4. Tangential equivalence of lens spaces

Throughout this section p will denote a fixed odd prime.

We have already mentioned Folkman's ~~result, which states that if two lens spaces of the same sufficiently large dimension are tangentially homotopy equivalent, then they are diffeomorphic.~~ result on tangentially homotopy equivalent lens spaces of sufficiently high dimension. In this section we shall derive a stronger version of Folkman's conclusion when the fundamental groups of the lens spaces are isomorphic to $\mathbb{Z}_p$:

PROPOSITION 4.1. *Let M and N be $(2k-1)$-dimensional lens spaces that are (stably) tangentially homotopy equivalent, and assume* *that* $k \geq p-1$. *Then* M *and* N *are diffeomorphic.*

PROOF. The result of Folkman [ ] ~~show~~ yield that conclusion when $k \geq 2p-1$, so it is only necessary to prove the result when $k \leq 2(p-1)$, so that $(p-1) \leq k \leq 2(p-1)$.

Let V and W be free representations of $\mathbb{Z}_p$ such that the associated lens spaces $L(V)$ and $L(W)$ are homotopy equivalent; assume that V and W are at least 4 dimensional. The free $\mathbb{Z}_p$-actions on the universal coverings $S(V)$ and $S(W)$ determine isomorphisms λ_V and λ_W from $\pi_1(L(V))$ and $\pi_1(L(W))$ to $\mathbb{Z}_p$, and if h is a homotopy equivalence from $L(V)$ to $L(W)$ then we obtain an automorphism $\beta = \lambda_W h_* \lambda_V^{-1}$ of $\mathbb{Z}_p$. If $\tilde{h} : S(V) \to S(W)$ is the associated map of universal covering spaces, then $\tilde{h}$ satisfies the semi-equivariance ~~condition~~ identity

$$\tilde{h}(g\cdot v) = \beta(g)\cdot\tilde{h}(v)$$

and if we define a new representation V' with the same underlying vector space as V and a group action given by $g * v = \beta^{-1}(g)\cdot v$, then

~~the map $\tilde{h}: S(V') \to S(W)$ [illegible]~~

we may view $\tilde{h}$ as a $\mathbb{Z}_p$-equivariant homotopy equivalence from $S(V')$, since $L(V') = L(V)$, which equals $S(V)$ as a set, to $S(W)$. This means that we might as well assume the representations V and W are chosen so that β is the identity and $\tilde{h}$ is equivariant. This is important for computational purposes because it yields the following commutative diagram, where the map from $RO(G)$ to $KO_G(X)$ sends a representation V to the class of the trivial $\mathbb{Z}_p$-vector bundle $[V] = X \times V$:

$$\begin{tikzcd}
RO(\mathbb{Z}_p) \arrow[r] \arrow[d, "="] & KO_{\mathbb{Z}_p}(S(W)) \arrow[r, "\cong"] \arrow[d, "\tilde{h}^*", "\cong"'] & KO(L(W)) \arrow[d, "h^*"] \\
RO(\mathbb{Z}_p) \arrow[r] & KO_{\mathbb{Z}_p}(S(V)) \arrow[r, "\cong"] & KO(L(V)).
\end{tikzcd}$$

The horizontal arrows on the right are the standard isomorphisms $KO_G(X) \cong KO(X/G)$ for a free G-space X.

