TANGENTIAL THICKNESS OF MANIFOLDS

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1, 3 or 2k where k ≥ 1 is arbitrary

ABSTRACT. A notion of tangential thickness of a manifold is introduced. An extensive calculation within the class of lens and fake lens spaces leads to complete classification of such manifolds with thickness 1, 2 or 3. On the other hand, calculations of tangential thickness in terms of the dimension of the manifold and the rank of the fundamental group show very interesting and quite surprising correlations between these invariants.

One or two examples or a forward reference to this paper would be useful

1. INTRODUCTION

Given two nonhomeomorphic topological spaces, X and Y, it is often interesting and important to specify necessary or sufficient conditions for $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ to be homeomorphic, where $\mathbb{R}$ denotes the real line. More generally, it is also useful to have criteria for determining whether $X \times \mathbb{R}^k$ and $Y \times \mathbb{R}^k$ are homeomorphic for some $k \geqslant 1$. If X and Y are closed manifolds the following result, due to B. Mazur in the smooth and piecewise linear categories [], provides an abstract answer; in (No ¶ break)

the statement of this result ~~below~~, CAT refers to the category of smooth, piecewise linear, or topological manifolds, and a CAT-isomorphism is a diffeomorphism, piecewise linear homomorphism or homeomorphism, respectively.

Stable Equivalence Theorem: Let M and N be closed CAT-manifolds. Then $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are CAT-isomorphic for some $k \geqslant 1$ if and only if M and N are tangentially homotopy equivalent (i.e., there is a homotopy equivalence $f : M \to N$ such that the pullback of the stable tangent bundle/tangent microbundle of N is the stable tangent bundle/tangent microbundle of M).

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In fact, if f exists, then for some k the map $f \times \mathrm{Id}_{\mathbb{R}^k}$ if properly homotopic to a CAT-isomorphism. The topological version of this result followa from [], [].

Given two manifolds M and N satisfying the conditions of the Stable Equivalence Theorem, it is natural to ask the following:

Optimal Value Question: For a given tangential homotopy equivalence $f : M \to N$, what is the least value of $k \geqslant \cancel{1}\,0$ such that $f \times \mathrm{Id}_{\mathbb{R}^k}$ is properly homotopic to a CAT-isomorphism?

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[HW]

q+1 (q = dim M = dim N)

The Whitney Embedding and Tubular Neighborhood Theorems imply that ~~dim$N$+~~1 is a~~n~~ universal upper bound for k in the smooth category. Standard results for piecewise linear manifolds, and results of Siebenmann [] imply the analog in the piecewise linear and topological categories, respectively.

In [], the Optimal Value Question was considered for linear spherical space forms. In particular, it was shown that if M and N are linear space forms such that $M \times \mathbb{R}^2$ is homeomorphic to $N \times \mathbb{R}^2$, then M and N are diffeomorphic. Furthermore, examples of fake lens spaces M and N were constructed in [] such that M and N are homeomorphic but $M \times \mathbb{R}^3$ and $N \times \mathbb{R}^3$ are not diffeomorphic. These results already reflect the relative complexity of this problem. Some of the techniques and ideas of this paper were applied in [] and [] when studying and classifying open complete manifolds of nonnegative curvature. By the results of Cheeger and Gromoll [CG], such manifolds are diffeomorphic to the total space of a normal bundle to a compact locally geodesic submanifold called a soul. This, of course, leads to an obvious variation on the notion of tangential thickness of the soul.

Explain this term

We first consider linear lens spaces.

Theorem 1. *Let $f : M \to N$ be a tangential homotopy equivalence of lens spaces with prime order fundamental groups. Then $f \times \mathrm{Id}_{\mathbb{R}^3}$ is properly homotopic to a homeomorphism.*

I think this is redundant

We would like to point out that the above theorem is directly related to a remarkable ~~(but surprisingly poorly publicized)~~ theorem of J. Folkman []. Namely,

Theorem (Folkman): Let m be a power of a prime. If two k-dimensional lens spaces with fundamental group $\mathbb{Z}_m$ have the same tangential homotopy type, ~~and if~~ where k is greater than a certain function of m, then the two must actually be isometric (diffeomorphic).

