[CONTINUATION OF SECTION 5]

More precisely, we need to extend our observations about the map $[B\mathbb{Z}_p, SG] \longrightarrow [B\mathbb{Z}_p, G/Top]$ into an effective analysis of all objects and morphisms in the following commutative diagram:

$$\begin{array}{ccccc}
[B\mathbb{Z}_p, SG] & \longrightarrow & [B\mathbb{Z}_p, G/Top] & \xrightarrow{\cong} & \widetilde{KO}(B\mathbb{Z}_p) \\
\downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} \\
[L, SG] & \longrightarrow & [L, G/Top] & \xrightarrow{\cong} & \widetilde{KO}(L)
\end{array}$$

The results of [KAMBE] show that the induced map in reduced KO-theory is surjective and yield an explicit description of its kernel; the following result on the summand $E_0K_{(p)}$ is a straightforward consequence of the methods and conclusion of [ ]:

PROPOSITION 5.3. *If L is a $(2n-1)$-dimensional $\mathbb{Z}_p$ lens space, then the Atiyah-Hirzebruch spectral sequence for $E_0K_{(p)}(L)$ collapses, and this group is cyclic of order p^m, where*

$$m = \left[\frac{n-1}{p-1}\right]$$

(and [—] is the greatest integer function).

SKETCH OF PROOF. The spectral sequence collapses because the analogous spectral sequence for $\widetilde{KO}_{(p)}(L)$ collapses (see [ATIYAH] or [KAMBE]), and the cyclic nature of the group follows because $\widetilde{KO}(B\mathbb{Z}_p) \longrightarrow KO_{(p)}(L)$ is onto (see [KAMBE]) and $E_0K_{(p)}$ is a direct summand of KO. ∎

In contrast to this result, the map from $[B\mathbb{Z}_p, SG]$ to $[L, SG_{(p)}]$ is not necessarily onto, but we shall show that the ~~latter splits in'~~ image is a natural direct summand ~~homomorphically~~ which maps, with easily described kernel, onto $E_0 K_{(p)}(L) \subseteq KO_{(p)}(L) \cong [L, G/Top_{(p)}]$ and the complementary summand of $[L, SG_{(p)}]$ maps to zero in ~~the latter group~~ $[L, G/Top_{(p)}]$. These summands of $SG_{(p)}$ are given by the splitting $SG_{(p)} \simeq J_p \times Cok\, J_p$; ~~the~~ results of [MAY E_∞] imply this decomposition comes from a splitting of infinite loop spaces.

Most of what we need to know about $[B\mathbb{Z}_p, J_p] \longrightarrow [L, J_p]$ is contained in the following results.

PROPOSITION 5.4. $[B\mathbb{Z}_p, \mathrm{Cok}\, J_p] = 0$ and ~~hence~~ $[B\mathbb{Z}_p, SG] \cong [B\mathbb{Z}_p, SG_{(p)}]$ is isomorphic to $[B\mathbb{Z}_p, J_p]$.

SKETCH OF PROOF. Propositions 5.1 and 5.2 imply that the composite

On one line

$$[B\mathbb{Z}_p, SG_{(p)}] \cong [B\mathbb{Z}_p, J_p] \times [B\mathbb{Z}_p, \mathrm{Cok}\, J_p] \xrightarrow{\text{inclusions}} [B\mathbb{Z}_p, SG_{(p)}] \times [B\mathbb{Z}_p, SG_{(p)}] \longrightarrow$$

$$[B\mathbb{Z}_p, G/Top_{(p)}] \times [B\mathbb{Z}_p, G/Top_{(p)}] \xrightarrow{\oplus} [B\mathbb{Z}_p, G/Top_{(p)}]$$

is split injective, and since the composite $\mathrm{Cok}\, J_p \to SG_{(p)} \longrightarrow G/Top_{(p)}$ is nullhomotopic the displayed composite can be rewritten more simply as

$$[B\mathbb{Z}_p, SG_{(p)}] \xrightarrow{\text{proj}} [B\mathbb{Z}_p, J_p] \longrightarrow [B\mathbb{Z}_p, G/Top_{(p)}].$$

It follows that the projection map induces a split injection from $[B\mathbb{Z}_p, SG_{(p)}]$ to $[B\mathbb{Z}_p, J_p]$.