The KO-groups in the diagram are given by results of T. Kambe [ ]; in that paper the complex K-groups are computed, but one can extract ~~the~~ computations for KO because the reduced KO groups of $L(V)$ and $L(W)$ are finite p-primary abelian groups, which implies that $\widetilde{KO}(L(V))$ and $\widetilde{KO}(L(W))$ are isomorphic to the self-conjugate elements in $\widetilde{K}(L(V))$ and $\widetilde{K}(L(W))$. For our purposes, the most important aspects of the computations are that the maps from $RO(\mathbb{Z}_p)$ to $KO(L(V))$ and $KO(L(W))$ are onto, and if $A \subseteq RO(\mathbb{Z}_p)$ denotes the common kernel of these maps, then A is contained in $p \cdot RO(\mathbb{Z}_p) \subseteq RO(\mathbb{Z}_p)$. Hence we can adjoin the following commutative square to the preceding diagram, and in the expanded diagram the composites $RO(\mathbb{Z}_p) \longrightarrow KO(L(V)) \longrightarrow RO(\mathbb{Z}_p) \otimes \mathbb{Z}_p$, $RO(\mathbb{Z}_p) \longrightarrow KO(L(W)) \longrightarrow RO(\mathbb{Z}_p)$ are the canonical maps induced by the ~~projection~~ mod p reduction map $\mathbb{Z} \longrightarrow \mathbb{Z}_p$.

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~~Let r is a primitive root of unity in $\mathbb{Z}_{p^2}$, then theorem results of J.F. Adams [ ]~~

The preceding discussion yields the following consequences. Since the stable tangent bundles of $L(V)$ and $L(W)$ are the images of V and W under the maps from $\widetilde{RO}(\mathbb{Z}_p)$ to $\widetilde{KO}(L(V))$ and $\widetilde{KO}(L(W))$, the tangential homotopy equivalence h yields the condition $V-W \in A$, and hence it follows that $V-W \in p\cdot\widetilde{RO}(\mathbb{Z}_p)$.

On the other hand, since h is a homotopy equivalence the bunde $[V]-[W]$ is (stably) fiber homotopically trivial; since $\widetilde{KO}(L(V))$ and $\widetilde{KO}(L(W))$ are p-primary results of Adams [ ] imply that $[V]-[W]$ lies in the kernel of $\psi^r - 1$, where ψ^r is the r-th Adams operation. Since the imply that an element of $\widetilde{KO}(L(V))$ or $\widetilde{KO}(L(W))$ represents a stably fiber homotopically trivial bundle if and only if it lies in the kernel of $\psi^r - id$, where ψ^r is the usual Adams operation. Since the nontrivial irreducible representations of $\mathbb{Z}_p$ are 2-dimensional and given by the 1-dimensional unitary representations ρ_k sending (g, z) to $g^k z$, where $1 \le k \le \frac{1}{2}(p-1)$, it follows that ψ^r sends ρ_k to ρ_{kr} (where ρ_{kr} is computed mod p and $\rho_j = \rho_{-j}$). Therefore the action of ψ^r on $KO(L(V))$ and $KO(L(W))$ pass to the self map of $RO(\mathbb{Z}_p) \otimes \mathbb{Z}_p$ sending ρ_k to ρ_{kr} for each k. Since r is a primitive root of unity mod p^2, it follows that the images of the stably fiber homotopically trivial elements in

$RO(\mathbb{Z}_p)/RO(1)$ are precisely the linear combinations of the form $m \cdot \sum \rho_j$, where the summation is over all j such that $1 \le j \le \frac{1}{2}(p-1)$.

Express V and W as direct sums of irreducible representations $\sum a_j \rho_j$ and $\sum b_j \rho_j$, where $a_j, b_j \geq 0$. Since $\dim V = \dim W \le 4p-4$, it follows that $\sum a_j = \sum b_j \le 2p-2$. By the preceding discussions we know that $a_j - b_j \equiv 0\ (p)$ for all j and that $a_j \equiv a_1 \equiv b_1 \equiv b_j\ (p)$ for all j. The final step is to show ~~thate~~ there are very few sequences a_j, b_j satisfying all these conditions unless $a_j = b_j$ for all j, and in all such cases the or ~~all but one~~ ~~there are~~ lens spaces $L(V)$ and $L(W)$ are diffeomorphic.
~~j_1 and j_2 such that $a_{j_1} = b_{j_2} = 4p$ and $a_j = 0$,~~
~~$b_j = 0$ otherwise.~~

