3. Tangential thickness and normal invariants

Suppose that M and N are closed ~~pieceus~~ n-manifolds and $k \geq 3$ is such that $n+k \geq 6$. Surgery theory then yields the following criterion for $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ to be homeomorphic:

PROPOSITION 3.1. *If M and N (and k) are as above, then $M \times \mathbb{R}^k$ is homeomorphic to $N \times \mathbb{R}^k$ if and only if the compact bounded manifolds $M \times D^k$ and $N \times D^k$ are h-cobordant in the following sense: There is a compact manifold with boundary X^{n+k+1} and a compact manifold with boundary $W^{n+k} \subseteq \partial X$ such that the following hold:*

(i) ∂W^{n+k} is homeomorphic to a disjoint union of $M \times S^{k-1}$

and $N \times S^{k-1}$

(ii) $\partial X^{m+k+1} \cong M \times D^k \cup W \cup N \times D^k$, where $M \times D^k \cap W = M \times S^{k-1}$ and $N \times D^k \cap W = N \times S^{k-1}$.

(iii) The inclusion of pairs $(M \times D^k, M \times S^{k-1}) \subseteq (\partial X, W) \subseteq (X, W)$ and $(N \times D^k, N \times S^{k-1}) \subseteq (\partial X, W) \subseteq (X, W)$ are homotopy equivalences of pairs.

Sketch of proof. This is fairly standard. If $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are homeomorphic, then the homeomorphism maps $M \times D^k$ into some ~~large~~ subset $N \times r \cdot D^k$, where $r \cdot D^k$ is the disk of radius r for some very large value of r. Let W be the bounded manifold $N \times r \cdot D^k -$ Interior$(M \times D^k)$, and take X to be

$N \times D^k \times [0,1]$. The decomposition of ∂X in (iii) is then given by identifying $M \times D^k$ with $M \times D^k \times \{0\}$, W with $W \times \{0\}$, and $N \times D^k$ with $N \times rD^k \times \{1\} \cup N \times \partial(rD^k) \times [0,1]$. It is then fairly straightforward to check that the inclusions in (iii) are homotopy equivalences of pairs. Conversely, if we are given X as in the theorem, ~~then since $k \geq 3$ and $n+k \geq 6$ one can apply~~ theorem, then it follows that $X - \text{Interior}(W)$ is a proper h-cobordism from $M \times \text{Interior}(D^k) \cong M \times \mathbb{R}^k$ to $N \times \text{Interior}(D^k) \cong N \times \mathbb{R}^k$, in the sense of [ ] Siebenmann ∞ simple types, and by the proper h-cobordism theorem of [ ] it follows that $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are homeomorphic. ∎

~~This result allows us to analyze the homo-~~

COMPLEMENT 3.2. Similar results are true in the categories of piecewise linear (PL) or smooth manifolds if we stipulate that all manifolds lie in the given category and the homeomorphisms are (respectively) PL homeomorphisms or diffeomorphisms.

This is true because one has analogs of the ~~proper topological~~ proper h-cobordism theorem in the PL and smooth categories (~~indeed~~ in fact, they predate the topological version). In the smooth category there are some issues ~~involving co~~ about rounding corners in a product of two bounded smooth manifolds, but there are standard ways of addressing such points

CONNER DIFF PDIC MAPS DOUADY-HERAULT

(e.g., see [§I.3] or the treatment in [ ]). ■

SECTION SIGN

These results lead to the use of surgery-theoretic structure sets; the latter are defined for closed manifolds in [RANICKI HAUPTVERMUTUNG, pp. 17–18], and one can ~~study~~ treat the bounded case using maps and homotopy equivalences of pairs as in Chapter 10 of Wall's book [ ]. INSERT 3.5A TO 3.5D ~~Specifically, Proposition 3.1 translates into the following statement.~~

PROPOSITION 3.2. Let M and N ~~and k~~ be closed connected PL or topological manifolds, let $k \geq 3$ and $\dim M = \dim N \geq 5$, and let $f : M \to N$ be a homotopy equivalence. Then $M \times D^k$ is PL homeomorphic (resp., homeomorphic) to $N \times D^k$ if and only if the normal invariant in $[N, G/PL]$ (resp., $[N, G/Top]$) lies in the image of $[N, G_k]$ under