Since the projection is onto by construction, it follows that the map $[B\mathbb{Z}_p, SG_{(p)}] \to [B\mathbb{Z}_p, J_p]$ is an isomorphism, proving one assertion in the proposition. To see that $[B\mathbb{Z}_p, \mathrm{Cok}\, J_p] = 0$, notice that if this group is nonzero then the projection $[B\mathbb{Z}_p, SG_{(p)}] \to [B\mathbb{Z}_p, J_p]$ would not be injective. ■

We are now ready to analyze objects like $[L, SG_{(p)}]$ and its summands, where L is a lens space as above.

PROPOSITION 5.5. Let L be a $(2n-1)$-dimensional $\mathbb{Z}_p$ lens space, let $\eta : L \to B\mathbb{Z}_p$ denote its classifying map, and let $q : L \to S^{2n-1}$ be a map of degree 1 (which is unique up to homotopy). Then $[L, J_p]$ is the sum of the images of $\eta^* : [B\mathbb{Z}_p, J_p] \longrightarrow [L, J_p]$ and $q^* : \pi_{2m-1}(J_p) \longrightarrow [L, J_p]$. The image of η^* is cyclic of order p^m, where $m = \left[\frac{n}{p-1}\right]$, the map q^* is injective, and the structures of $[L, J_p]$, and the map $[L, J_p] \longrightarrow [L, G/Top_{(p)}] \cong \widetilde{KO}(L)$ are given as follows:

(i) Suppose that $n \not\equiv 0 \bmod p-1$, so that $\pi_{2m-1}(J_p) = 0$. Then $[B\mathbb{Z}_p, J_p] \longrightarrow [L, J_p]$ is onto and $[L, J_p] \longrightarrow [L, G/Top_{(p)}] \cong \widetilde{KO}(L)$ is split injective with image corresponding to $E_0 K_{(p)}(L)$. Furthermore, the latter also equals the image of $[L, SG_{(p)}] \rightarrow [L, G/Top_{(p)}]$, and this map is a split surjection.

(ii) Suppose that $n = p^{\nu}(p-1)r$ where r is prime to p, so that $\pi_{2n-1}(J_p) \cong \mathbb{Z}_{p^{\nu}}$. Then the images of η^* and q^* intersect in a subgroup of order p; the image of q^* is the kernel of the map $[L, SG_{(p)}] \longrightarrow [L, G/Top_{(p)}]$, and the image of the latter is given by $E_0 K_{(p)}(L)$. Furthermore the latter is also equal to the images of $[L, J_p] \longrightarrow [L, G/Top_{(p)}]$ and $[L, G/O_{(p)}]$, and ~~the kernel of the map from~~ ~~$[L, J_p]$ has order p, and this map~~ the kernel of this map has order p.

Note. Since the group $[L, \mathrm{Cok}\, J_p]$ is usually nontrivial, the map $[B\mathbb{Z}_p, SG_{(p)}] \longrightarrow [L, SG_{(p)}]$ is usually not onto; furthermore, since the homotopy groups of $\mathrm{Cok}\, J_p$ are given by largely unknown factors in the stable homotopy groups of spheres, the groups $[L, \mathrm{Cok}\, J_p]$ are usually not easy to describe explicitly. This leads to major ~~and consequently the image of~~ $\cancel{[L, SG_{(p)}]}$ ~~in~~ $\cancel{[L, G/O_{(p)}]}$ ~~is~~ complications in studying the notion of smooth tangential thickness for lens spaces.

BOLD **Notation.** Given a CW complex X and ~~any~~ arcwise connected space Y, the skeletal filtration of the [X,Y] set of homotopy classes is the family of subsets

BOLD ~~$\underline{XF}_k(X) \subseteq [X,Y]$ such that~~

$$\underline{XF}_k([X,Y]) = \{u \in [X,Y] \mid u|X_k \text{ is trivial}\}.$$

Clearly if $f: X' \to X$ is a cellular map, then the map $f^*: [X,Y] \to [X',Y]$ is filtration preserving. Similarly, if $g: Y \to Y'$ is continuous, then $g_*: [X,Y] \to [X,Y']$ is filtration preserving. If Y is a double loop space, set

$$\underline{FF}_k([X,Y]) = \underline{XF}_k([X,Y]) / \underline{XF}_{k-1}([X,Y])$$

and note that (i) this has a natural abelian group structure, ~~&~~ (ii) ~~if $g: Y \to Y'$ is a morphism of double loop spaces then~~ $\underline{FF}_k([X,\Omega^2 W])$ is functorial in the second variable W.

PROOF. The first step is an analysis of the Atiyah-Hirzebruch spectral sequences for $[B\mathbb{Z}_p, J_p]$ and $[L, J_{(p)}]$.

