5. Normal invariants for tangential homotopy lens spaces

INSERT 5.1A

If $f: M \to N$ is a homotopy equivalence of compact ~~closed~~ topological manifolds (possibly with boundary) and $\eta(f) \in [N, G/Top]$ is its normal invariant, then f is a tangential homotopy equivalence if and only if a canonical map from $[N, G/Top]$ to $[N, BSTop]$ sends $\eta(f)$ to zero (e.g., see [MADSEN-TAYLOR-WILLIAMS]), and by the exactness of the fibration sequence $SG \longrightarrow G/Top \longrightarrow BSTop$ the image vanishes if and only if $\eta(f)$ lies in the image of the associated map from $[N, SG]$ to $[N, G/Top]$. In this section we shall describe this image when N is a $\mathbb{Z}_p$ lens space.

Throughout this section p will denote a fixed odd prime.

RETURN

Our analysis is based upon fundamental ~~[illegible]~~
results on the structure of the localized spaces
$G/O_{(p)}$ $BSO_{(p)}$,
$SG_{(p)}$, $G/Top_{(p)}$, $BSTop_{(p)}$ and similar objects;
MADSEN MILGRAM Chapter V
some ~~one~~ basic references are ~~is~~ [ ], and ~~we shall~~
MAY, E_∞ RING.. of [ ], and
~~also need some results from~~ Lecture 4 of
ADAMS – GEN. COHOMOLOGY
[ ]. ¶We are particularly interested in
the structure of $SG_{(p)}$ and $G/Top_{(p)}$. Results
of Sullivan (compare [MM]) imply that
the localized spaces $BSO_{(p)}$ and $G/Top_{(p)}$ are
homotopy equivalent and that $G/O_{(p)}$ is
homotopy equivalent to ~~the~~ $BSO_{(p)} \times Cok J_{(p)}$
for some space $Cok J_{(p)}$ (see [MM] for the
definition of the latter). ~~Let~~ Furthermore, if
J_p is defined as the fiber of $\psi^k - 1 : BSO_{(p)} \to BSO_{(p)}$, where

r is a primitive root of unity mod p^2 and ψ^r is the Adams operation in K-theory, then there is a homotopy equivalence from $SG_{(p)}$ to $J_p \times \mathrm{Cok}\, J_{(p)}$ such that the following diagram is (homotopy) commutative:

$$\begin{array}{ccccc} SG_{(p)} & \longrightarrow & G/O_{(p)} & \longrightarrow & G/\mathrm{Top}_{(p)} \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ J_p \times \mathrm{Cok}\, J_{(p)} & \xrightarrow{\beta\times 1} & BSO_{(p)} \times \mathrm{Cok}\, J_{(p)} & \xrightarrow{\varphi} & BSO_{(p)} \end{array}$$

In this diagram $\beta : J_p \to BSO_{(p)}$ is the homotopy fiber of $\psi^r - 1$ and φ factors as ~~a composition~~ follows:

$$BSO_{(p)} \times \mathrm{Cok}\, J_{(p)} \xrightarrow{\text{proj}} BSO_{(p)} \xrightarrow{\varphi'} BSO_{(p)}$$

Since $[N, \cancel{\mathrm{Cok}}\, J_p]$ ~~~~ can often be computed fairly directly but $[N, \mathrm{Cok}\, J_{(p)}]$ generally cannot, ~~it follows that~~ these splittings are often ~~~~ very helpful for describing the image of $[N, SG_{(p)}]$ in $[N, G/Top_{(p)}]$.

In addition to the splittings described above, there are also splittings of $BSO_{(p)}$ that will be useful in this section. The results of [, Lecture 4] (compare [MADSEN MILGRAM, Lemma 5.8, p.104] and [,] PETERSON) imply

that ~~complex K theory~~ localized ~~splits~~ $K_{(p)}$ splits into a sum

that the localized complex K-theory spectrum $K_{(p)}$ splits into a sum of $(p-1)$ periodic spectra $E_\alpha K_{(p)}$, where α runs through the elements of $\mathbb{Z}_{p-1}$. Each of these spectra is periodic of period $2p-2$, and the coefficient groups $E_\alpha K_{(p)}(S^n)$ are given by $\mathbb{Z}_{(p)}$ if $n \equiv 2\alpha \mod 2p-2$ and zero otherwise.

