BETWEEN LOWER AND HIGHER DIMENSIONS

(in the work of Terry Lawson)

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There are several approaches to summarizing a mathematician's research accomplishments, and each has its advantages and disadvantages. This article is based upon a talk given at Tulane that was aimed at a fairly general audience, including faculty members in other areas and graduate students who had taken the usual entry level courses. As such, it is meant to be relatively nontechnical and to emphasize qualitative rather than quantitative issues; in keeping with this aim, references will be given for some standard topological notions that are not normally treated in entry level graduate courses.

Since this was an hour talk, it was also not feasible to describe every single piece of published mathematical work that Terry Lawson has ever written; in particular, some papers like [42] and [50] would require lengthy digressions that are not easily related to the central themes in his main lines of research. Instead, we shall focus on some ways in which Terry's work relates to an important thread in geometric topology; namely, the passage from studying problems in a given dimension to studying problems in the next dimensions. Qualitatively speaking, there are fairly well-developed theories for very low dimensions and for all sufficiently large dimensions, but between these ranges there are some dimensions in which the answers to many fundamental questions are extremely unclear. Much of Terry's work, and most of his best known results and papers, are directly related to such questions.

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1. Lower versus higher dimensions

Of course, the concept of *dimension* is central to many geometrical questions, and in the physical world one can have objects of dimension n for $n = 0, 1, 2, 3$. During the nineteenth century, several mathematicians recognized that the methods of coordinate geometry lead to a theory of n-dimensional geometrical objects, where n is an arbitrary nonnegative integer. In particular, the vector space structure on $\mathbb{R}^n$, including the standard inner product, provide a setting in which one can describe an n-dimensional analog of classical Euclidean plane or solid geometry. Higher dimensional objects are more than just intellectual curiosities, for they have multiple uses in many contexts, including a many areas in the mathematical sciences, several branches of physics, and even in other subjects like mathematical economics.

Many important n-dimensional geometrical objects are examples of *topological n-manifolds*; formally, these are Hausdorff topological spaces in which every point has an open neighborhood which is homeomorphic to $\mathbb{R}^n$. Objects of this sort were introduced in the middle of the nineteenth century, and as noted above they arise naturally in a wide range of topics, both within the mathematical sciences and in their applications to other fields. We shall deal mainly with topological manifolds in this article, but in some cases we must restrict attention to *differential* or *smooth* n-manifolds (see [46] or [33]), which have the additional structure needed to discuss differentiation and integration on the space.

In classical Euclidean geometry, clearly some things become more complicated when one passes from line geometry to plane geometry or from plane geometry to solid geometry, and it is normal to expect a similar pattern when one goes from n-dimensional objects to $(n+1)$-dimensional objects. This is true in many cases, but one also has the following somewhat unanticipated fact:

> *Sometimes the answers to basic geometrical questions become simpler if the dimension n is sufficiently large. In other words, there are instances where general patterns of results exist if one excludes finitely many exceptional dimensions.*

AN EXAMPLE FROM EUCLIDEAN GEOMETRY. The classification of (solid) regular polyhedra in Euclidean n-space up to similarity illustrates this phenomenon fairly well. If $n = 2$ then the possibilities are given by the usual regular k-gons, where k is an arbitrary integer ≥ 3. On the other hand, if $n = 3$ then the theory is simpler in some ways but more complicated in others. There are only finitely many possibilities, and they are given by the classical *Platonic solids*; namely, the regular triangular pyramid (or tetrahedron), the cube, the regular octahedron (which can be constructed by taking the centers of the six faces of a cube), the regular dodecahedron and the regular icosahedron (compare [15] and [32]).

If we pass to higher dimensions, then purely algebraic considerations show that for every $n \geq 4$ one can construct a *hypercube* given by all $\mathbf{x} \in \mathbb{R}^n$ whose coordinates lie between 0 and 1, an *n-simplex* which is analogous to an equilateral triangle or regular tetrahedron, and a third object which is dual to the hypercube, with vertices given by the centers of the faces of the hypercube; such objects are analogous to the regular octahedron in 3 dimensions. Further information on these figures can be found in either [32] or [15].

One immediate question is whether there are any other examples, and this was answered by results of Ludwig Schläfli [94] which date back to the mid-nineteenth century. In particular, he showed that there are *three additional examples* if $n = 4$, but *no additional examples* if $n \geq 5$. All but one of the examples for $n = 4$ are analogs of Platonic solids (again see [32] or [15]).

