TANGENTIAL THICKNESS OF MANIFOLDS

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ABSTRACT. A notion of tangential thickness of a manifold is introduced. An extensive calculation within the class of lens and fake lens spaces leads to complete classification of such manifolds with thickness 1, 3 or $2k$, for $k \geq 1$. On the other hand, calculations of tangential thickness in terms of the dimension of the manifold and the rank of the fundamental group show very interesting and quite surprising correlations between these invariants.

1. INTRODUCTION

Given two nonhomeomorphic topological spaces, X and Y, it is often interesting and important to specify necessary or sufficient conditions for $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ to be homeomorphic, where $\mathbb{R}$ denotes the real line. More generally, it is also useful to have criteria for determining whether $X \times \mathbb{R}^k$ and $Y \times \mathbb{R}^k$ are homeomorphic for some $k \geq 1$ (*cf.* [13] or [40]). If X and Y are closed manifolds the following result, due to B. Mazur in the smooth and piecewise linear categories [47], provides an abstract answer; in the statement of this result below, CAT refers to the category of smooth, piecewise linear, or topological manifolds, and a CAT-isomorphism is a diffeomorphism, piecewise linear homomorphism or homeomorphism, respectively.

Stable Equivalence Theorem: *Let M and N be closed* CAT-*manifolds. Then $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are* CAT-*isomorphic for some $k \geq 1$ if and only if M and N are tangentially homotopy equivalent (i.e., there is a homotopy equivalence $f : M \to N$ such that the pullback of the stable tangent bundle/microbundle of N is the stable tangent bundle/microbundle of M).*

In fact, if f exists, then for some k the map $f \times \mathrm{Id}_{\mathbb{R}^k}$ is properly homotopic to a CAT-isomorphism. The topological version of this result follows from [36].

Given two manifolds M and N satisfying the conditions of the Stable Equivalence Theorem, it is natural to ask the following:

*The author has been supported by the BOR award LEQSF(2008-2011)-RD-A-24.

Optimal Value Question: *For a given tangential homotopy equivalence $f : M \to N$, what is the least value of $k \geq 0$ such that $f \times Id_{\mathbb{R}^k}$ is properly homotopic to a* CAT-*isomorphism?*

The Whitney Embedding and Tubular Neighborhood Theorems imply that $\dim N + 1 = \dim M + 1$ is a universal upper bound for k in the smooth category. Standard results for piecewise linear manifolds [28] and results of Kirby-Siebenmann [36] imply the analog in the piecewise linear and topological categories, respectively.

In [39], the Optimal Value Question was considered for linear spherical space forms. In particular, it was shown that if M and N are linear space forms such that $M \times \mathbb{R}^2$ is homeomorphic to $N \times \mathbb{R}^2$, then M and N are diffeomorphic. Furthermore, examples of fake lens spaces (*i.e.*, quotients of S^n by a free *nonlinear* action of a finite cyclic group) M and N were constructed in [39] such that M and N are homeomorphic but $M \times \mathbb{R}^3$ and $N \times \mathbb{R}^3$ are not diffeomorphic. These results already reflect the relative complexity of this problem. Some of the techniques and ideas of this paper were applied in [12] when studying and classifying open complete manifolds of nonnegative curvature. By the results of J. Cheeger and D. Gromoll [18], such manifolds are diffeomorphic to the total space of a normal bundle to a compact locally geodesic submanifold called a **soul**. An obvious variation on the notion of the Optimal Value Question in this case leads to a notion of *twisted* tangential thickness and is briefly discussed in Section 8.

We first consider linear lens spaces.

Theorem 1. *Let $f : M \to N$ be a tangential homotopy equivalence of lens spaces with prime order fundamental groups. Then $f \times \mathrm{Id}_{\mathbb{R}^3}$ is properly homotopic to a homeomorphism.*

Remark. Techniques of S. Cappell and J. Shaneson in [17] imply that a result analogous to Theorem 1 remains true for $\mathbb{Z}_{2^r}$ lens spaces.

We would like to point out that the above theorem is directly related to a remarkable theorem of J. Folkman [26]:

Theorem (Folkman): *Let m be a power of a prime. If two k-dimensional lens spaces with fundamental group $\mathbb{Z}_m$ have the same tangential homotopy type, where k is greater than a certain function of m, then the two must actually be isometric (diffeomorphic).*

This is a truly startling result which probably deserves much more attention and publicity. We hope to discuss this and some ideas from [26] in a future paper. However, at this time we shall restrict ourselves to the following partial extension of Folkman's theorem.