This is a truly startling result which probably deserves much more attention and publicity. We hope to discuss this and some of the ideas contained in [] in a future paper. We will, however, restrict ourselves at this time to the following extension of Folkman's theorem.

Theorem 2. *Let $f : M^{2n-1} \to N^{2n-1}$ be a stably tangential homotopy equivalence of lens spaces with $\pi_1(M^{2n-1}) \cong \pi_1(N^{2n-1}) \cong \mathbb{Z}_p$, for p an odd prime. Then M and N are isometric (diffeomorphic) if $n \geqslant p-1$.*

Most [well over 50% – "remainder suggests << 50%]

~~The remainder~~ of the paper (and its main purpose) is devoted to a study of the Optimal Value Question for fake lens spaces. We will concentrate on the case of (odd) prime order fundamental groups, although many of our results hold without this restriction.

bold font

Qualitatively, ~~on~~ one can describe our results in terms of a concept we shall call tangential thickness, defined as follows:

HW A. Haefliger and C. T. C. Wall, Piecewise linear bundles in the stable range. Topology 4 (1965), 209–214.

CG J. Cheeger and D. Gromoll, On the structure of complete manifolds of nonnegative curvature, Ann. of Math. 96 (1972), 413–433

Definition 1. *Two CAT manifolds M and N are said to have tangential thickness $\leqslant k$ if and only if $M \times \mathbb{R}^k$ is CAT-isomorphic to $N \times \mathbb{R}^k$. Given two tangentially homotopy equivalent manifolds M and N, the tangential thickness of the pair $\{M, N\}$ is the least k such that the tangential thickness of M and N is $\leqslant k$.*

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Given a manifold M, let $\mathcal{TT}_k(M)$ denote the isomorphism classes of manifolds N such that the pair $\{M, N\}$ has tangential thickness $\leqslant k$. One then has an increasing sequence of sets:

$$\{M\} = \mathcal{TT}_0(M) \subseteq \mathcal{TT}_1(M) \subseteq \ldots \subseteq \mathcal{TT}_k(M) \subseteq \ldots \subseteq \mathcal{TT}(M)$$

where $\mathcal{TT}(M)$ is the set of isomorphism classes of manifolds tangentially homotopy equivalent to M. As noted earlier, ~~This~~ this ~~finite~~ sequence stabilizes for $k \geqslant \dim M+1$:

LEAVE MORE SPACE

$$\mathcal{TT}_k(M) = \mathcal{TT}_{k+i}(M) = \mathcal{TT}(M) \text{ for } i = 1, 2, \ldots$$

In particular, given a manifold M, the classification of all manifolds having tangential thickness $tt(M) = k$ is equivalent to the computation of the set $\mathcal{TT}_k(M) \setminus \mathcal{TT}_{k-1}(M)$. We are now ready to state the results of this paper pertinent to tangential thickness.

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Theorem 3. *Let M^{2n-1}, $n \geqslant 3$, be a fake lens space (arbitrary fake spherical space form). Then $\mathcal{TT}_1^{TOP}(M^{2n-1})$ consists of manifolds h-cobordant to M^{2n-1}. These manifolds are classified by $Wh(\pi_1(M^{2n-1}))$ via realization of Whitehead torsion by h-cobordisms (i.e., the action of $Wh(\pi_1(M^{2n-1}))$ on M^{2n-1} is free).*

Theorem 4. *Let M^{2n-1}, $n \geqslant 3$, be a fake lens space. Then N^{2n-1} is in $\mathcal{TT}_2^{TOP}(M^{2n-1})$ if and only if $N^{2n-1} \times \mathbb{R}$ is properly h-cobordant to $M^{2n-1} \times \mathbb{R}$. The set $\mathcal{TT}_2^{TOP}(M^{2n-1}) \setminus \mathcal{TT}_1^{TOP}(M^{2n-1})$ is in one-to-one correspondence with $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[\pi_1(M^{2n-1})]))$. Moreover, all possible manifolds in $\mathcal{TT}_2^{TOP}(M^{2n-1}) \setminus \mathcal{TT}_1^{TOP}(M^{2n-1})$ are obtained by a free action of $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[\pi_1(M^{2n-1})]))$ on $M^{2n-1} \times \mathbb{R}$ via the realization of Whitehead torsion by proper h-cobordisms.*