The illustrates the earlier comment about simplifications for sufficiently large dimensions; if we agree that the 2- and 3-dimensional cases are understood, then we see that the 4-dimensional case is more complicated than the 3-dimensional situation and in all dimensions $n \geq 5$ there is a uniform pattern of behavior which is simpler to describe than in either dimension 3 or 4.

SIMILAR PATTERNS IN ALGEBRA. Such patterns also arise very often in group theory. For example, for each integer n consider the alternating group A_n of all even permutations on n letters. A basic result of group theory states that A_n has no nontrivial normal subgroups for all n except $n = 4$. For lower values of n there is no room to squeeze in any nonzero proper subgroups at all, while if $n \geq 5$ there is enough room to perform certain algebraic constructions which force a nontrivial normal subgroup to be the whole group.

Still further examples arise at deeper levels of group theory. In each case there is a very systematic conclusion provided one avoids a finite list of exceptional values; however, in general the latter are not contained in $\{1, 2, 3, 4\}$. For example, one can consider the automorphism group of the symmetric group Σ_n on n letters; one natural question is whether this group has automorphisms besides the standard inner automorphisms; in this case there are no other automorphisms unless $n = 6$, in which case there is an additional "outer" automorphism (for example, see [88]). Another illustration of systematic behavior with finitely many exceptions is the classification of compact simply connected Lie groups, which can be written down very directly provided a numerical invariant called the **rank** is greater than 8 [4] (a quick but accurate summary is available online at http://en.wikipedia.org/wiki/Compact_Lie_Group), and yet another such pattern is the classification of finite simple groups (see [109] for a summary and [102] for a more detailed discussion; this result involves 26 exceptional or *sporadic* examples — the orders of the latter are often astronomical, so the notion of "sufficiently large" is not in the very small ranges we have seen thus far).

COUNTERPARTS IN GEOMETRIC TOPOLOGY. Here is a basic question that is simple to formulate:

> *For a fixed value of n, which finite abelian groups can arise as the fundamental groups of compact (unbounded) n-manifolds?*

If $n \leq 2$ one can answer this using the well-known classification theory for manifolds in these dimensions (*e.g.*, see [70] for the 2-dimensional case); no finite groups can be realized if $n = 1$, and only finite groups of orders 1 and 2 can be realized if $n = 2$. Fundamental results of C. D. Papakyriakopoulos in 3-dimensional topology [85] imply that a finite abelian group G can be realized if $n = 3$ if and only if G is cyclic (see Chapter 9 of [31] for further information). On the other hand, if $n \geq 4$ then by results of A. A. Markov (see [74] or [75]) one has enough geometric "room" to show that *every* finite abelian group can be realized.

Similar patterns appear elsewhere in geometric topology. Often one sees that everything can be described fairly systematically if $n \geq M$ for some small value of M (which is generally equal to 4, 5 or 6), and for all sufficiently small values of n (usually $n \leq 2$) everything is fairly well understood but usually for entirely different reasons. In particular, if $n = 1$ everything is usually extremely straightforward (for example, see the relevant sections of [35]), and our understanding geometric topology in dimension 2 is fairly complete based upon advances from the first part of the twentieth century (compare [70], [95], or [112]). If $n = 3$, there are many new phenomena to consider (including some highly pathological ones as in [7] or [93] in addition to new regular patterns discussed in [31] and [79]), but it appears that 3-dimensional topology will be in a fairly definitive (but still incomplete) form within the next ten years.

As in the case of regular polyhedra (but for entirely different reasons), many basic phenomena in geometric topology become much easier to analyze if $n \geq 5$, As noted in a survey article by L. Siebenmann [99] several breakthroughs involving work from the nineteen forties to seventies have laid a very solid foundation for studying n-manifolds with a few loose ends remaining if $n = 5$ (see [40] for additional information; some later developments are covered in [87]). The results in [99] and [40] also imply that some basic results in higher dimensions cannot be extended to dimensions 3 and 4 (see [98]) Our present understanding of the case $n = 4$ is still only partial despite some revolutionary advances during the past three decades, particularly in the work of M. H. Freedman (see [24] and [25]) and S. K. Donaldson (see [16], [17], [18]); when R. Kirby compiled a list of open questions in 4-dimensional topology during the past decade [41], the result was a massive work of more than 350 pages. A good qualitiative description of the situation is given at the beginning of of A. Scorpan's long and very readable survey of 4-dimensional topology [97]: Dimension 4 has enough room for wild things to happen, but not enough room to tame and undo them.

