the relative complexity of this problem.

PROPOSITION 4·7. *Let L be an lens space of dimension $8k+1 \geqslant 9$ whose fundamental group has odd order. Then there is a smooth homotopy lens space L' such that L' is almost diffeomorphic to L (i.e., there is a homeomorphism that is a diffeomorphism on the complement of one point) but $L \times \mathbb{R}^3$ is not diffeomorphic to $L' \times \mathbb{R}^3$.*

We hope to discuss such results further in a subsequent paper.

5. *Exponential stabilization in higher dimensions*

We shall prove Theorem 1·2 first in the topological category for all dimensions $\geqslant 4$ and then indicate the purely formal modifications needed to prove the result in the smooth category for all dimensions $\geqslant 5$.

Given the lens space L with fundamental group π, let $\alpha \in L^s_4(\pi)$ have a ρ-invariant that cannot be expressed as $\rho(M) - \rho(L)$ for some lens space M and such that α lies in the image of the transfer map $L^s_4(\pi^*) \to L^s_4(\pi)$ where π^* is cyclic and $|\pi^*| = 2|\pi|$. Choose a lens space L^* such that L is a double covering of L^*, and let $\alpha^* \in L^s_4(\pi^*)$ be a class mapping to α under the transfer. Observe that α^* also is not a difference $\rho(M^*) - \rho(L^*)$ for some lens space M^*.

By the 3-dimensional version of the surgery sequence (compare Jahren–Kwasik [**18**]) there is a 4-dimensional normal cobordism $(F^* : W^* \to L^*)$ with surgery obstruction α^* whose restriction to $\partial_0 W^*$ is the identity and whose restriction to $\partial_1 W^*$ is a simple $\mathbb{Z}[\pi^*]$-homology equivalence $f^* : M^* \to L^*$. It follows immediately that the product map $f^* \times \mathrm{id}_{L^*}$ is normally cobordant to the identity and, in fact, the results and methods of [**25**] show that one can choose the normal cobordism of the covering map $f \times \mathrm{id}_{L^*}$ to be a simple homotopy equivalence; it might not be an s-cobordism because the map of fundamental groups associated to f is not known to be bijective (see the note at the end of the proof for further information).

Although f is not necessarily a simple homotopy equivalence, one can perform topological surgery on $1 \times f : S^1 \times M \to S^1 \times L$ to obtain a simple homotopy equivalence $f_1 : X \to L \times S^1$, and in fact one can do this so that the normal cobordism from $f \times 1$ to f_1 has zero surgery obstruction. Consider now the product map $f_1 \times \mathrm{id}_L : X \times L \to S^1 \times L \times L$. This map is normally cobordant to $\mathrm{id}_{S^1} \times f \times \mathrm{id}_L$ by a normal cobordism with vanishing surgery obstruction, and the latter map is in turn normally cobordant to the identity by a cobordism with vanishing surgery obstruction. Stacking the first normal cobordism on top of the second, we obtain an s-cobordism from $X \times L$ to $S^1 \times L \times L$.

The preceding shows that $X \times L \times S^1$ is homeomorphic to $T^2 \times L \times L$; since the higher ρ-invariants satisfy $\widetilde{\rho}(X) - \widetilde{\rho}(S^1 \times L) = \alpha$, it follows that X and $S^1 \times L$ are not homeomorphic; in fact, since there are infinitely many choices for α, there are infinitely many topologically distinct choices for X. However, we really want $X \times X$ to be homeomorphic to $T^2 \times L \times L$, and this requires a litle more work. Suppose now that $g : Z^m \to Y^m$ is a (simple) homotopy equivalence that is normally cobordant to the identity such that normal cobordism has surgery obstruction $\beta \in L^s_{m+1}(\pi_1(Y))$. Then the previous considerations and the machinery of Ranicki [**37**] imply that $g \times g$ is normally cobordant to the identity by a normal cobordism whose surgery obstruction is $\beta \times \sigma^*(Y) + \sigma^*(Y) \times \beta \in L^s_{2m+1}(\pi_1(Y) \times \pi_1(Y))$, where $\sigma^*(Y)$ is the image of Y in Ranicki's **symmetric** Wall group $L_s^{2m+1}(\pi_1(Y))$; the crucial point is that $\sigma^*(Y)$ depends only on the simple homotopy type of Y. The first summand is merely the surgery obstruction for a normal cobordism from the identity to $g \times \mathrm{id}_Y$, and the formula shows that the surgery obstruction associated to $g \times g$ will vanish if the surgery obstruction associated to $g \times \mathrm{id}_Y$ does. If we now take g to be the map f_1 constructed above, the previous paragraph implies that we can take the associated surgery obstruction β to be zero, and it follows that $f_1 \times f_2$ is normally cobordant to the identity by a normal cobordism with trivial surgery obstruction. In other words, $f_1 \times f_1$ is s-cobordant to the identity and thus homotopic to a homeomorphism. This proves the result for cartesian squares; *i.e.,* the case $m = 2$. If we can also prove the result for $m \geqslant 3$, then the general result will follow because each $m \geqslant 4$ can be written as a positive linear combination of 2 and 3. Consider the factorization

$$f_1 \times f_1 \times f_1 = (f_1 \times f_1 \times \mathrm{id}) \circ (\mathrm{id} \times \mathrm{id} \times f_1) \ .$$

Since $f_1 \times f_1$ and $\mathrm{id} \times f_1$ are both homotopic to homeomorphisms by the argument given above, it follows that both $(f_1 \times f_1 \times \mathrm{id})$ and $(\mathrm{id} \times \mathrm{id} \times f_1)$ are homotopic to homeomorphisms, which in turn implies that $f_1 \times f_1 \times f_1$ is also homotopic to a homeomorphism.

The preceding argument uses the methods of [**45**] to do topological surgery in dimensions $\geqslant 5$ along with the results of [**12**] on topological surgery in dimension 4. The former extend directly to the smooth category but the latter do not, and this is why the results for smooth manifolds are only stated in dimensions $\geqslant 5$. However, if we take products with T^2 rather than S^1 in the smooth category, then we are in a situation where smooth surgery works well, and therefore in this case we obtain X' simple homotopy equivalent to $T^2 \times L$ such that $\Pi^m X'$ is diffeomorphic to $\Pi^m \left(L \times T^{n-3}\right)$ for all $m \geqslant 2$ but the higher ρ-invariants of X' and $T^2 \times L$ are distinct; in fact, one has an infinite family of such X' with pairwise distinct higher ρ-invariants. To obtain examples in dimensions $n \geqslant 6$, take the product of X' with the torus T^{n-5}.

Remark. If L is a lens space and $\alpha \in L_4^s(\pi)$ has a ρ-invariant that is not of the form $\rho(M) - \rho(L)$ for two lens spaces, then results of S. Gadgil [**13**] show that if $W \to L$ is a normal cobordism whose restriction to $\partial_0 W$ is the identity and whose surgery obstruction is α, then the natural surjection from $\pi_1(\partial_1 W)$ to $\pi_1(L)$ must have a nontrivial kernel. The preceding result could be proved far more easily if it were possible to take $\partial_1 W \to L$ to be a simple homotopy equivalence.

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