6. Desuspension results

If X is a connected finite complex, it is well known that the standard "loop sum with identity" bijection from $\{X, S^0\}$ to $[X, SG]$ is ~~usually~~ not necessarily a homomorphism with respect to the loop sum structure on the domain and the composition/direct sum structure on the codomain; specifically, if we view $\{X, S^0\}$ as a ring using the standard smash product ring spectrum structure on the spectrum for S^0, then the composition/direct sum structure is given by

$$a \oplus b = a + b + ab$$

(i.e., the Perlis circle operation — see [KAPLANSKY, p.85] or [Polcino Milies-Sehgal, Def. 9.4.1, p. 298]). Fortunately, one can often show that these two algebraic structures are similar in key respects (for example, the are equal if X is a suspension [G. Whitehead, Elts of Htpy Theory, Thm. 5.21, pp. 124–125]). In particular, we have the following:

PROPOSITION 6.1. Let r be an arbitrary positive integer, and let $\sigma : \{B\mathbb{Z}_p, S^0\} \longrightarrow [B\mathbb{Z}_p, SG]$ be the standard set-theoretic isomorphism such that $\sigma^{-1}(u \oplus v) = u + v + uv$. Then $y \in \{B\mathbb{Z}_p, S^0\}$ is divisible by p^r with respect to the loop sum operation if and only if $\sigma(y)$ is divisible by p^r with respect to the composition or direct sum (or circle) operation.

PROOF. By construction the standard map from $\Omega_0^\infty S^\infty$ to SG ~~has~~ induces a ~~map~~ set-theoretic bijection from $\{B\mathbb{Z}_p, S^0\}$ to $[B\mathbb{Z}_p, SG]$ which is skeletal filtration preserving. The sets in

these skeletal filtrations are subgroups with respect to the standard binary operations on the respective sets. Therefore the sets $XF_k(\{B\mathbb{Z}_p, S^0\})$ are subgroups with respect to both the loop sum and the circle operation corresponding to the operation on $[B\mathbb{Z}_p, SG]$. Furthermore, it follows that each subquotient $FF_k(\{B\mathbb{Z}_p, S^0\})$ has group structures given by each binary operation. These subquotients ~~either~~ have order equal to 1 or p; since
SWITCH
$[B\mathbb{Z}_p, SG]$ and $\{B\mathbb{Z}_p, S^0\}$ are both isomorphic to $\mathbb{Z}_{(p)}^{\wedge}$, this means that the classes in $FF_k(\{B\mathbb{Z}_p, S^0\})$ are precisely those which are divisible by the same prime power p^t with respect to each operation. ∎

We shall need the following dualization of Proposition 6.1 for lens spaces:

PROPOSITION 6.2. *Let $\mathbb{T}(L) \subseteq \{L, S^0\}$ denote the image of $\{B\mathbb{Z}_p, S^0\}$ in $\{L, S^0\}$, so that $\mathbb{T}(L)$ corresponds to ~~the~~ cyclic subgroup $[L, J_p]$ of order p^m in $[L, SG]$, where* ~~[illegible]~~

DISPLAY

$$m = \left[\frac{n}{p-1}\right].$$

NEW LINE *Then $\mathbb{T}(L)$ is a cyclic subgroup of $\{L, S^0\}$ with respect to the loop sum, and for all positive integers t a class $x \in \mathbb{T}(L)$ has order p^t with respect to the loop sum if and only if it has order p^t with respect to the circle operation.*

PROOF. The assertion that $\mathbb{T}(L)$ is a finite cyclic group follows because the image of $\{B\mathbb{Z}_p, S^0\}$ is a subgroup with respect to the loop sum, the group

$\{L, S^0\}$ is finite, and a finite quotient of $\hat{\mathbb{Z}}_{(p)} \cong \{B\mathbb{Z}_p, S^0\}$ must be cyclic. As before the sets in the skeletal filtration are subgroups with respect to both binary operations, and the subquotients either have order 1 or p.
Since the set of all elements of ~~order~~ exponent p^t in $\mathbb{Z}_{p^m}$ is cyclic of order p^t, it follows that there is some k such that $XF_k(\{B\mathbb{Z}_p, S^0\})$ has order p^t and the latter contains all elements of exponent p^t with respect to both operations. Likewise, there is some $k' > k$ such that $XF_{k'}(\{B\mathbb{Z}_p, S^0\})$ has order p^{t-1}, and the latter contains all elements of exponent p^{t-1} with respect to either operation. Therefore $XF_k - XF_{k'}$ is the set of all elements

~~of~~ with order p for each operation. ∎

In view of the results from Section 3, we are interested in determining how far one can desuspend the classes in $T(L)$, and here is the main result:

PROPOSITION 6.3. Let k be an integer such that $1 \leq k \leq m-1$, where L, n and m are given as ~~stated~~ above. Then a class in $T(L)$ desuspends to $[S^{2k+1}L, S^{2k+1}]$ if and only if its order divides p^k.