REMARK. Since $n = 4$ is exceptional in both geometric topology and the structure of alternating groups, it seems worthwhile to stress that the similarities are qualitative and (presumably) the appearance of the same number 4 in both contexts is basically coincidental.

Much of Terry's mathematical work has been devoted to issues involving the relation of 4-manifold theory to the theory of manifolds in higher dimensions. I shall concentrate on two themes runing through many of his papers; the first mainly involves work up to the early nineteen eighties, and the second mainly involves work after that point.

2. Higher dimensional shadows: Stabilization and bisection

We have already noted one basic fact from higher dimensional topology which in fact holds for all $n \geq 4$ (all finite abelian groups arise as fundamental groups of compact n-manifolds). During the nineteen sixties it was known that reasonably simple modifications of certain other basic results for $n \geq 5$ were also true if $n = 4$, and one recurrent (but often unstated) motivation for much of the research during the sixties and seventies was to see how much insight into 4-dimensional topology could be obtained using the methods and results from higher dimensions (*cf.* [72]).

We shall be particularly interested in the following problem, which is important for its own sake and has many far-reaching implications throughout the topology of manifolds:

CYLINDER RECOGNITION QUESTION. *Suppose that we are given a compact connected unbounded n-manifold M^n. Can one describe elementary criteria under which a topological space X is equivalent to the cylinder $M^n \times [0,1]$?*

If $n = 1$ this question has a very elegant answer given by the classical theory of surfaces. The first step is to generalize the concept of n-manifold to include *manifolds with boundaries*. For example, the unit disk in $\mathbb{R}^n$ should be an n-manifold whose boundary is the $(n-1)$-dimensional unit sphere, and a **standard cylinder** $M^n \times [0,1]$ should be an $(n+1)$-manifold whose boundary is two disjoint copies of M; more generally, an $(n+1)$-manifold with boundary W will then have a closed subset ∂W (called the *boundary* of W) such that ∂W is an n-manifold without boundary and the *interior* $W - \partial W$ is an $(n+1)$-manifold without boundary. More information on manifolds with boundary can be found in the standard textbooks by S. Lang [46] and M. Hirsch [33].

Standard results in classical surface theory (see [70]) imply that *a compact connected 2-manifold with boundary W is topologically equivalent to the standard circular cylinder $S^1 \times [0,1]$ if and only if*

- (i)**:** *the boundary of W has two components, say V_0 and V_1,*
- (ii)**:** *the inclusion of either boundary component is a homotopy equivalence.*

More generally, manifolds with boundary that satisfy these properties are called *h-cobordisms*, and the following *h-cobordism Theorem*, which was shown by S. Smale [101] around 1960, is one of the cornerstones of high-dimensional geometric topology. The standard source for the proof in the category of smooth manifolds is Milnor's book [77]; the first proof in the topological case was given a few years later by E. H. Connell [14] and predates the results presented in [40].

Theorem 1. *Let $n \geq 5$, and let W be a simply connected compact $(n+1)$-manifold with boundary $V_0 \amalg V_1$ such that conditions (i) and (ii) above are satisfied. Then W is topologically equivalent to the cylinders $V_0 \times [0,1]$ and $V_1 \times [0,1]$.*

This result extends to manifolds with free abelian fundamental groups, but it does not extend to the general case. Instead, one has the following result, known as the *s-cobordism Theorem* [43] (original sources include [89] and [40]):

Theorem 2. *Let $n \geq 5$, and let W be a connected compact $(n+1)$-manifold with boundary $V_0 \amalg V_1$ such that conditions (i) and (ii) above are satisfied. Then W is topologically equivalent to the cylinders $V_0 \times [0,1]$ and $V_1 \times [0,1]$ if and only if a* Whitehead torsion invariant *$\tau(W, V_0)$ in the algebraically defined* Whitehead group *$\mathrm{Wh}(\pi_1(V_0))$ is equal to zero.*

