EQUIVARIANT SMOOTHINGS OF FAKE $\mathbb{R}^4$'s

For the sake of simplicity, assume we are working with semifree and locally linear actions of a finite group G on $\mathbb{R}^4$.

PROPOSITION 0. Such an action is equivariantly smoothable except possibly if $G \cong \mathbb{Z}_2$ and the fixed set is 1-dim'l.

<u>Notation</u> $\mathcal{G}$ = Gompf's end sum

$\mathcal{R}$ = commutative monoid of oriented diffeo classes of smoothings of $\mathbb{R}^4$.

(1) U = Freedman-Taylor example
$\Longrightarrow [U] \in \mathcal{R}$ is the zero element
$(zx = xz = z \quad \forall\, x \in \mathcal{R})$.

(2) $[\mathbb{R}]$ is the unit element

(3) There is a nice countable sum operation
$\coprod_{i=1}^{\infty} x_i$

④ The unit is the only invertible elt.
("Eilenberg swindle": $x = x x^{-1} x x^{-1} \ldots = 1$).

Variations of smooth structure

Question 1 Is there a 1-parameter family $x_t \in \mathcal{R}$ s.t. $t \neq s \Rightarrow x_t^n \neq x_s^n \ \forall\, n \geq 1$?

If so, it is easy to resmooth any smooth o.pres. action on $\mathbb{R}^4$ in uncountably many ways: If $|G| = n$ and the orientation is preserved, then a positive answer to Q1 would yield actions on the 1-parameter family x_t^n. INSERT 2A HERE
Unfortunately, it is not clear if Q1 has a positive answer.

Question 2 Does every locally linear action on $\mathbb{R}^4$ come from a smooth action on $\mathbb{R}^4$?

In any case, it comes from a smooth action on $[U] = [U]^n = [\cancel{\ }^n U]$

The following simple observation is crucial:

Proposition 00. Suppose G acts smoothly and orientation preservingly on $\mathbb{R}^4$, and let V be a smooth oriented fake $\mathbb{R}^4$. Then the iterated Gompf sum $\natural^{|G|} V$ supports a smooth G-action which is TOP equivalent to the given one.

There is an analog if G does not act orientationpreservingly, where the ambient fake $\mathbb{R}^4$ is $\natural^{|G|/2}(V \natural -V)$.

(Q: What do we know about counting fake $\mathbb{R}^4$'s that are not o. reversingly self diffeomorphic, beyond Gompf's examples in [Gompf 1983]?)

<u>Partial result:</u> Suppose G acts smoothly on $\mathbb{R}^4$ in a given fashion. Then there are at least countably many smooth structures on $\mathbb{R}^4$ s.t. the action is smooth w.r.t. these structures.

<u>Sketch</u> It is enough to find $x \in \mathcal{R}$ s.t. $1, x, x^2, \ldots$ are distinct. Then one has action on the classes $x^{k|G|}$ all $k \geq 1$.

[Taylor 1997] There is an $x \in \mathcal{R}$ s.t. $\forall\, n \geq 2$, $k \geq 1$ the sequence x^{n^k} has distinct values for n fixed & k variable.
(See Taylor, last ¶ on page 73)

<u>Algebraic Lemma</u> S = abelian semigroup $x \in S$ s.t. $\forall\, n \geq 2$ the sequence $x, x^n, x^{n^2}, \ldots$ has no repeated values. Then the sequence $x, x^2, \ldots$ has no repeated values.

<u>Proof.</u> Say $x^p = x^q$ for $q > p$. Then $\forall\, m > 0$

we have $x^p = x^{q+m(q-p)}$. Choose n s.t.

$n \geq p$

$q-p \mid n$. Then $q-p \mid n^2-n$, so

$x^n = x^{n^2}$ ($x^n = x^{n-p}x^p = x^{n-p}x^p x^{k(q-p)}$, where $n^2-n = k(q-p)$.

Hence the classes $x, x^2, \ldots \in \mathcal{R}$ are distinct.

<u>Further examples</u> Suppose the action on $\mathbb{R}^4$ has a shell decomposition $U_1 \subseteq U_2 \cup \ldots$ $U_i \cong$ open disk s.t. on each U_i, the action near ∞ is collared ($\Rightarrow$ linear on $S^3 \times \mathbb{R}$).