Theorem 5. *Let M^{2n-1}, $n \geqslant 3$, be a fake lens space with $\pi_1(M^{2n-1}) \cong \mathbb{Z}_p$, for p an odd prime. Then the set $\mathcal{TT}_3^{TOP}(M^{2n-1})$ is the set of homeomorphism classes of manifolds normally cobordant to M^{2n-1}. The set $\mathcal{TT}_3^{TOP}(M^{2n-1}) \setminus \mathcal{TT}_2^{TOP}(M^{2n-1})$ is in one-to-one correspondence with the free abelian group $\mathbb{Z}^{\frac{p-1}{2}}$. A manifold N^{2n-1} is in $\mathcal{TT}_3^{TOP}(M^{2n-1}) \setminus \mathcal{TT}_2^{TOP}(M^{2n-1})$ if it is obtained from M^{2n-1} by the action of $\widetilde{L}_{n+1}^h(\mathbb{Z}_p)$ on M^{2n-1}. Moreover, the action is given by the difference of ρ-invariants, $\rho(M^{2n-1}) - \rho(N^{2n-1})$.*

EXPLAIN

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results exhibit — see next page

Our final result exhibits a remarkable correlation between the three crucial parameters: the dimension of the fake lens space, the order of its fundamental group and its tangential thickness.

I'd also prefer more space between the \ symbol and adjacent stuff.

ρ- invariant reference

W. Browder, T. Petrie and C. T. C. Wall, The classification of free actions of cyclic groups of odd order on homotopy spheres, Bull. AMS 77 (1971), 455-459.

Note that the ρ-invariant is BPW's Δ-invariant

Let p be an odd prime, let $n \geqslant 3$, and

Theorem 6. *~~Let~~ M^{2n-1}, ~~$n \geqslant 3$~~, be a fake lens space with $\pi_1(M^{2n-1}) \cong \mathbb{Z}_p$, ~~for p an odd prime~~. Then ~~the following~~*

I'd prefer to split this into separate statements (i.-iv.) with connecting sentences

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Example 6 [1st (iv) then (i)]

7 [First part of (ii) (iii) Last part of (ii)]

(i.) *if $n \geqslant p$, then $TT_3^{TOP}(M^{2n-1}) \neq TT_4^{TOP}(M^{2n-1})$*

(ii.) *if $n \geqslant p$, then the inclusions $TT_{2k}^{TOP}(M^{2n-1}) \subseteq TT_{2k+2}^{TOP}(M^{2n-1})$ are proper for $2 \leqslant k \leqslant \left\lfloor \dfrac{n-1}{p-1} \right\rfloor$, where $\lfloor - \rfloor$ denotes the greatest integer function, and*

$$TT_{2k}^{TOP}(M^{2n-1}) = TT^{TOP}(M^{2n-1}) \text{ for } k \geqslant \left\lfloor \frac{n-1}{p-1} \right\rfloor$$

(iii.) *for each individual $k \geqslant 2$, either $TT_{2k+1}^{TOP}(M^{2n-1}) = TT_{2k}^{TOP}(M^{2n-1})$ or else $TT_{2k+1}^{TOP}(M^{2n-1}) = TT_{2k+2}^{TOP}(M^{2n-1})$*

(iv.) *if $n \leqslant p-1$, then $TT_3^{TOP}(M^{2n-1}) = TT^{TOP}(M^{2n-1})$*

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2. Proofs

This section contains proofs of our results. The main techniques used in these proofs are surgery, geometric topology and homotopy theory. We start with a proof of Theorem 3.