Elements of the Whitehead group are represented by invertible matrices over a certain ring associated to $\pi_1(M)$, and the Whitehead torsion can be defined entirely in terms of algebraic topology (see [13]); the Whitehead group is trivial if $\pi_1(V_0)$ is a free abelian group, and it follows from our previous remarks that the s-cobordism theorem is also true if $n = 1$; thus the result is true provided $n \neq 2, 3, 4$. It is not known whether the result remains true for arbitrary topological manifolds if $n = 4$, but the analogous result for smooth 5-dimensional h-cobordisms was shown to be false in the nineteen eighties by S. Donaldson [18]. If a basic statement about 3-manifolds known as the *Thurston Geometrization Conjecture* [80] is true (as most workers in the area expect), then the s-cobordism Theorem will also hold if $n = 2$, but if $n = 3$ then there are s-cobordisms that are not cylinders (the first examples are described in [11]). Finally, we should note that

> *the topological h-cobordism Theorem for* simply connected *manifolds is true in* EVERY *positive dimension.*

If $n = 4$ this follows from the work of Freedman [25] in the nineteen eighties, if $n = 2$ this follows from the recent solution of the 3-dimensional Poincaré Conjecture by G. Perelman [80], and if $n = 3$ this follows by combining Perelman's result with certain parts of Freedman's work.

The techniques which prove the s-cobordism theorem yield weak analogs of the latter if $n = 4$ by results of D. Barden [5] and C. T. C. Wall [107]. In particular, Wall's results are part of a general pattern.

> *Many basic results concerning manifolds of dimension ≥ 5 have "stabilized" analogs in dimension 4.*

Roughly speaking, the advantage of stabilization is that it provides some extra room in which to make key constructions. The alternating groups A_n provide a simple but fundamentally important example of **algebraic stabilization**. One crucial step in proving the simplicity of A_n for $n \geq 5$ is showing that it is generated by cyclic permutations of three letters. If $n = 4$, then there is not enough room in A_4 to express some even permutations in this manner, but if one stabilizes by passing to A_5 then there is enough working room to write an even permutation of four letters as a product of such cyclic permutations.

There are several ways of viewing the **geometric stablization** process. Given a manifold M^n, one can adopt the viewpoint of algebraic geometry and "blow up" a finite number of points topologically in a suitable manner (the mental picture is the nonexplosive inflation of a balloon). More precisely, one finds a manifold N^n and a map $f : N^n \to M^n$ such that f is a homeomorphism (or diffeomorphism of smooth manifolds) on some set $f^{-1}[A]$, where A is a finite subset of M, and the inverse images of points in A all have some prescribed topological type (the classical process of blowing up points is described in detail, with extensive illustrations, on pages 286–290 of [97]). For one of Wall's result when $n = 4$, these exceptional sets are all homeomorphic to unions of two 2-dimensional spheres with exactly one point in common; alternatively, one can view these stabilizations as connected sums [92] with finitely many copies of $S^2 \times S^2$, and if there are k exceptional points we shall say that N^4 is a k-fold stabilization of M^4 by $S^2 \times S^2$.

One then has the following analog of the s-cobordism theorem when $n = 4$ for finite stabilizations by $S^2 \times S^2$ (see [107]).

Theorem 3. *Let $n = 4$, and let W be a connected compact smooth $(n+1)$-manifold with boundary $V_0 \amalg V_1$ such that conditions (i) and (ii) above are satisfied (hence W is an h-cobordism). Then there is some $k \geq 0$ such that the k-fold stabilizations of V_0 and V_1 by $S^2 \times S^2$ are diffeomorphic.*

There are also several interesting and important results involving stabilizations by other 4-manifolds (*e.g.*, see [73] or page 151 of [97]), but for our purposes it will suffice to consider only stabilizations by $S^2 \times S^2$.

Numerous other results involving stabilizations by $S^2 \times S^2$ were obtained by many topologists during the nineteen sixties and seventies (for example, [9], [10], [23], and [100]), and Terry was also one of the contributors ([51], [52], [53], [54], [55], [58]). In some instances his work also shed light on related questions about higher dimensional manifolds; for example, his paper with A. Hatcher [30] proves a strong analog of Wall's result in higher dimensions and also gives a very nice 1-parameter analog. The latter can also be viewed as one aspect of Terry's work on fiber bundles (see [48], [49], [52]), which contains several interesting results but was not covered in my talk at the miniconference due to time constraints.

One of the more important and easily stated contributions in Terry's work is his extension of Wall's result to a *stabilized h-cobordism theorem* [55] which gives deeper insight into the structure of a 5-dimensional h-cobordism and shows that such an object becomes a product if one performs a 1-parameter version of the stabilization construction described above.