CLAIM: The Atiyah-Hirzebruch spectral sequences for $[B\mathbb{Z}_p, J_p]$ collapses.

~~We shall first prove this for~~

It ~~will suffice~~ to prove ~~this for $B\mathbb{Z}_p$, the result for L will follow by naturality~~. The relevant E_2 terms are given by $\tilde{H}^i(B\mathbb{Z}_p, \pi_i(J_p))$; these groups are isomorphic to $\mathbb{Z}_p$ if $i = 2k(p-1)-1$ for some integer k and zero otherwise. We also know that $[B\mathbb{Z}_p, J_p]$ maps isomorphically to $E_0 K_{(p)}(B\mathbb{Z}_p)$, and the collapsing Atiyah-Hirzebruch spectral sequence for the latter has E_2

terms given by $\tilde{H}^i(B\mathbb{Z}_p; E_0 K_{(p)}(S^i))$, which are isomorphic to $\mathbb{Z}_p$ if $i \equiv 0 \mod 2(p-1)$ and zero otherwise. It is a fairly straightforward exercise to check that the bijectivity of $[B\mathbb{Z}_p, J_p] \longrightarrow E_0 K_{(p)}(B\mathbb{Z}_p)$ implies that the spectral sequence for $[B\mathbb{Z}_p, J_p]$ must also collapse. ■

Next, we shall use the naturality properties of the Atiyah-Hirzebruch spectral sequence to ~~those~~ and related objects
analyze
~~that the spectral sequence for~~ $[L, J_p]$ ~~also collapses~~.
Let $\pi : S^{2n-1} \longrightarrow L$ be the universal covering projection; then the mapping cone $\hat{L}$ of π can be viewed as the $2n$-skeleton of $B\mathbb{Z}_p$ and

the restriction $H^*(B\mathbb{Z}_p) \longrightarrow H^*(\hat{L})$ is an isomorphism in dimensions $\leq 2n$ for all coefficients. Therefore a naturality argument implies that the ~~map~~

$$\cancel{\mathbb{Z}_{(p)}{}^{\wedge} \cong [B\mathbb{Z}_p, J_p] \to [\hat{L}, J_p]}$$

~~is onto~~ Atiyah-Hirzebruch spectral sequence for $[\hat{L}, J_p]$ collapses and the ~~restriction~~ restriction

$$\mathbb{Z}_{(p)}{}^{\wedge} \cong [B\mathbb{Z}_p, J_p] \longrightarrow [\hat{L}, J_p]$$

is onto with image $\mathbb{Z}_{p^m}$ where

$$m = \left[\frac{n}{p-1}\right].$$

Since $\pi_{2n}(J_p) = 0$, the Barratt-Puppe exact sequence associated to

$$S^{2n-1} \xrightarrow{\lambda} L \longrightarrow \hat{L} \longrightarrow S^{2n} \longrightarrow \cdots$$

implies that the restriction map $[\hat{L}, J_p] \longrightarrow [L, J_p]$ is injective, and hence the image of

$[B\mathbb{Z}_p, J_p]$ in $[L, J_p]$ is also cyclic of order p^m (where m is given as above). To describe the entire group $[L, J_p]$, let $L_0 = L \smallsetminus \mathrm{Int}\, D$ where D is a smoothly embedded closed $(2n-1)$-disk; then L_0 may be viewed as a $(2n-2)$-skeleton for $B\mathbb{Z}_p$ and $H^*(B\mathbb{Z}_p) \to H^*(L_0)$ is an isomorphism in dimensions $\leq 2n-2$. As before, it follows that $[B\mathbb{Z}_p, J_p] \to [L_0, J_p]$ is onto. If we now consider the exact sequence for the Barratt-Puppe sequence

$$S^{2n-2} \subseteq L_0 \subseteq L \xrightarrow{q} S^{2n-1}$$

we see that if $y \in [L, J_p]$, the restriction of y to $[L_0, J_p]$ is the image of some class $z \in [B\mathbb{Z}_p, J_p]$ and therefore $y - \eta^* z \in [L, J_p]$ must lie in the image of q^*. Thus the images of q^* and η^* generate $[L, J_p]$.