Theorem 2. *Let $f : M^{2n-1} \to N^{2n-1}$ be a stably tangential homotopy equivalence of lens spaces with $\pi_1(M) \cong \pi_1(N) \cong \mathbb{Z}_p$, for p an odd prime. Then M and N are isometric (hence diffeomorphic) if $n \geq p-1$.*

Most of the paper (and its main purpose) is devoted to a study of the Optimal Value Question for fake lens spaces. We will concentrate on the case of (odd) prime order fundamental groups, although many of our results hold without this restriction.

Qualitatively, one can describe our results in terms of a concept we shall call **tangential thickness**, which is defined as follows:

Definition 1.1. *Two* CAT*-manifolds M and N are said to have tangential thickness $\leq k$ if and only if $M \times \mathbb{R}^k$ is* CAT*-isomorphic to $N \times \mathbb{R}^k$. Given two tangentially homotopy equivalent manifolds M and N, the tangential thickness of the pair $\{M, N\}$ is the least k such that the tangential thickness of M and N is $\leq k$.*

Given a compact unbounded manifold M, let $\mathcal{TT}_k^{\mathrm{Top}}(M)$ denote the equivalence classes of homotopy structures (N, f)—in other words, homotopy equivalences $f : N \to M$ where N is another manifold of the same dimension—such that $N \times \mathbb{R}^k$ is homeomorphic to $M \times \mathbb{R}^k$. One then has an increasing sequence of sets

$$\{M\} = \mathcal{TT}_0^{\mathrm{Top}}(M) \subseteq \mathcal{TT}_1^{\mathrm{Top}}(M) \subseteq \cdots \subseteq \mathcal{TT}^{\mathrm{Top}}(M)$$

where $\mathcal{TT}^{\mathrm{Top}}(M)$ is the set of all homotopy structures (N, f) such that f is a tangential homotopy equivalence. As noted earlier, this sequence stabilizes for $k \geq \dim M + 1$; *i.e.*, we have

$$\mathcal{TT}_{\dim M+1}(M) = \mathcal{TT}_{\dim M+i+1}(M) = \mathcal{TT}(M) \qquad \text{for } i \geq 1.$$

If the pair $\{N, M\}$ has tangential thickness $\leq k$, then one obtains a tangential homotopy equivalence $f : N \to M$ and an associated homotopy structure class (N, f) in $\mathcal{TT}_k^{\mathrm{Top}}(M)$. In general this class need not be well-defined, but if every tangential homotopy self-equivalence of M is homotopic to a homeomorphism, then $\{M, N\}$ determines a uniquely defined homotopy structure. Folkman's result implies that every tangential homotopy self-equivalence of a $\mathbb{Z}_p$ lens space is homotopic to a diffeomorphism (see the Addendum to Section 4), so if M is such a manifold then the topological classification of all N such that $\{N, M\}$ has tangential thickness $\leq k$ is in one-to-one correspondence with the elements of $\mathcal{TT}_k^{\mathrm{Top}}(M)$. It also follows that the classification of all pairs with tangential thickness k is given by the set-theoretic difference $\mathcal{TT}_k^{\mathrm{Top}}(M) \smallsetminus \mathcal{TT}_{k-1}^{\mathrm{Top}}(M)$.

Theorem 1 and the remark following it, when combined with the results in [39], show that each odd prime p all $\mathbb{Z}_{2^r}$ and $\mathbb{Z}_{p2^r}$ lens spaces have tangential thickness 3.

Our other main results on tangential thickness of fake lens spaces can now be stated as follows:

Theorem 3. *Let $n \geq 3$, and let M^{2n-1} be a fake lens space (arbitrary fake spherical space form). Then $\mathfrak{TT}_1^{\mathrm{Top}}(M)$ consists of manifolds h-cobordant to M. These manifolds are classified by $Wh(\pi_1(M))$ via realization of Whitehead torsion by h-cobordisms (i.e., the action of $Wh(\pi_1(M))$ on M is free).*