TWISTED DOUBLES AND OPEN BOOKS. Certain other results from around this time concern special structures on manifolds that are highly significant, both for the insights they yield into the structure theory of manifolds and for their usefulness in studying various sorts of flexible geometrical structures on manifolds. The underlying concept is given as follows:

Definition. Let W be a manifold with boundary V, and let $h : V \to V$ be a homeomorphism. The **twisted double** $W \cup_h W$ is the space formed by taking two disjoint copies W_1 and W_2 of W and gluing them together such that each point $x \in \partial W_1 \cong V$ is identified to the corresponding point $h(x) \in W_2 \cong V$. A result of M. Brown (the Collar Neighborhood Theorem [8]) implies that $W \cup_h W$ is a topological manifold without boundary. Furthermore, if W has a smooth structure and h is a diffeomorphism, then the twisted double has a smooth structure, and frequently other special properties of h translate into corresponding special properties of $W \cup_h W$.

For each positive integer n, the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ has a standard description as a twisted double, where W is the unit disk and the images of W_1 and W_2 correspond to northern and southern hemispheres, given by points for which the last coordinate x_n is either nonnegative or nonpositive. Of course, the common boundary corresponds to the equator, which is merely S^{n-1}, and in this case one can take h to be the identity map (*i.e.*, the sphere is an *untwisted* double). More generally, if W is any manifold with boundary we can form the untwisted double

$$\mathcal{D}(W) \;=\; W \,\cup_{\text{identity}}\, W\ .$$

The only compact 1-manifold (without boundary) is the circle S^1, and we have seen that the latter is an untwisted double. In the case of 2-manifolds, the theory of surfaces yields three important facts about twisted doubles.

Dependence on h: Different choices of h generally yield manifolds that are not homeomorphic (or even homotopy equivalent). For example, the 2-dimensional torus is homeomorphic to the untwisted double of $S^1 \times [0,1]$, but if one forms the twisted double using the homeomorphism of

$$\partial S^1 \times [0,1] \;=\; S^1 \times \{0\} \,\cup\, S^1 \times \{1\}$$

which sends (x, y, ε) to $(x, (-1)^{\varepsilon} y, \varepsilon)$, then one obtaines the Klein bottle [110].

Most surfaces are doubles: A compact unbounded 2-manifold is (homeomorphic to) a twisted double if and only if it is NOT homeomorphic to the real projective plane $\mathbb{RP}^2$ (see [81], p. 372), and every **oriented** surface is in fact (homeomorphic to) an untwisted double. (See pp. 234–236 of [29] for a discussion of orientations.)

Converse statement: The manifold $\mathbb{RP}^2$ is not (homeomorphic to) a twisted double.

In 3-dimensional topology, twisted double structures always exist (*cf.* Chapter 2 of [31]); the standard examples are called *Heegaard splittings* because the existence of such structures on arbitrary compact unbounded 3-manifolds was discovered (in the smooth case, at least) by P. Heegaard just before the end of the nineteenth century.

What happens in higher dimensions? There are systematic infinite families of manifolds in all even dimensions ≥ 2 which cannot be realized as twisted doubles (for example, the even-dimensional complex projective spaces $\mathbb{CP}^{2n}$, where $n \geq 1$; these are defined on pp. 90–93 of [6]). On the other hand, in **odd** dimensions such structures always exist, and for sufficiently large odd dimensions this was shown in the unpublished doctoral dissertations of D. Barden [5] and J. P. Alexander [2]. In the early nineteen seventies, H. E. Winkelnkemper [113] and (independently) I. Tamura ([104] and [105]) described a very special type of twisted double structure called an *open book decomposition* [27], which has proven to be extremely useful in the theory of foliations on manifolds (see [47]) and also in recent work on contact geometry. A detailed discussion of these matters would require substantial digressions (see the survey by Winkelnkemper [114] for more information, and see [115] for a purely algebraic approach to the 3-dimensional case). For our purposes it will suffice to state the Open Book Theorem for simply connected manifolds as follows:

Theorem 4. *Let $n \geq 6$, and let M be a simply connected compact smooth n-manifold (without boundary). Then M has an open book decomposition if and only if either n is* NOT *divisible by 4 or if n is divisible by 4 and an integer valued invariant called the* **signature** *of M (see [78]) is equal to zero.*

Terry's results establish a nontrivial extension of the Open Book Theorem to arbitrary odd-dimensional manifolds in dimensions ≥ 7 [56], and in another paper the existence of twisted double structures for 5-manifolds is shown [57]. If one combines these results with the previous remarks on low-dimensional cases, then one has the following unified conclusion.