In order to translate Proposition 3.1 and Complement 3.2 into the language of structure sets, we need to ~~use~~ work with certain function spaces. Following ITERATED SUSPENSION James [ ], we shall denote the identity ~~of~~ component of the continuous function space $\mathcal{F}(S^{k-1}, S^{k-1})$ by SG_{k}, and SF_{k-1} will denote the subspace of base point preserving maps. By the results of [JAMES] and [~~DAVID~~ STASHEFF], there is a Serre fibration $SF_{k-1} \longrightarrow SG_k \longrightarrow S^{k-1}$ and a corresponding classifying space fibration $S^{k-1} \longrightarrow BSF_{k-1} \longrightarrow BSG_k$. The space of degree zero basepoint preserving self-maps is homeomorphic to the iterated loop space $\Omega^{k-1} S^{k-1}$ and the map $W: \Omega^{k-1} S^{k-1} \longrightarrow SF_{k-1}$ sending $f: S^{k-1} \longrightarrow S^{k-1}$ to the composite

PUT ON A SINGE LINE

$$S^{k-1} \xrightarrow{\text{PINCH}} S^{k-1} \vee S^{k-1} \xrightarrow{f \vee id} S^{k-1} \vee S^{k-1} \xrightarrow{\text{FOLD}} S^{k-1}$$

is a homotopy equivalence; it is important to note that this homotopy equivalence does not send the loop sum on $\Omega^{k-1} S^{k-1}$ to SF_{k-1}.

~~Conversely, the Hopf construction~~ [Whitehead, Elts of Homotopy Theory, Section XI.4]

The unreduced suspension functor defines continuous homomorphisms $SG_k \to SF_{k+1}$, and if $\Omega^{k-1}S^{k-1} \longrightarrow \Omega^k S^k$ is the suspension map induced by the suspension adjoint $\sigma: S^{k-1} \longrightarrow \Omega S^k$, then we have the following homotopy commutative diagram, in which $\Omega^m_0 Y$ is the path component of the constant map in $\Omega^m Y$:

(3.3)
$$\begin{tikzcd}
\Omega_0^{k-1} S^{k-1} \arrow[rr, "\Omega^{k-1}\sigma"] \arrow[d, "w_{k-1}"'] & & \Omega_0^k S^k \arrow[d, "w_k"] \\
SF_{k-1} \arrow[r] & SG_k \arrow[r] & SF_k .
\end{tikzcd}$$

The preceding chain of ~~inclusions~~ maps can be extended by adjoining SG_{k+1} on the right, and if we take limits we obtain a topological monoid that is denoted by SG or SF.

~~With this preparation, we can restate Proposition 3.1 in the piecewise linear PL~~

With this preparation, we can restate Proposition 3.1 and Complement 3.2 in the piecewise linear (PL) and topological categories as follows:

RETURN TO 3.5.

under the map induced by the composite

$G_k \longrightarrow G \longrightarrow G/PL$ (resp., $G_k \longrightarrow G \longrightarrow G/Top$).

There is an analog of this result in the smooth category, but the proof is longer and we shall not need the smooth version of Proposition 3.3.

PROOF. We begin with the case of the PL category since the argument is simpler ~~and~~ but also contains the ideas to be employed in the topological category. Given a homotopy equivalence $f : M \longrightarrow N$, we want to consider the homotopy structure on $N \times D^k$ given by the product map $f \times id(D^k)$. Standard properties of normal invariants imply that $\eta[f \times id(D^k)] = p^* \eta(f)$, where $\eta[\ldots]$ denotes the normal invariant and

$p^*: [N, G/PL] \longrightarrow [N \times D^k, G/PL]$ is induced by the coordinate projection map $p: N \times D^k \to N$; since D^k is contractible, the map p^* is an isomorphism. Since $k \geq 3$ it follows that the maps $\pi_i(M \times S^{k-1}) \longrightarrow \pi_i(M \times D^k)$ are isomorphisms for $i = 0$ or 1, and hence the π-π Theorem of [WALL BOOK, Chapter 4] implies that the normal invariant map

$$\mathcal{S}(M \times D^k) \longrightarrow [M \times D^k, G/PL]$$

is 1-1 and onto. By the Embedding Theorem ~~[ROURKE]~~ of Browder, Casson, Haefliger, Sullivan and Wall [ROURKE, (8.10), p. 161] there is a k-dimensional block bundle [ ] over N, say ξ, and a PL homeomorphism φ from its total space $E(\xi)$ to $M \times D^k$ such that

$[f \times id(D^k)] \circ \varphi$ is homotopic to the identity.