Suppose that $\{a_j\}$ and $\{b_j\}$ satisfy all the conditions given above. Since $\sum a_j = \sum b_j \leq 2p-2$, there is at most one j_1 such that $a_{j_1} \geq p$ and at most one j_2 such that $b_{j_2} \geq p$. Since $a_j \equiv b_j\ (p)$, this ~~forces the cor~~ implies that either $a_j = b_j$ for all j or else we have $a_{j_1} - b_{j_1} = p$, $b_{j_2} - a_{j_2} = p$, and $a_j = b_j$ otherwise. Furthermore, since $a_j \equiv a_{j_1}$ (p) and $b_j \equiv b_{j_2}\ (p)$ for all j, it follows that there ~~are constants~~ is nonnegative some $c \geq 0$ such that $c < p$, $c = b_{j_1} = a_{j_2}$, and $a_j = b_j = c$ for $j \neq j_1, j_2$. It follows that

$$\sum a_j = \sum b_j = \frac{c(p-1)}{2} + p.$$

Twice this the number is equal to $\dim V = \dim W$, and therefore the inequality $\dim V = \dim W \leq 4p-4$ implies that $c = 0$ or $c = 1$. Also since the right hand side is $\geq p$, it follows that we have eliminated the case where $\dim V = \dim W = 2(p-1)$; in other words, $a_j = b_j$ in this case, so that $L(V)$ is diffeomorphic to $L(W)$.

We shall extend the definitions of a_j and b_j to all nonzero elements of $\mathbb{Z}_p$ by setting $a_{-j} = a_j$ and $b_{-j} = b_j$. With these conventions the complexifications of V and W are given by

$$V \otimes \mathbb{C} = \sum a_j t^j, \quad W \otimes \mathbb{C} = \sum b_j t^j$$

where t^j is the 1-dimensional unitary representation sending (z, v) to $z^j v$. Note that V and W are equivalent orthogonal representations if and only if $V \otimes \mathbb{C}$ and $W \otimes \mathbb{C}$ are equivalent unitary representations. The standard diffeomorphism criterion for lens spaces

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(see [ ]) implies that $L(V)$ and $L(W)$ are diffeomorphic if there is some $s \neq 0$ in $\mathbb{Z}_p$ such that $a_j = b_{sj}$ for all j.

Let $c = 0$ or 1, and assume that V and W satisfy the previous conditions on the coefficients a_j and b_j. Then $b_{j2} = a_{j1} = p+c$ implies that $b_{-j2} = a_{-j1} = p+c$; if we choose s to be the unique element of $\mathbb{Z}_p$ such that $j_2 = s j_1$, then it follows that

$$a_{j1} = b_{sj1} = p+c = b_{-sj1} = a_{-j1}$$

and also

$$a_j = b_{sj} = c, \quad \text{where } j \neq \pm j_1.$$

Therefore the diffeomorphism criterion in the previous paragraph implies that $L(V)$ is diffeomorphic to $L(W)$. ∎

PROOF OF THEOREM 2. All we need to do is check that the lens space diffeomorphisms are isometries Riemannian. This fact is contained in Folkman's work if $n > 2(p-1)$. In the remaining cases, the discussion ~~of~~ in the second paragraph of the proof for 4.1 shows that it suffices to consider cases where the map $L(V) \to L(W)$ lifts to a $\mathbb{Z}_p$-equivariant map $S(V) \to S(W)$, and, in the diffeomorphism criterion from [COHEN], the sufficient condition in fact implies that the lens spaces are isometric (see [COHEN,]). ∎

PROOF OF THEOREM 1. If $\dim M = \dim N \geq 2p-1$, then M and N are diffeomorphic by Theorem 2, so it is only necessary to consider cases where the dimension $2n-1$ is $\leq 2p-3$. Given f as in the Theorem, the normal invariant ~~of~~ $\eta(f)$ of f lies in $[N, G/O]$, and by the π-π Theorem the map $f \times id_{D^3}$ is h-cobordant to a diffeomorphism if and only if $\eta(f)$ is trivial. $\boxed{\text{INSERT 4.14A}}$