Theorem 5. *If n is an odd positive integer, then every compact n-manifold can be realized as a twisted double.*

In addition to its intrinsic interest and applications, this result reflects a relationship between $2k$-manifolds and $(2k+1)$-manifolds that plays a central role in the classification theory of manifolds; for example, in Wall's theory of nonsimply connected surgery [108] one has a parallel relationship between the surgery obstruction groups in dimensions $2k$ and $2k+1$ (in more technical terms, the common thread is that $(2k+1)$-dimensional objects correspond to automorphisms of $2k$-dimensional objects that are represent zero in some appropriate group of equivalence classes).

Incidentally, if one has a nonsimply connected $2k$-manifold, the existence of an open book structure implies additional numerical conditions beyond the vanishing of the signature, and further work is needed. Subsequent work of F. Quinn [86] gives a definitive formulation of the necessary conditions and shows that they are also sufficient for the existence of open book decompositions on arbitrary $2k$-dimensional compact manifolds if $k > 2$.

From the preceding discussion it is clear that Terry's work on some of these problems during the nineteen seventies is closely related to the research of several other topologists, and in fact there are cases of overlapping, independently obtained results; we shall not try to tabulate such instances for the sake of relative brevity (in particular, there is no conscious effort to ignore or denigrate the contributions of others). In cases where there is overlap with the contributions of others, usually Terry's work is particularly noteworthy because (i) he always added some fresh insights of his own, (ii) he was very effective at writing up his results in a clear and thorough form. At the time, geometric topology was an extremely active field with an enormous amount of competition, and in the rush for recognition many pieces of work were written up too hastily (or never even published!) and did not always meet the high standards for mathematical writing that are implicit in Terry's papers (related concerns are stated emphatically and but perhaps excessively in [83]).

STABILIZATION REVISITED. The work of Terry described above was done during the nineteen seventies. However, during the nineteen eighties he wrote one more paper on the subject, and it reflected some important breakthroughs that had taken place in 4-dimensional topology during the intervening years and yielded the following results on 5-dimensional h-cobordisms.

Theorem 6. *Let W be a simply connected compact 5-manifold with boundary $V_0 \cup V_1$ that is an h-cobordism. Then W is topologically equivalent to the cylinders $V_0 \times [0,1]$ and $V_1 \times [0,1]$. However, there are examples of* smooth *simply connected compact 5-dimensional h-cobordisms that are* NOT *smoothly equivalent to cylinders because V_0 and V_1 are not diffeomorphic.*

The first part of this follows from the work of M. Freedman [25], while the second follows from the work of S. Donaldson [18]. Further work of many topologists and geometers yielded large families of examples similar to Donaldson's (see [26] for a survey of the earliest examples, and [97] for an extensive survey of work through the middle of 2004), and one particularly noteworthy family involves a class of objects related to algebraic geometry which are called *Dolgachev surfaces* (see pp. 310-316 of [97]). By Wall's earlier work, if such 4-manifolds are h-cobordant then certain stabilizations of them are diffeomorphic, and the central question in [65] concerns the number of stabilizations that are needed. We know that this number must be positive, and [65] gives simple conditions on Dolgachev surfaces for which one or two stabilizations will suffice. In some cases this yielded new classification theorems for smooth h-cobordisms between nondiffeomorphic Dolgachev surfaces.

The preceding results reflect the emergence of gauge theory as an important tool for studying questions about smooth 4-manifolds, and as such they provide a natural transition to the second theme in Terry's work to be discussed here.

3. Gauge theory and surfaces in 4-manifolds

Gauge theory was first studied by physicists, and in the late nineteen seventies mathematicians began to discover some striking results on the relationship of gauge theories to geometry [3]. In the early nineteen eighties the potential of gauge theory to be a powerful tool in topology became undeniably obvious in monumental work of Donaldson (see [16] and [17]), including his totally unanticipated discovery of smooth manifolds that are homeomorphic to ordinary Euclidean 4-space but not smoothly equivalent to it. We shall not attempt to discuss the details of gauge theory here, for our emphasis will be on its applications to topological questions in Terry's work during the nineteen eighties and nineties. Much of the work involves questions regarding smooth nonsingular surfaces embedded in a smooth 4-manifold.