If we view the localized real K theory spectrum $KO_{(p)}$ as the direct summand given by the self-conjugate part of $K_{(p)}$, then $KO_{(p)}$ corresponds to the sum of the spectra $E_\beta K_{(p)}$, where ~~[illegible]~~ β runs through all the even elements of $\mathbb{Z}_{p-1}$. We shall be particularly interested in $E_0 K_{(p)}$, which is just ~~also called~~ the first (p-local) Morava K-theory $K(1)$ with $K(1)(S^n) = \mathbb{Z}_{(p)}$ if $n \equiv 0 \bmod 2p-2$ and 0 otherwise (see [WÜRGLER] for background on Morava K-theories).

If L is a $\mathbb{Z}_{(p)}$-lens space, then there is a canonical map k_L from L to the classifying space $B\mathbb{Z}_{(p)}$ and our analysis of the image of $[L, SG] \to [L, G/Top]$ begins with a study of the analogous problem with $B\mathbb{Z}_p$ replacing L. Both $[B\mathbb{Z}_p, SG]$ $\cong \{B\mathbb{Z}_p, S^0\}$ and $[B\mathbb{Z}_p, BSO] \cong \widetilde{KO}(B\mathbb{Z}_p)$ are well understood; results of D.W. Anderson [ ] imply that the latter is

is isomorphic to the completion of the ideal $IO(\mathbb{Z}_p)$ in the real representation ring $RO(\mathbb{Z}_p)$ given by all 0-dimensional virtual representations (this also follows directly from [ATIYAH]), while the proof of the Segal Conjecture for $\mathbb{Z}_p$ (see [RAVENEL, SEGAL CONJ] and [ADAMS-GUNAWARDENA-MILLER]) implies that $\{B\mathbb{Z}_p, S^0\}$ is isomorphic to the completion of the ideal $IA(\mathbb{Z}_p)$ in the Burnside ring $A(\mathbb{Z}_p)$ given by all virtual finite $\mathbb{Z}_p$-sets with virtual cardinality 0. Although the set theoretic isomorphism from $\{B\mathbb{Z}_p, S^0\}$ to

$[B\mathbb{Z}_p, SG]$ is not additive, one can prove that the latter is also isomorphic to the completion of $IA(\mathbb{Z}_p)$ using the methods of [ATIYAH-SEGAL, EXP ISOS]; and this is explained in [LAITINEN]; INSERT 5.7A. It is fairly straightforward to prove that the completion $IA(\mathbb{Z}_p)^\wedge$ is topologically and additively isomorphic to the additive p-adic integers $\widehat{\mathbb{Z}_{(p)}}$ and $IO(\mathbb{Z}_p)^\wedge$ is similarly isomorphic to a sum of $\frac{1}{2}(p-1)$ copies of $\widehat{\mathbb{Z}_{(p)}}$. One can describe these groups and their interrelationships more precisely as follows:

~~PROPOSITION 5.1. The complex K-group $\tilde{K}(B\mathbb{Z}_p) \cong \hat{I}(\mathbb{Z}_p)$, is a free $\hat{\mathbb{Z}}_{(p)}$-module on $(p-1)$ generators e_a, where a runs through the nonzero elements of $\mathbb{Z}_p$, and the complexification~~

~~given by all~~

the ideal $IA(\mathbb{Z}_p)$ is infinite cyclic, and it turns out that the image of its generator in the completed ideal $IA(\mathbb{Z}_p)^\wedge \cong \{B\mathbb{Z}_p, S^0\}$ corresponds to the reduced stable homotopy-theoretic transfer $B\mathbb{Z}_p \longrightarrow S^0$ associated to the (see [ ]) KAHN-PRIDDY standard p-fold covering $E\mathbb{Z}_p \longrightarrow B\mathbb{Z}_p$, whose total space is contractible.