These data ~~determine~~ correspond to a unique homotopy class $\alpha \in$ $[N, G_k/\widetilde{PL}_k]$ with the following properties:

(i) The image of α in $[N, G/PL]$ under a canonical stabilization map $G_k/\widetilde{PL}_k \longrightarrow G/PL$ (which is a homotopy equivalence) is the normal invariant $\eta(f)$.

(ii) The image of α in $[N, B\widetilde{PL}_k]$ under a canonical map $G_k/\widetilde{PL}_k \longrightarrow B\widetilde{PL}_k$ classifies the block bundle ξ.

By the
~~The classifying class in $[N, B\widetilde{PL}_k]$ determined by ξ is the trivial class if and only if~~

~~the construct~~
In the theory of block bundles, a block bundle ξ

is trivial if and only if $E(\xi)$ is PL homeomorphic to $N \times D^n$. Therefore $M \times D^n$ is PL homeomorphic to $N \times D^n$ if and only if the image of α in $[N, BPL_k]$ is trivial; since the latter is true if and only if α lies in the image of the map $[N, G_k] \longrightarrow [N, G_k/PL_k]$, the result follows in the PL category.

The proof in the topological category is similar, but ~~the~~ one must replace the theory of PL block bundles with a corresponding theory of topological regular neighborhoods as in [ROURKE SANDERSON TOP NEIGHBORHOODS] ~~and [EDWARDS TOP REG NBHDS]. As noted in [RANICKI , p. 174], the some stabilization map $G_k/Top_k \to G/Top$ is a homotopy equivalence if $k \geq 3$~~

EDWARDS
TOP REG NBHDS

and [ ]. One crucial step in the PL proof uses the fact that the stabilization map $G_k/\widetilde{PL}_k \to G/PL$ is a homotopy equivalence if $k \geq 3$; the corresponding fact for the map $G/\widetilde{Top}_k \to G/Top$ is contained in [ ]. ∎

ROURKE SANDERSON
TOP NBDS

If X is a connected finite cell complex, then diagram (3.3) yields an isomorphism of sets from the stable cohomotopy group $\{X, S^0\}$ to $[X, SG]$, and under this isomorphism the image of $[X, SG_k] \to [X, SG]$ is sandwiched between the images of the iterated suspensions $[S^{k-1}X, S^{k-1}] \to \{X, S^0\}$ and $[S^kX, S^k] \to \{X, S^0\}$. The results of [ ] show that the images of $[X, SG_k] \to [X, SG]$

corresponds to the image of $[S^kX, S^k] \longrightarrow \{X, S^0\}$ if $\dim X \leq 2k-2$. We shall also need the following criteria for ~~lift~~ determining whether a class in $[X, SG]$ lifts back to $[X, SG_k]$:

PROPOSITION 3.5. *Let* X *be a connected finite complex, and let* $\alpha \in [X, SG]$ *be a class such that* α *lifts to* $[X, SG_3]$; *take the group structures on these spaces induced by the composition products on the function spaces* $\mathcal{F}(S^3, S^3)$ *and* $\lim_{m\to\infty} \mathcal{F}(S^m, S^m)$. *Then* $\alpha = \alpha_1\alpha_2$ *where* α_2 *lies in the image of* $[X, SO] \longrightarrow [X, SG]$ (*where* SO *is the group* $\lim_{m\to\infty} SO_m$) *and* α_1 *corresponds to an element in the image of*

$[S^2 X, S^2] \longrightarrow \{X, S^0\}$.