Questions about embedded surfaces play important roles in the structure theory of n-manifolds if $n \neq 1$ (in which case everything can be worked our directly). The reasons for this may be summarized as follows.

$n = 2$: The quickest justification is that "a surface IS a surface."

$n \geq 5$: Fundamental methods due to H. Whitney [111] show it is possible to construct embedded surfaces which can be used to replace certain geometric configurations with much simpler ones (in fact, this property essentially characterizes topological manifolds in sufficiently large dimensions [87]).

$n = 3$: The work of Papakyriakopoulos [85] (see also Chapter 4 of [31] and later results of other topologists (*e.g.*, W. Haken [28], F. Waldhausen [106], K. Johannson [38], W. Jaco and P. Shalen [37]) show that one can often detect embedded surfaces from relatively weak algebraic data, and these surfaces can often be used to cut a 3-manifold into relatively manageable pieces.

$n = 4$: Under suitable restrictions, the work of Freedman yields *locally flat* topological surfaces (see [93], p. 33) which behave like Whitney's surfaces when $n \geq 5$.

In several respects, our understanding of 4-manifolds is limited by our lack of understanding embedded surfaces. The first example of a breakdown was discovered by M. Kervaire and J. Milnor around 1960 [39], and it concerns smoothly embedded copies of S^2 in $S^2 \times S^2$. Up to homotopy, continuous mappings from S^2 to $S^2 \times S^2$ are classified by an ordered pair of integers known as the *degrees of the projections onto the factors* (see Hatcher's book [29] for the concept of degree). It is not difficult to show that a degree pair (a, b) can be realized if either a or b is equal to 0 or ± 1 (the other can be arbitrary). In contrast, the result of Kervaire and Milnor showed that the pair $(2, 2)$ cannot be realized by a smoothly embedded sphere (however, one can realize every pair by a **piecewise smooth** embedded sphere). Several further results on nonembeddings of surfaces in 4-manifolds were obtained by others before the emergence of gauge theory in the early nineteen eighties Their methods and results were extended by others (*e.g.*, see W.-C. Hsiang and R. Szczarba [36]; in a somewhat different direction see [12]). One early application of gauge theory was a complete determination of the pairs (a, b) that could be realized by a theorem first published by K. Kuga [45] (see also [103]):

Theorem 7. *A pair of integers (a, b) is realized by a smooth embedding of S^2 into $S^2 \times S^2$ if and only if one of a or b is equal to 0 or ± 1.*

This result also illustrates one of the many ways in which the structure theories of topological and smooth 4-manifolds differ, for it is known that many ordered pairs of integers (a, b) can be realized by locally flat topologically embedded spheres; if a and b are nonzero and relatively prime, this is true by Corollary 1 of [24], and the results of [71] provide considerably more detailed information for other ordered pairs. Incidentally, there is a much closer relationship between smooth and locally flat embeddings in higher dimensions (*cf.* Theorem 2 in [96]).

More generally, for every compact, unbounded, smooth, simply connected 4-manifold M and every continuous mapping from S^2 to M, one can assign a *multidegree* — *i.e.*, a sequence of k integers $(d_1, \cdots, d_k)$, where k depends upon the underlying topological space of M — which generalizes the notion of degree pair when $M = S^2 \times S^2$, and one can then ask which multidegrees are realized by smooth embeddings of S^2.

For the most basic choices of M, there are relatively short lists of multidegrees which can be realized by well-known constructions. The preceding theorem implies that no others can be realized if $M = S^2 \times S^2$, and a similar conclusion holds for the complex projective plane $\mathbb{CP}^2$. In [64] Terry considered some of the next few cases from a somewhat different viewpoint involving results of R. Fintushel and R. Stern [21], and he obtained new results for the manifolds $M(1,1)$ and $M(1,2)$ given by taking connected sums of $\mathbb{CP}^2$ with 1 or 2 copies of the oppositely oriented manifold $\overline{\mathbb{CP}^2}$ (in the previously used language of algebraic geometry [26], this corresponds to blowing up one or two points). The results for $M(1,1)$ are complete, while the results for $M(1,2)$ apply to exactly half of the possible multidegrees.