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the complex representation ring

Let $I(\mathbb{Z}_p)$ be the ideal in $R(\mathbb{Z}_p)$ given by the kernel of the virtual dimension map from $R(\mathbb{Z}_p)$ to $\mathbb{Z}$. Then $\tilde{K}(B\mathbb{Z}_p)$ is isomorphic to the completion $I(\mathbb{Z}_p)^\wedge$ by [ATIYAH], and hence it is a free $\hat{\mathbb{Z}}_{(p)}$-module on $(p-1)$ generators.

[1] We can choose these free generators to have the form $\underset{\sim}{e}_a$, where a runs through the nonzero elements of $\mathbb{Z}_p$, and if r is a primitive root of unity mod p^2 then the Adams operation ψ^r on $\tilde{K}(B\mathbb{Z}_p)$ sends $\underset{\sim}{e}_a$ to $\underset{\sim}{e}_{ra}$; furthermore, if θ is the additive automorphism of $\mathbb{Z}_p$ sending $a \in \mathbb{Z}_p$ to ra, ~~then the map~~ and $B\theta$ is ~~the~~ an induced self-map of $B\mathbb{Z}_p$ which induces θ on the fundamental group level (so $B\theta$ is unique up to

$\underset{\sim}{e}$ in BOLDFACE

homotopy), then the induced automorphism $B\varphi^*$ in K-theory also sends e_a to e_{ra}. Furthermore, the ~~under~~ complexification map from $\widetilde{KO}(B\mathbb{Z}_p)$ to $\widetilde{K}(B\mathbb{Z}_p)$ is split injective, and its image is the free submodule whose generators have the form $e_a + e_{-a}$, where a runs through all nonzero elements of $\mathbb{Z}_p$ (note that there are $\frac{1}{2}(p-1)$ elements of this form). With this background, we can describe ~~the map~~ a canonical homomorphism from $[B\mathbb{Z}_p, SG]$ to $[B\mathbb{Z}_p, BSO]$ as follows:

PROPOSITION 5.1. *Let $F : SG_{(p)} \longrightarrow BSO_{(p)}$ be the composite*

$$SG_{(p)} \longrightarrow G/O_{(p)} \simeq BSO_{(p)} \times \mathrm{Coker}\, J_p \longrightarrow BSO_{(p)}$$

where the final arrow is coordinate projection.

Then the image of $[B\mathbb{Z}_p, SG] \cong [B\mathbb{Z}_p, SG_{(p)}]$ in $[B\mathbb{Z}_p, BSO] \cong [B\mathbb{Z}_p, BSO_{(p)}]$ corresponds to the split free submodule of $\tilde{K}(B\mathbb{Z}_p) \cong \bigoplus^{p-1} \hat{\mathbb{Z}}_{(p)}$ generated by the sum of the basic elements $\sum_a e_a$; ~~Under the isomorphism~~ $\widetilde{KO}(B\mathbb{Z}_p)$ ~~...~~ and the image corresponds to the direct summand $E_0 K_{(p)}(B\mathbb{Z}_p)$ in $KO(B\mathbb{Z}_p) \cong KO_{(p)}(B\mathbb{Z}_p)$.