Since f is tangential, the normal invariant lifts back to $[N, SG] \cong \{N, S^0\}$. ~~Furthermore, since the universal coverings of M and N are both diffeomorphic to S^{2n-1}, it follows that the pullback of $\eta(f)$ under the universal covering map $S^{2n-1} \to N$ is trivial.~~

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Consider the Atiyah-Hirzebruch spectral sequence for $\{N, S^0\}$. Its E_2 terms have the form $\tilde{H}^i(N; \pi_i)$ where π_i is the i-th stable stem $\pi_i(SG)$. These

Therefore the analog of Proposition 3.1 in the smooth category will imply that $f \times id_{\mathbb{R}^3}$ is properly homotopic to a diffeomorphism if and only if $\eta(f)$ is trivial, and accordingly it will suffice to prove the latter.

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groups contain no p-torsion if $i < 2p-3$ [ ],
and since $\tilde{H}_i(N;\mathbb{Z}) \cong \mathbb{Z}_p$ (i odd) or 0 (i even) when
$i \leq 2n-2$, it follows that $\tilde{H}^i(N;\pi_i) = 0$
if $i \leq 2n-2$ and $\tilde{H}^i(N;\pi_i) \cong \pi_i$. This
means that the degree 1 collapsing map from N to S^{2n-1}
induces an isomorphism from π_{2n-1} to $\{N, S^0\}$.
But the composite $S^{2n-1} \longrightarrow N \longrightarrow S^{2n-1}$ has
degree p, so that the composite ~~map~~ $\pi_{2n-1} \longrightarrow \{N, S^0\} \longrightarrow \pi_{2n-1}$
is also multiplication by p.

If $n \leq p-2$, then π_{2n-1} has order
prime to p and hence the map $\{N, S^0\} \longrightarrow \pi_{2n-1}$ is
an isomorphism. Consider now the following
commutative diagram:

$$\begin{tikzcd}
\pi_{2n-1} \arrow[r] \arrow[d] & \{N, S^0\} \arrow[r] \arrow[d] & \pi_{2n-1} \arrow[d] \\
\pi_{2n-1}(G/O) \arrow[r] & {[N, G/O]} \arrow[r] & \pi_{2n-1}(G/O)
\end{tikzcd}$$

We already know that $\eta(f)$ lifts to a class x_0 in $\{N, S^0\}$ and hence lifts to a class x in π_{2n}. If y denotes the image of x in $\pi_{2n-1}(G/O)$, then y maps to $\eta(f)$. We also know that $\eta(f)$ maps to zero in $\pi_{2n-1}(G/O)$, and from this we may conclude that $p\,y = 0$ in $\pi_{2n-1}(G/O)$.

~~Since the ord~~

Now $\pi_{2n-1}(G/O)$ has no nontrivial elements of order p, and therefore $y = 0$ so that $\eta(f) = 0$ and we are done if $n \leq p-1$.

We are left with the case $n = p-1$, in which $\pi_{2n-1} = \pi_{2p-3}$ is the direct sum of $\mathbb{Z}_p$ and a group of order prime to p; we also know that the p-torsion maps to zero in $\pi_{2p-3}(G/O)$. Therefore, if we consider the same diagram as before we again obtain a class $y \in \pi_{2p-3}(G/O)$ which maps to $\eta(\xi)$, but in this case we can conclude that the order of y is prime to p, regardless of whether or not p divides the order of x. One can now reason as before to conclude that y and $\eta(f)$ must be trivial. ■

NOTE. A closer examination of results due to J. Ewing, S. Moolgavkar, R. Stong and L. Smith [EMSS] shows that for each $n \geq 2$ there are infinitely many primes p for which one has homotopy equivalent but nondiffeomorphic lens spaces which are stably parallelizable, so ~~that~~ for each n there are many examples of ~~distinct~~ nondiffeomorphic lens spaces M, N such that $M \times \mathbb{R}^3$ and $N \times \mathbb{R}^3$ are diffeomorphic [ ].

Do we want a reference or should we insert the construction in this paper?