To see that q^* is injective, we begin by noting that $\pi_{2n-1}(J_p) = 0$ unless $n \equiv 0 \bmod p-1$, and ~~in those~~ if $n = p^\nu (p-1) r$, where $\nu \geq 1$ and r is prime to p, then $\underset{J(X)-I}{\pi_{2n-1}}(J_p)$ is cyclic of order p^ν (see [ ]). If $n \equiv 0 \bmod p-1$, then our computations of $[L, J_p]$ and $[\hat{L}, J_p]$ show that q^* maps a generator for the p-torsion in $\pi_{2n-1}(J_p) \cong \mathbb{Z}_{p^\nu}$ to a class of order p in the image of η^*: $[B\mathbb{Z}_p, J_p] \longrightarrow [L, J_p]$. In particular, q^* maps the p-torsion injectively, and hence it must map the entire cyclic p-group $\pi_{2n-1}(J_p)$ injectively. Observe that if $n \equiv 0 \bmod p-1$

the preceding argument and skeletal filtration considerations ~~also~~ show that the intersection of Image η^* and Image q^* is a cyclic subgroup of order p.

We must now describe the image of $[L, J_p]$ in $[L, G/Top_{(p)}]$. First of all, we claim that the map $[L, J_p] \longrightarrow [L, G/Top_{(p)}]$ is trivial on the image of $\hat{q}^*$. More or less by construction the composite maps of homotopy groups $\pi_*(SO) \longrightarrow \pi_*(SG) \longrightarrow \pi_*(J_p)$ are onto in all dimensions, and since the ~~therefore~~ ~~it follows~~ composite in the commutative diagram

$$\begin{array}{ccccc} SO & \longrightarrow & SG_{(p)} & \longrightarrow & G/O_{(p)} \\ & & \downarrow & & \downarrow \\ & & J_p & \longrightarrow & G/Top_{(p)} \end{array}$$

is homotopically trivial, and therefore the

the composite

SWITCH

$$\begin{tikzcd} \pi_{2m-1}(SO)_{(p)} \arrow[r, "\text{onto}"] & \pi_{2m-1}(J_p) \arrow[r] \arrow[d, "q^*"] & \pi_{2m-1}(G/Top)_{(p)} \arrow[d, "q^*"] \\ & {[L, J_p]} \arrow[r] & {[L, G/Top_{(p)}]} \end{tikzcd}$$

must be zero.

The final step is to check that the morphism ~~map~~ from the cyclic p-group $[L, J_p]$ to the cyclic p-group $[L, G/Top_{(p)}]$ ~~maps~~ onto the summand $E_0K_{(p)}(L)$ and the kernel is precisely the image of q^*. It is convenient to split the discussion into two cases, depending upon whether or not $n \equiv 0 \bmod p-1$. In both cases the argument uses the commutative diagram

$$\begin{array}{ccc} [L, J_p] & \longrightarrow & [L, G/Top_{(p)}] \\ \downarrow & & \downarrow \\ [L_0, J_p] & \longrightarrow & [L_0, G/Top_{(p)}] \end{array}$$

in which the vertical arrow on the right is an isomorphism by standard results on $\widetilde{KO}(L)$ and $\widetilde{KO}(L_0)$ which follow from the collapsing of their Atiyah-Hirzebruch spectral sequences.

CASE (i). If $n \not\equiv 0 \mod p-1$, then the restriction map from $[L, J_p]$ to $[L_0, J_p]$ is an isomorphism, and the restrictions $[B\mathbb{Z}_p, J_p] \to [L_0, J_p]$ $[B\mathbb{Z}_{(p)}, G/Top_{(p)}] \longrightarrow [L_0, G/Top_{(p)}]$ are onto with isomorphic images. A diagram chase now shows that $[L_0, J_p] \longrightarrow [L_0, G/Top_{(p)}]$ is a split injection whose image is $E_0 K_{(p)}(L)$, which is what we wanted to prove.

CASE (ii). If $n \equiv 0 \mod p-1$, then the kernel of the restriction map from $[L, J_p]$ to $[L_0, J_p]$ is the image of q^*, and the kernel of the ~~restriction~~ map from Image β^* to $[L_0, J_p]$ has order p. As in the preceding case, the map $[L_0, J_p] \longrightarrow [L_0, G/Top_{(p)}]$ is an isomorphism, so the conclusion in this case also follows from a diagram chase. $\blacksquare$

To see the statements about the images of $[L, SG_{(p)}]$ and $[L, J_p]$ in $[L, G/Top_{(p)}]$, note that these images are equal by the splitting $SG_{(p)} \simeq J_p \times Cok\, J_p$ and the homotopic triviality of $\mathrm{Coh}\, J_p \longrightarrow G/Top_{(p)}$. ■