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PROOF. If $V: G/O \to BSO$ is the homotopy fiber of $BSO \to BSG$, then by construction the composite

$$BSO_{(p)} \xrightarrow{\text{SLICE}} BSO_{(p)} \times \mathrm{Cok}\, J_p \simeq G/O_{(p)} \xrightarrow{V_{(p)}} BSO_{(p)}$$

is given by $\Psi^r - 1$. Now $V_{(p)}$ is trivial on $\mathrm{Cok}\, J_p$, and for all connected CW complexes the image of $[X, SG_{(p)}]$ in $[X, G/O_{(p)}]$ will be the kernel of the map $V_{(p)*} : [X, G/O_{(p)}] \longrightarrow [X, BSO_{(p)}]$; if we combine these we see that the kernel of $V_{(p)*}$ ~~is generated~~ ~~consists of~~ is generated by $[X, \mathrm{Cok}\, J_p]$ and the kernel of $\Psi^r - 1$ on $\widetilde{KO}_{(p)}(X)$. If we let $X = B\mathbb{Z}_p$, then the localized and unlocalized groups are isomorphic, and if we expand an element ξ of $\widetilde{KO}_{(p)}(B\mathbb{Z}_p)$ as $\sum c_a e_a$ for suitable coefficients c_a (note that $c_a = c_{-a}$), then ~~Ψ^r~~ ξ lies in the kernel of $\Psi^r - 1$ if and only if $c_{ra} = c_a$ for all a. We claim this happens if and only if the coefficients c_a are all equal. Sufficiency is obvious; on the other hand, ~~since we~~ it follows by induction that $c_{r^k a} = c_a$ for all k and a, and since the powers of r^k exhaust the nonzero elements of

$\mathbb{Z}_p$ we must have $c_a = c_b$ for all $a, b \neq 0$.

If we now denote the image of $[B\mathbb{Z}_p, SG_{\dots}]$ in $\widetilde{KO}(B\mathbb{Z}_p)$, the preceding discussion shows that M is a direct summand of $\widetilde{KO}(B\mathbb{Z}_p)$ which is isomorphic to $\mathbb{Z}_{(p)}^{\wedge}$ and the complementary summand $M^{\perp}$ is a free $\mathbb{Z}_{(p)}^{\wedge}$-module on $(p-2)$ generators. In particular $M^{\perp} \cong \widetilde{KO}(B\mathbb{Z}_p)/M$ is torsion free.

CLAIM: M is contained in the summand $E_0 K_{(p)}(B\mathbb{Z}_p)$. — To see this, let $E_0^{\perp} K_{(p)}$ denote the sum of the other cohomology theories $E_i K_{(p)}$, and let $\overline{M}$ denote the projection of M onto $E_0^{\perp} K_{(p)}$ with respect to the splitting $\widetilde{KO}(B\mathbb{Z}_p) \cong E_0 K_{(p)}(B\mathbb{Z}_p) \oplus E_0^{\perp} K_{(p)}(B\mathbb{Z}_p)$.

We know that $\Psi^r - 1$ restricted to M is trivial, but we also know that $\Psi^r - 1$ restricted to $E_0^{\perp} K_{(p)}(B\mathbb{Z}_p)$ is injective (compare [MM $\to$ Lemma 5.8, p. 904]), and these combine to imply that $\overline{M}$ is trivial, so that M must be contained in $E_0 K_{(p)}(B\mathbb{Z}_p)$.

The results of [KAMBE] imply that the summand $E_0 K_{(p)}(B\mathbb{Z}_p)$ of $\tilde{K}(B\mathbb{Z}_p)$ must also be isomorphic to $\mathbb{Z}_{(p)}\hat{}$, and the complementary summand $E_0^{\perp} K_{(p)}(B\mathbb{Z}_p)$ must be torsion free. Therefore the quotient $\widetilde{KO}(B\mathbb{Z}_p)/M$ is isomorphic the direct sum of $E_0^{\perp} K_{(p)}(B\mathbb{Z}_p)$ and a quotient $M_1 \cong \mathbb{Z}_{(p)}\hat{}/M$, where M is also isomorphic to $\mathbb{Z}_{(p)}\hat{}$. If M is a proper subgroup of $\mathbb{Z}_{(p)}\hat{}$, then the quotient M_1 must be a nontrivial finite cyclic p-group, ~~since~~ and therefore the quotient M_1 has nontrivial elements of finite order. Since M is a direct summand of $\widetilde{KO}(B\mathbb{Z}_{(p)}^{\wedge})$ and the latter is a direct sum of $\frac{1}{2}(p-1)$ copies of $\mathbb{Z}_{(p)}^{\wedge}$, this cannot happen and hence we must have $M = E_0K_{(p)}(B\mathbb{Z}_p)$. ■