<u>Prop.</u> Shell decomposition $\Rightarrow$ if we smooth the action using the Freedman–Taylor examples then the induced G-smoothings of the U_i are on nondiffeomorphic manifolds.

<u>Cor.</u> If the action on $\mathbb{R}^4$ extends to a smooth action on S^4, then it is smoothable in countably infinitely many ways.

<u>Note</u> Giffen–Gordon–Summers gives many examples of the Cor. One has further examples in the Prop.

by taking an infinite equivariant connected sum of examples coming from S^4 ($2^{\aleph_0}$ many
probably, but needs to be checked).
TOP. TYPES

<u>Continuous families</u> Look at actions extending to smooth actions on S^4 (collared near ∞)

<u>Claim</u> If $N_t \subseteq_G \mathbb{R}^4$ = open disk of radius t, then there is a G-smoothing V of $\mathbb{R}^4$ s.t. $\forall t$ suff large the G-inv^t smooth manifolds $V_t \longleftrightarrow N_t$ are distinct.

In fact, if W = original Donaldson-Freedman fake $\mathbb{R}^4$, (see [Gompf 1983]), then we can take V to be the $|G|$-fold Gompf sum $\natural^{|G|} W$.

<u>Sketch of proof.</u> This requires a closer look at the arguments in [Gompf 1983] and [Taubes 1987].
§1 §1 + 1st ¶s in §2

Here are the main ingredients:

Thm. 1 Let M_{16} be the closed 4-manifold s.t. $\pi_1 = \{1\}$ and the intersection form is $E_8 \oplus E_8$, and consider the smoothing of $M_{16} - \{pt.\}$ as an open subset of the Kummer surface K with end $\cong$ the end of a Donaldson–Freedman fake $\mathbb{R}^4$, say W. Then for each $n > 0$ (or 1) there is a smoothing of $(\#^n M_{16}) - \{pt\}$ with end $\cong \natural^n W$.

(To be proved below)

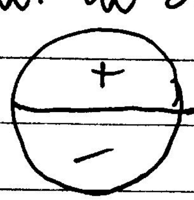
Thm. 2 Let $\varphi : \mathbb{R}^4 \to \natural^n W$ be a homeomorphism, and let X_t = image of $\{|v| < t\}$. Then for all t sufficiently large the X_t's are inequivalent smoothly and orientation preservingly.

Thm. 2 follows from Thm. 1 and the methods in [Taubes 1987]. If the conclusion were false, then $\natural^n W$ would have a periodic end, and likewise for some smooth structure on $(\#^n M_{16})$ − pt. Since $H_2((\#^n M_{16})) \cong 2n$ copies of E_8, this would contradict Taubes' results.

To get a continuum of group actions for G, let $\varphi : \mathbb{R}^4 \to \natural^{|G|} W$ be the standard equivariant homeomorphism, so that G acts on $\natural^{|G|} W$ and each X_t is G-invariant.

<u>Proof of 1</u> (sketch) Follow §1 of [Gompf 1983]. $X = \#^3 S^2 \times S^2$ − open disk

Note that one can choose the embedding in Thm. 1.2 (p. 319) so that it is smooth near a point in ∂X s.t.



4 dim'l $\longrightarrow$ $K - i[X]$

3 dim'l $\longrightarrow$ ———

$\longrightarrow$ Int $i[X]$

Use this disk and its copies to construct an embedding of

$$\#^n X \text{ in } \#^n K.$$

By construction, the end of $\#^n K - \#^n X$ is smoothly equivalent to the end of $\natural^n W$, where $W = \mathbb{R}^4_{DF}$ in Gompf's notation. $\boxtimes$

Why can one choose the embedding as indicated? Take any embedding of X and take a smooth embedding of D^4 in a small nbhd of X in K. Join these topologically with a tube, obtaining an embedding of $X \# D^4 \cong X$ with the indicated properties.

References

[Taubes 1987]
C. Taubes, Gauge theory on asymptotically periodic 4-manifolds, JDG 25 (1987), 363-430.

[Gompf 1983]
R. Gompf, Three exotic $\mathbb{R}^4$'s and other anomalies, JDG 18 (1983), 317-328.

[Taylor 1997]
L. Taylor, An invariant of smooth 4-manifolds, Geometry & Topology 1 (1997), 71-89.

[Freedman Taylor]
M. Freedman & L. Taylor, A universal smoothing of four-space, JDG 24 (1986), 69-78.