Several other papers by Terry address further questions involving the methods of Fintushel and Stern as well as the applications of their techniques. To describe this work, we first recall that gauge theory analyzes topological questions by first constructing certain associated "moduli spaces of instantons" whose elements are equivalence classes of appropriate types of geometric structures, and then studying the properties of such spaces. Especially in the early work, compactness questions involving such spaces played a fundamental role, and a pair of Terry's papers ([20] and [66]) — one of which was joint with Fintushel — show that earlier compactness results of Fintushel and Stern [22] could be generalized extensively.

In some related papers such as [63] and [68], Terry considered another question arising from work of Fintushel and Stern [21]. It is known that every compact 3-manifold M^3 bounds a smooth compact manifold W^4, and a central problem in low-dimensional topology is to make W^4 as simple as possible. The results of [63] yield lower limits on the amount of simplification that can be done for certain fundamental 3-manifolds called *Seifert homology 3-spheres* (see [84]), and the precise conclusions are stated in terms of certain trigonometric expressions. Terry extended the the earlier results of [21] on such questions in two ways, using his compactness results and analyzing the trigonometric expressions by number-theoretic methods from work of W. Neumann and D. Zagier [82].

An entirely different class of contributions appear in [62], which consider smooth embeddings of the real projective plane $\mathbb{RP}^2$ into simply connected 4-manifolds. Terry's interest in such issues was already evident in earlier papers about embeddings of $\mathbb{RP}^2$ in S^4 ([59], [60], [61]). In general, if we are given a smooth embedding of $\mathbb{RP}^2$ into a simply connected 4-manifold, then there is an integer called the *twisted Euler number* which describes small neighborhoods of the embedded submanifold, and the goal of [62] is to describe the possible Euler numbers for certain choices of M. When $M = S^4$, the answer to this question was found in the late nineteen sixties [76]. Using the methods described above for the given 4-manifolds, Terry proves a numerical congruence mod 4 and determines a lower bound for the twisted Euler number in a substantially more general situation; there is also a natural conjecture for the upper bound, but this remains an open question.

In all these cases, Terry's results yielded strong new results on questions that had seemed totally beyond reach in 1980 (the beginning of the decade when the papers were written). Equally important, his work was also significant because it provided models for applying the recently developed machinery of gauge theory in a systematic manner that did not require extensive work with the deep and complicated details of gauge theory itself. Terry's work marked a major step in reducing many topologists' apprehensiveness about the powerful and effective new methods that had already made such an enormous impact on the subject.

The results of [63] on Seifert homology 3-spheres led to some highly original joint work with S. Kwasik [44] on symmetries of certain compact 4-manifolds with boundary. References are given for several specialized terms which appear in the statement of the main result.

Theorem 8. *There are infinitely many finite group actions* [91] *on compact, smooth, contractible* 4*-manifolds with boundary* W^4 *(see* [81], *p.* 330*) such that*
(i) *each action is free* [90] *on the complement of a single fixed point in the interior of* W^4*,*
(ii) *the restrictions of each action to the interior and boundary are smoothable,*

(*iii*) *none of these actions are globally smoothable.*

The results of [44] also yielded some new implications about the differences between the structure theories for smooth and topological 4-manifolds which are unique to dimension 4.

During the nineteen nineties, gauge theory underwent some major changes that were motivated by work in theoretical physics due to N. Seiberg and E. Witten (*e.g.*, see [19]). This new and improved version of gauge theory depends strongly on geometric properties called **Spin** and **Spin**^{c} structures, which are essentially higher order analogs of orientations on a manifold. In [1], written jointly with D. Acosta, the role of these conditions in the case of 4-manifolds is analyzed carefully, and the result is a clear description of issues which, as noted in the summary of [1] in *Mathematical Reviews*, "can be confusing even to the initiated."

Finally, no discussion of Terry's papers on gauge theory would be complete without mentioning is two excellent and very highly regarded survey articles of results on smoothly embedded surfaces in compact simply connected 4-manifolds. The first of these [68] deals with embedded spheres, while the second [69] concerns more general oriented surfaces and lower bounds for a basic numerical invariant (the **genus**) of such a smoothly embedded surface.

4. Closing remarks

Terry Lawson has worked productively on a variety of problems that really matter in geometric topology, he has been willing and able to move with the subject, and he has done an excellent job of presenting both his results and related material. Each of these qualities is indispensable for the successful development of a mathematical subject, and I have very much appreciated Terry's contributions in all these directions.

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