PROOF. Fundamental results of F. Cohen, J. C. Moore, and J. Neisendorfer [CMN two papers] imply that if ~~an~~ p-primary element α in the stable homotopy groups of sphere desuspends to $\pi_{m+2k+1}(S^{2k+1})$, then the orders of the element α and its preimage have orders dividing p^k. In fact their methods immediately yield a far more general conclusion:

LEMMA 6.4. *Let X be a finite complex, and let α be a p-primary element of the stable cohomotopy group* $\{X, S^0\}_{(p)}$ *which desuspends to* $[S^{2k+1}X, S^{2k+1}]_{(p)}$. *Then α and its preimage have orders dividing p^k.*

~~PROOF OF LEMMA 6.4.~~

The "only if" part of Theorem 6.3 is an immediate consequence of this result.

PROOF OF LEMMA 6.4. As noted in [NEISENDORFER, Cor 11.8.2, p. 461], if $I_p : \Omega^2 S^{2r+1} \longrightarrow \Omega^2 S^{2r+1}$ is is the double looping of the degree p self-map for S^{2r+1}, then $\Phi_p = \sigma \circ \pi$, where $\sigma : S^{2r-1} \longrightarrow \Omega^2 S^{2r+1}$ is adjoint to the identity. Therefore, if Y is a connected finite complex, ~~[illegible]~~ then the H-space structure on S^{2r+1} and the square lemma (see the previous citatation from [WHITEHEAD]) imply that if $\beta \in [S^{2r+1}Y, S^{2r+1}]_{(p)}$, then $p \cdot \beta$ desuspends to $[S^{2r-1}Y, S^{2r-1}]_{(p)}$. One can now proceed by induction as in [CMN + NEISENDORFER] to conclude that $p^r\beta = 0$ (e.g., see [NEIS, Cor. 11.8.3, p. 462]). ∎

PROOF OF NECESSITY IN PROPOSITION 6.3. By Lemma 6.4 and Proposition 6.2, it will suffice to show that a generator τ of $T(L)$ desuspends to S^{2t+1}, where $t = \left[\frac{n}{p-1}\right]$. ~~Finally~~ Since τ has order p^t by Proposition 6.2, we can use Lemma 6.4 to conclude that τ cannot double desuspend any further. Similarly, if $r < t$, then it will follow that $p^r \tau$ must desuspend to $S^{2(t-r)+1}$ but can not double desuspend any further. The conclusion in Proposition 6.3 follows because ~~a multiple of class $k\tau$ (where $k \in \mathbb{Z}$) is zero in $B(L)$ if and only if~~ a multiple $a\tau$ of τ satisfies $p^k(a\tau) = 0$ if and only if a is divisible by p^{t-k}.

It is well known that the localized

stabilization maps

$$S^{2m+1}_{(p)} \longrightarrow Q_0(S^{2m+1}) = \lim \Omega^{2m+1}_0 S^{2m+1}_{(p)}$$

are very highly connected. In fact, using the fibration sequences in (1.5.3) and (1.5.5) of [RAVENEL BOOK, p. 25], and an inductive argument, ~~then~~, one can prove that the localized stabilization map is $(2(m+1)(p-1)-3)$-connected (a related statement appears in [RAVENEL BOOK, p. 26]). Therefore, if $2n-1 \leq 2(m+1)(p-1)-3$, then τ (and its loop sum multiples) will automatically desuspend to S^{2m+1}. In particular, if the preceding inequality holds when

$$m = \left[\frac{n}{p-1}\right]$$

then τ will desuspend to S^{2m+1}, and hence the conclusion of Proposition 6.3 will follow.

Write $n = j(p-1)+s$, where $0 \leq s \leq p-1$, so that $j = [\frac{n}{p-1}]$. With this notation the dimension versus connectivity inequality reduces to

$$m \geq \frac{n+1}{p-1} - 1 = j + \frac{s+1}{p-1} - 1.$$

and as indicated in the preceding paragraph we ~~need~~ want to verify this holds when $m = j$. To see this, note that $0 \leq s \leq p-2$ implies

$$-1 < \frac{s+1}{p-1} - 1 \leq \frac{p-1}{p-1} - 1 = 0$$

and therefore we do have

$$j \geq j + \frac{s+1}{p-1} - 1$$

which is what we ~~needed~~ wanted to verify. ■