Proof. (Theorem 3) * see below

Let M^{2n-1}, $n \geqslant 3$, be a fake spherical space form and $f : N^{2n-1} \to M^{2n-1}$ a tangential homotopy equivalence. Suppose $N^{2n-1} \times \mathbb{R}$ and $M^{2n-1} \times \mathbb{R}$ are isomorphic. Then, it follows that N^{2n-1} and M^{2n-1} are h-cobordant. The action of the Whitehead group $Wh(\pi_1(M^{2n-1}))$ on M^{2n-1} is free by the main result of [].

On the other hand, if $(W^{2n}; N^{2n-1}, M^{2n-1})$ is an h-cobordism between N^{2n-1} and M^{2n-1}, then $W^{2n} \times S^1$ is an s-cobordism between $N^{2n-1} \times S^1$ and $M^{2n-1} \times S^1$. Thus, $M^{2n-1} \times S^1$ is isomorphic to $N^{2n-1} \times S^1$ and hence $M^{2n-1} \times \mathbb{R}$ and $N^{2n-1} \times \mathbb{R}$ are isomorphic as well. □

Proof. (Theorem 4) * see below

Let ~~G~~ π $\cong \pi_1(M^{2n-1})$ be the fundamental group of M^{2n-1}. If $N^{2n-1} \in TT_2(M^{2n-1})$, then N^{2n-1} is a fake lens space and there exists a homeomorphism $h : N^{2n-1} \times \mathbb{R}^2 \to M^{2n-1} \times \mathbb{R}^2$. This yields an h-cobordism W^{2n+1} between $N^{2n-1} \times S^1$ and $M^{2n-1} \times S^1$ (cf. []). By taking the infinite cyclic covering, one gets a proper h-cobordism $\widetilde{W}^{2n+1}$ between $N^{2n-1} \times \mathbb{R}$ and $M^{2n-1} \times \mathbb{R}$.

Since we shall use G for something else later.

Conversely, if there is a proper h-cobordism V^{2n+1} between $N^{2n-1} \times \mathbb{R}$ and $M^{2n-1} \times \mathbb{R}$, then $V^{2n+1} \times S^1$ is a product cobordism between $N^{2n-1} \times \mathbb{R} \times S^1$ and $M^{2n-1} \times \mathbb{R} \times S^1$. In particular, $N^{2n-1} \times \mathbb{R} \times S^1 \approx M^{2n-1} \times \mathbb{R} \times S^1$ and hence $N^{2n-1} \times \mathbb{R} \times \mathbb{R} \approx M^{2n-1} \times \mathbb{R} \times \mathbb{R}$ (i.e. $N^{2n-1} \times \mathbb{R}^2 \approx M^{2n-1} \times \mathbb{R}^2$).

italic

*I'd prefer BOLD FACE: Proof of Theorem X.

COMMENT

This is a point at which the subsequent material should be described in outline form. Here is my tentative outline:

2. Results in low codimensions
3. Tangential thickness and normal invariants
4. Tangential equivalence of lens spaces
5. Normal invariants for fake lens spaces
6. Desuspension results
7. Proof of Theorems 5 and 6 [or split up pieces].

And here is a preliminary attempt to describe the paper's contents:

The proofs of these results will appear in Sections 2 - 7 below. In Section 2 we use surgery-theoretic methods as in [KwS, Toral & exponential... (2004)] to prove Theorems 3 and 4. Section 3 ~~establishes~~ ~~key~~ gives a surgery-theoretic criterion for two manifolds to have tangential thickness $\leq k$, where $k \geq 3$; most of this material is surely well known, but we include it since it is fundamental to our work and difficult to extract from the literature. ~~We shall~~ In the case of odd-dimensional $\mathbb{Z}[\frac{1}{2}]$ homology spheres, these results ~~that are can~~ will be restated very simply in terms of desuspending classes in the stable cohomotopy groups of ~~the~~ such ~~smooth~~ manifolds (see Proposition 3.Y). In Section 4 we shall use the ideas of Section 3 and results on the K-theory of lens spaces [Kam] to prove Theorems 1 and 2. ~~A more detailed analysis of the results in Section 2 app~~ We shall then specialize the general setting of Section 3 to fake lens spaces in Section 5; ~~using~~ this uses a variety of ~~to~~ results ~~from stable homotopy theory and~~ about the structure of the classifying spaces for surgery theory (see ~~compare~~ [Madsen-Milgram, Ann. Math Studies]). ~~The previously mentioned~~ In Section 6 we shall analyze the cohomotopy desuspension question from Section 2 ~~for~~ the case of $\mathbb{Z}_p$ lens spaces ~~and we shall prove that the machinery of F. Cohen~~ using the work of F. Cohen, J. C. Moore and J. Neisendorfer (e.g., see [Neis]) ~~on~~ exponents of homotopy groups. Finally, we shall bring everything together in Section 7 to prove Theorems 5 and 6 [or whatever]