We also have a similar conclusion regarding the image of $[B\mathbb{Z}_p, SG]$ in $[B\mathbb{Z}_p, G/TOP] \cong [B\mathbb{Z}_p, G/TOP_{(p)}]$; as noted before, results of Sullivan show the latter is isomorphic to $\widetilde{KO}(B\mathbb{Z}_p)$.

PROPOSITION 5.2. *The image of the composite*

$$[B\mathbb{Z}_p, SG] \longrightarrow [B\mathbb{Z}_p, G/TOP] \cong \widetilde{KO}(B\mathbb{Z}_p)$$

is the image of $E_0K_{(p)}(B\mathbb{Z}_p)$, and the kernel of this map is trivial.

PROOF. To prove the first statement, we need to analyze the image of the ~~map following composite~~ map

$$\cancel{[B\mathbb{Z}_p, BSO_{(p)}]}$$

$$E_0 K_{(p)}(B\mathbb{Z}_p) \to \widetilde{KO}(B\mathbb{Z}_p) \subseteq [B\mathbb{Z}_p, G/O_{(p)}] \to [B\mathbb{Z}_p, G/Top_{(p)}] \cong \widetilde{KO}(B\mathbb{Z}_p).$$

On one line

and the composite of this ~~latter~~ map with the splitting retraction $\widetilde{KO}(B\mathbb{Z}_p) \to E_0 K_{(p)}(B\mathbb{Z}_p)$. We claim this ~~map~~ composite is an isomorphism. Since the composite is given by a natural transformation of cohomology theories, it suffices to show that this transformation induces an isomorphism of cohomology theories, and the latter in turn reduces to showing that the induced self maps of the localized homotopy groups $\pi_{2k(p-1)}(BSO)_{(p)}$ are isomorphisms.

We have the following commutative diagram, in which each group except $\pi_{2k(p-1)}(BJ_p)$ is ~~the~~ a direct sum of $\mathbb{Z}_{(p)}$ and a finite abelian p-group:

$$\begin{tikzcd}
\pi_{2k(p-1)}(BSO)_{(p)} \arrow[r, "\text{splitting}", "\text{injection}"'] \arrow[d] & \pi_{2k(p-1)}(G/O)_{(p)} \arrow[r, "\beta_*^O"] \arrow[d] & \pi_{2k(p-1)}(BSO)_{(p)} \arrow[d] \\
\pi_{2k(p-1)}(BSO)_{(p)} \arrow[r, "\cong", "\text{Sullivan}"'] & \pi_{2k(p-1)}(G/Top)_{(p)} \arrow[r, "\beta_*^{Top}"] & \pi_{2k(p-1)}(BSTop)_{(p)}
\end{tikzcd}$$

CONTINUE DIAGRAM ON SAME LINE

$$\begin{tikzcd}
{} \arrow[r, "e_*", "\text{onto}"'] & \pi_{2k(p-1)}(J_p) \arrow[d, "="] \\
{} \arrow[r] & \pi_{2k-1}(J_p)
\end{tikzcd}$$

As noted in [MM, p. 117], modulo torsion the two horizontal arrows on the left are multiplication by

. The maps β_*^O and β_*^{Top} ~~and~~ give the underlying bundles, and the first line ~~last three terms~~