PROOF. It will suffice to show that the images of $[X, SG_3]$ and $[X, SF_2]$ in $[X, G/O]$ are equal, for this implies that the image of $[X, SG_3]$ in $[X, SG]$ is generated by $[X, SF_2]$ and $[X, SO]$, and by (3.3) the image of $[X, SF_2]$ in $[X, SG]$ corresponds to the image of $[S^2 X, S^2]$ in $\{X, S^0\}$.

We begin with the following commutative diagram whose rows are equivalently fibrations:

$$\begin{tikzcd}
SO_2 \arrow[r] \arrow[d] & SO_3 \arrow[r] \arrow[d] & S^2 \arrow[r] \arrow[d, "="] & BSO_2 \arrow[r] \arrow[d] & BSO_3 \arrow[d] \\
SF_2 \arrow[r] & SG_3 \arrow[r] & S^2 \arrow[r] & BSF_2 \arrow[r] & BSG_3
\end{tikzcd}$$

It follows that the fibers of $BSO_2 \longrightarrow BSF_2$ and $BSO_3 \longrightarrow BSG_3$, which are SF_2/SO_2 and

SG_3/SO_3, are homotopy equivalent. Since

~~SG_3~~

the map $SO_3 \to SO$ is well known to be

2-connected and $SG_3 \to SG$ is also 2-connected

by [James], it follows that $SG_3/SO_3 \to G/O$ is

also 2-connected; and therefore $\pi_1(SG_3/SO_3) \cong$

$\pi_1(G/O) \cong \{0\}$, so that SF_2/SO_2 is also simply

connected. Furthermore, since SO_2 is aspherical

it follows that the composite of the universal

covering space projection $\widetilde{SF_2} \to SF_2$ and the

canonical map $SF_2 \to SF_2/SO_2$ is a homotopy

equivalence. Thus we have shown that the images

~~of $[X, G/O$~~ ~~$[X, G_3/O_3]$~~

of $[X, SG_3]$ and $[X, \widetilde{SF_2}]$ in $[X, G/O]$ are equal.

Finally, since the image of $[X, \widetilde{SF_2}]$ lies between the images of $[X, \widetilde{SF}_2]$ and $[X, G_3]_\Lambda$, it follows that the images of all three of these groups in $[X, G/O]$ must coincide. $\boxtimes$

GO TO 3.13.

~~the image of j_*. Furthermore, since α_1 lies in the image of $[X, SF_2]$, and by (3.3) it follows that α_1 corresponds to an element in the image of $[S^2X, S^2] \longrightarrow \{X, S^0\}$.~~

PROPOSITION 3.6. *Let p be an odd prime, let $k \geq 2$, and let $\alpha \in [X, SG]$ be an element of order p^r for some $r > 0$. Then α lies in the image of $[X, SG_{2k}] \longrightarrow [X, SG]$ if and only if α corresponds to an element in the image of $[S^{2k-1}X, S^{2k-1}] \longrightarrow \{X, S^0\}$.*

PROOF. We shall work with p-localizations in the sence of Sullivan [SULLIVAN 1970 MIT]. Since connected H-spaces all have good localizations at p, it is ~~we can~~ meaningful to ~~talk~~ discuss these spaces $SG_{(p)}$, $SG_{2k(p)}$,

$SF_{2k-1(p)}$, $S^{2k-1}_{(p)}$, ~~and~~ $SO_{2k(p)}$, $\Omega^\infty_0 S^\infty_{(p)}$ and $\Omega^{2k-1}_0 S^{2k-1}_{(p)}$, where $\Omega^m_0 Y$ denotes the path component of the constant map in the iterated loop space $\Omega^m Y$. Note that if W is an twice connected H-space whose homotopy groups are all finite, then W is naturally homotopy equivalent to ~~the~~ weak product of its localization $W_{(q)}$ at all primes q; ~~since~~ in particular, this applies to the H-spaces $SG \simeq \Omega^\infty_0 S^\infty$ and ~~SG_{2k}~~ $SF_{2k-1} \simeq \Omega^{2k-1}_0 S^{2k-1}$.

Recall that we have a fibration

$$SO_{2k-1} \longrightarrow SO_{2k} \longrightarrow S^{2k-1}$$

and the tangent bundle of $T(S^{2k})$ corresponds to a map $S^{2k-1} \longrightarrow SO_{2k}$ such that the composite $S^{2k-1} \longrightarrow SO_{2k} \longrightarrow S^{2k-1}$ has degree 2.