depending on how Thm 6 might be split.

2. Results in low codimensions

In this section and the next, we shall ~~stress~~ the basic surgery-theoretic conditions for determining ~~whether a~~ the tangential homotopy equivalences ~~[illegible]~~ $h : M \to N$ such that $h \times id(\mathbb{R}^k)$ is homotopic to a homeomorphism. As ~~is often the case~~ in many other situations within geometric topology, the cases ~~for which~~ with codimension $k \geq 3$ ~~are~~ differ greatly from the cases where $k = 1$ or 2, and within ~~so we~~ this section ~~that~~ we shall dispose of the latter cases. Continue as on page 4

References

T. Kambe, The structure of $K\Lambda$-rings of the lens space and their applications, J. Math. Soc. Japan 18 (1966), 135-146

Madsen-Milgram should be easy to find

J. Neisendorfer, Algebraic Methods in Unstable Homotopy Theory, ~~New~~ Cambridge New Mathematical Monographs No 12. Cambridge University Press, Cambridge (UK) 2010.

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Now, let $\tau_0 \in Wh(\widetilde{W}^{2n+1}, M^{2n-1} \times \mathbb{R}) \cong \widetilde{K}_0(\mathbb{Z}[G])$ (cf. []) be a proper Whitehead torsion of this proper h-cobordism. Analogously as in the compact case (cf. [], []) there is an involution on $Wh(\widetilde{W}^{2n+1})$ and duality between $\tau_0 \in Wh(\widetilde{W}^{2n+1}, M^{2n-1} \times \mathbb{R})$ and $\tau_1 \in Wh(\widetilde{W}^{2n+1}, M^{2n-1} \times \mathbb{R})$ given by $\tau_1 = (-1)^{\dim(M^{2n-1}\times\mathbb{R})}\tau_0^*$. Hence, $\tau_1 = \tau_0^*$.

Let $f : N^{2n-1} \times \mathbb{R} \to M^{2n-1} \times \mathbb{R}$ be a proper homotopy equivalence given by

cf. in italics

$$N^{2n-1} \times \mathbb{R} \xrightarrow{i} \widetilde{W}^{2n+1} \xrightarrow{r} M^{2n-1} \times \mathbb{R}$$

where i is the inclusion and r is the proper retraction. It follows that $\tau(f) = \tau_0^* - \tau_1$. However, f is properly homotopic to a map $f_0 \times \mathrm{Id}_{\mathbb{R}} : N^{2n-1} \times \mathbb{R} \to M^{2n-1} \times \mathbb{R}$ (cf. [], p. 61), with $f_0 : N^{2n-1} \to M^{2n-1}$. In particular, as $f_0 \times \mathrm{Id}_{S^1} : N^{2n-1} \times S^1 \to M^{2n-1} \times S^1$ is a simple homotopy equivalence (cf. []), so must be $f_0 \times \mathrm{Id}_{\mathbb{R}}$. As a consequence, $\tau_1 = \tau_0^*$ and $f : N^{2n-1} \times \mathbb{R} \to M^{2n-1} \times \mathbb{R}$ is a proper simple homotopy equivalence.