~~in the first line exact at the second~~

5.17

is exact β_*^0 and e_*; furthermore, up to units in $\mathbb{Z}_{(p)}$ the map β_*^0 is multiplication by the order of the image of the J-homomorphism, and this order is divisible by p. On the other hand, up to torsion and units in $\mathbb{Z}_{(p)}$ the maps $\pi_{2k(p-1)}(G/O)_{(p)} \longrightarrow \pi_{2k(p-1)}(G/Top)_{(p)}$ and $\pi_{2k(p-1)}(BSO)_{(p)} \longrightarrow \pi_{2k(p-1)}(BSTop)_{(p)}$ are multiplication by

$$c_{k,1} \equiv \left(2^{k(p-1)-1} - 1\right) \cdot \mathrm{NUM}\left(B_{k(p-1)}/2kp-2\right)$$

where NUM denotes the numerator of a fraction reduced to least terms and $B_{k(p-1)}$ is the appropriate Bernoulli number (see [MM, p. 117] for the first map and [BRUMFIEL,] for the second). Therefore, modulo torsion the right hand square is given by

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d, "c"'] & \mathbb{Z}_{p^m} \arrow[d, "="] \\ \mathbb{Z}_{(p)} \arrow[r] & \mathbb{Z}_{p^m} \end{tikzcd}$$

where both horizontal arrows are epimorphisms and the top arrow is the standard quotient projection.

Such a diagram can exist only if c is relatively prime to p, and therefore the ~~map~~ vertical arrow at the left is an isomorphism, which is what we wanted to ~~verify~~ prove.

Finally, we need to check that the image of [~~H~~, $[B\mathbb{Z}_p, SG]$ in $[B\mathbb{Z}_p, G/Top]$ corresponds to $E_0 K_{(p)}(B\mathbb{Z}_p)$ and that ~~the~~ $[B\mathbb{Z}_p, SG]$ is mapped isomorphically onto its image. By the preceding discussion we know that this image is a direct summand of $[B\mathbb{Z}_p, G/Top] \cong \widetilde{KO}(B\mathbb{Z}_p)$ and is isomorphic to $\mathbb{Z}_{(p)}^{\wedge}$. Thus the map from

$[B\mathbb{Z}_p, SG]$ to its image is given by a surjective homomorphism from $\mathbb{Z}_{(p)}^{\wedge}$ to itself. Since every such surjection is an isomorphism, we see that $[B\mathbb{Z}_p, SG]$ must be mapped isomorphically to its image. To prove that the image in $[B\mathbb{Z}_p, G/Top] \cong \widetilde{KO}(B\mathbb{Z}_p)$ is $E_0 K_{(p)}(B\mathbb{Z}_p)$, we can use the reasoning in the proof of Proposition 5.1 to ~~show~~ reduce the question to checking that the image of $[B\mathbb{Z}_p, SG]$ in $[B\mathbb{Z}_p, G/Top] \cong \widetilde{KO}(B\mathbb{Z}_p)$ is contained in the kernel of $\psi^r - 1$. We have already noted that on $\widetilde{KO}(B\mathbb{Z}_p)$ one has $\psi^r = B\theta^*$ for some automorphism θ of $\mathbb{Z}_p$, so everything reduces to showing that the map

$$\widetilde{KO}(B\mathbb{Z}_p) \subseteq [B\mathbb{Z}_p, G/O_{(p)}] \longrightarrow [B\mathbb{Z}_p, G/Top] \cong \widetilde{KO}(B\mathbb{Z}_p)$$

sends the kernel of $B\psi^*-1$ to itself. Since the displayed map ~~in question~~ arises from some self-map of $BSO_{(p)}$, it follows immediately that this mapping does send the kernel to itself, proving the remaining assertions in the proposition. $\blacksquare$

Now let L be a $(2n-1)$-dimensional $\mathbb{Z}_p$ lens space, and let $m_L : L \longrightarrow B\mathbb{Z}_p$ be its classifying map. We may assume that $B\mathbb{Z}_p$ is constructed so that its $(2n-1)$-skeleton is L, and we shall do so henceforth. Our next objective is to ~~study~~ derive analogs of Propositions 5.1 and 5.2 in which $B\mathbb{Z}_p$ is replaced by L.