~~Therefore the map~~
If we compose the map $S^{2k-1} \to SO_{2k}$ with the inclusion of SO_{2k} in SG_{2k}, ~~then~~ ~~the~~ and the ~~fibration~~ $SG_{2k} \to S^{2k-1}$, the resulting composite also has degree 2 and therefore the map

$$SF_{2k-1} \times S^{2k-1} \longrightarrow SG_{2k} \times SG_{2k} \xrightarrow{\text{mult}} SG_{2k}$$

becomes a homotopy equivalence when localized at the odd prime p. Since $S^{2k-1} \to SO_{2k} \to SO$ is nullhomotopic, it follows that the image of $[X, SG_{2k\,(p)}]$ in $[X, SG_{(p)}] \cong [X, SG]_{(p)}$ is equal to the image of $[X, SF_{2k-1\,(p)}]$, which corresponds to the image of $[S^{2k-1}X, S^{2k}_{(p)}] \cong [S^{2k-1}X, S^{2k-1}]_{(p)}$ in $\{X, S^0\}_{(p)}$; note that the ~~latter image~~ codomain is the Sylow p-subgroup of $\{X, S^0\}$ with respect to the loop sum, and likewise the domain is the Sylow p-subgroup of $[S^{2k-1}X, S^{2k-1}]$. ∎ These observations imply that if $\alpha \in [X, SG]$ is p-primary with respect to the composition product (which is homotopy abelian) and lifts to $[X, SG_{2n}]$, then α corresponds to an element of $\{X, S^0\}$ which desuspends to $[S^{2k-1}X, S^{2k-1}]$. ∎

We shall also need the following result:

PROPOSITION 3.7. *If $\alpha \in [X, SG]$ has odd order and lies in the image of $[X, SG_3]$, then* ~~α is trivial.~~ *the image of α in $[X, G/O]$ is trivial.*

insert 3.16A

PROOF. By the proof of Proposition 3.5 we may assume that α lies in the image of $[X, \widetilde{SF}_2]$, and since the homotopy groups of $\widetilde{SF}_2$ and $\Omega^2_0 S^{2n}$ are finite, we can say

Since the finite abelian group $[X, SG]$ splits into a product of the groups $[X, SG]_{(q)}$ where q runs through all odd primes, it will suffice to prove the result when the order of α is a power of some odd prime p.

NEW ¶ — RETURN.

that α corresponds to a class in $\{X, S^0\}_{(p)}$ which lies in the image of $[S^2X, S^2]_{(p)}$.

If $h : S^3 \to S^2$ is the Hopf map, whose fiber is S^1, then composition with h defines a homotopy equivalence from $\Omega_0^2 S^3$ to $\Omega_0^2 S^2$, where, as before, $\Omega_0^m Y$ denotes the arc component of the constant map in $\Omega^m Y$. Therefore it follows that α corresponds to a class in $\{X, S^0\}_{(p)}$ which lies in the image of the composite

$$h_* : [S^2X, S^3]_{(p)} \longrightarrow [S^2X, S^2]_{(p)}$$

and hence α ~~may be written~~ factors homotopically as a composite $\bar{h} \circ \beta$, where β lies in $\{X, S^1\}_{(p)}$ and $\bar{h}$ denotes the image of h in the stable group $\{S^3, S^2\}_{(p)} \cong \{S^1, S^0\}_{(p)}$.

But since p is odd, the group $\{S^1, S^0\}_{(p)}$ is trivial, and therefore α corresponds to the trivial element of $\{X, S^0\}$. ∎

Finally, since $\{S^1, S^0\} \cong \mathbb{Z}_2$ ~~at~~ we have $\{S^1, S^0\}_{(p)} = \{0\}$, and hence α corresponds to the trivial element of $\{X, S^0\}_{(p)}$, ~~which~~ where we ~~may view~~ interpret the latter as a subgroup of $\{X, S^0\}$. ∎

COROLLARY 3.8. <u>The same conclusion holds if we replace</u> G/O <u>by</u> G/PL or G/Top. ∎