The standard construction shows that elements in $Wh(\widetilde{W}^{2n+1})$ of the form $\rho + \rho^*$ can be realized by inertial proper h-cobordism. Consider

$$\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G])) = \left\{ \frac{\tau = \tau^*}{\tau + \tau^*} \right\}.$$

Claim 1. *Realization of elements in $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G]))$ via proper h-cobordisms starting with $M \times \mathbb{R}$ yields manifolds of the form $N \times \mathbb{R}$ on the other end.*

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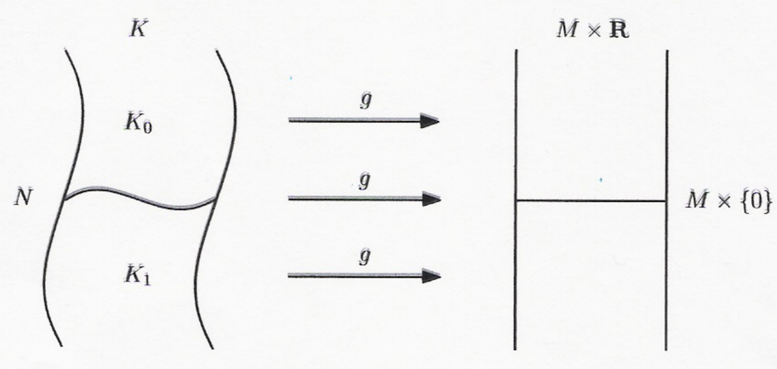
Proof. To see this, let $(\overline{W}; M\times\mathbb{R}, K)$ be a proper h-cobordism with $\tau_0 \in Wh(\overline{W}, M\times\mathbb{R})$, $\tau_0 \in \widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G]))$. Then there is a proper homotopy equivalence $f : K \hookrightarrow \overline{W} \to M\times\mathbb{R}$ which is simple. By the one-sided splitting theorem for proper maps and noncompact manifolds (cf. [], p.), f is properly homotopic to a map g with $g^{-1}(M \times \{0\}) = N \subset K$, and $g|_N : N \to M$ a homotopy equivalence. We have a splitting



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with

$$
\begin{aligned}
g|_{K_0} &: K_0 \to M \times [0, \infty) \\
g|_{K_1} &: K_1 \to (-\infty, 0] \times M
\end{aligned}
$$

proper homotopy equivalences. Now, the Collaring theorem of Siebenmann (cf. []) implies $K_0 \approx N \times [0, \infty)$ and $K_1 \approx (-\infty, 0] \times N$, and hence $K \approx N \times \mathbb{R}$. $\square$

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It remains to show that the action of $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G]))$ yields all manifolds in $\mathcal{TT}_2^{TOP}(M^{2n-1}) \setminus \mathcal{TT}_1^{TOP}(M^{2n-1})$. To see this, we use a proper surgery theory of Maumary-Taylor. Consider the long Wall-Sullivan exact sequence for proper surgery theory:

citations, including online access from Ranicki's site.

\mathbb{S}?

$$
\cdots \longrightarrow L_{*+1}^{s,open}(M \times \mathbb{R}) \to \mathcal{S}_{TOP}^{s}(M \times \mathbb{R}) \to [M \times \mathbb{R}; G/TOP] \to L_{*}^{s,open}(M \times \mathbb{R}) \longrightarrow \cdots
$$

We have $L_{*+1}^{s,open}(M^{2n-1} \times \mathbb{R}) \cong L_{even}^{h}(G) \cong L_{even}^{p,s}(G) \oplus \widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G]))$ (cf. []). By using the equivariant Hopf theorem (cf. [] analogously as in []), one shows that the action of $L_{even}^{h}(G)$, and hence the action of $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G]))$, is not through self-homotopy equivalences of $M^{2n-1} \times \mathbb{R}$. This, combined with the description of $\mathcal{TT}_1^{TOP}(M^{2n-1})$, shows the one-to-one correspondence between $\mathcal{TT}_2^{TOP}(M^{2n-1}) \setminus \mathcal{TT}_1^{TOP}(M^{2n-1})$ and [the nonzero elements of] $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[G]))$. $\square$

cf. should be set in italics [true for all Latin abbreviation of this type]

[end of Section 2].

Online references

Maumary: http://www.maths.ed.ac.uk/~aar/papers/ ~~surgery~~ maumary0.pdf

Taylor: DITTO except instead of papers/maumary0.pdf insert ~~taylor.pdf~~ surgery/taylor.pdf