Theorem 4. *Let M^{2n-1}, $n \geq 3$, be a fake lens space. Then N^{2n-1} is in $\mathfrak{TT}_2^{\mathrm{Top}}(M)$ if and only if $N \times \mathbb{R}$ is properly h-cobordant to $M \times \mathbb{R}$. The set $\mathfrak{TT}_2^{\mathrm{Top}}(M) \setminus \mathfrak{TT}_1^{\mathrm{Top}}(M)$ is in one-to-one correspondence with $\widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]))$. Moreover, all possible manifolds in $\mathfrak{TT}_2^{\mathrm{Top}}(M) \setminus \mathfrak{TT}_1^{\mathrm{Top}}(M)$ are obtained by a free action of $\widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]))$ on $M \times \mathbb{R}$ via the realization of Whitehead torsion by proper h-cobordisms.*

Theorem 5. *Let M^{2n-1}, $n \geq 3$, be a fake lens space with $\pi_1(M) \cong \mathbb{Z}_p$, for p an odd prime. Then the set $\mathfrak{TT}_3^{\mathrm{Top}}(M)$ is the set of homeomorphism classes of manifolds normally cobordant to M. The set $\mathfrak{TT}_3^{\mathrm{Top}}(M) \setminus \mathfrak{TT}_2^{\mathrm{Top}}(M)$ is in one-to-one correspondence with the free abelian group $\mathbb{Z}^{\frac{p-1}{2}}$. A manifold N^{2n-1} is in $\mathfrak{TT}_3^{\mathrm{Top}}(M) \setminus \mathfrak{TT}_2^{\mathrm{Top}}(M)$ if it is obtained from M by the action of $\widetilde{L}_{2n}^h(\mathbb{Z}_p)$ on M via realization of elements of the surgery obstruction group by normal cobordism starting with M (cf. [15], [74]). Moreover, the action is given by the difference of the Atiyah-Singer ρ-invariants: $\rho(M) - \rho(N)$*

Note. The ρ-invariant is the invariant denoted by σ in [11].

Our remaining results on $\mathfrak{TT}_k^{\mathrm{Top}}(M)$ for $k \geq 3$ are most conveniently stated in terms of the normal invariant map

$$\eta : \mathbb{S}(M) \longrightarrow [M, G/\mathrm{Top}]$$

(see [74]) in the Sullivan-Wall surgery exact sequence.

Theorem 6. *Let M^m be a compact unbounded topological manifold of dimension $m \geq 5$, and let $k \geq 3$. Assume that the image of the normal invariant map η is a subgroup of $[M, G/\mathrm{Top}]$, where the group operation on the latter is given by taking direct sums. Then, there is an increasing sequence of subgroups $\theta_k([M, G/\mathrm{Top}])$, defined for all $k \geq 3$, with the following properties:*

(i) *If $k \geq m+1$, then $\theta_k([M, G/\mathrm{Top}]) = \theta_{k+1}([M, G/\mathrm{Top}])$.*
(ii) *If $f : N \to M$ is a homotopy equivalence of manifolds, then $f \times \mathrm{Id}_{\mathbb{R}^k}$ is properly homotopic to a homeomorphism if and only if $\eta(f) \in \theta_k([M, G/\mathrm{Top}])$.*

In view of the first conclusion in this theorem, it is meaningful to write $\theta([M, G/\mathrm{Top}]) = \theta_k([M, G/\mathrm{Top}])$ if $k \geq m+1$.

Note. We are assuming that the image of η is a subgroup with respect to direct sum in order to avoid possible problems with the nonadditivity of the surgery obstruction map $\sigma : [M, G/\mathrm{Top}] \to L^h_m(\pi_1(M^m), w_1)$, where the operation on the domain is given by taking direct sums. One easy way to ensure that the image of η is a subgroup is to assume that $L^h_m(\pi_1(M^m), w_1) = 0$ so that η must be onto. This condition holds if $\pi_1(M^m)$ has odd order and m is odd [73], and therefore Theorem 6 is valid for the examples of primary interest in this paper.

Theorem 6 implies that the difference sets $\mathfrak{TT}^{\mathrm{Top}}_k(M) \setminus \mathfrak{TT}^{\mathrm{Top}}_{k-1}(M)$ are the inverse images of the set-theoretic differences $\theta_k([M, G/\mathrm{Top}]) \setminus \theta_{k-1}([M, G/\mathrm{Top}])$ with respect to the normal invariant map η. For example, Theorem 5 translates into the equation $\theta_3([M, G/\mathrm{Top}]) = 0$ if M satisfies the hypotheses in that result. More generally, this allows us to characterize the differences between $\mathfrak{TT}^{\mathrm{Top}}_k(M)$ and $\mathfrak{TT}^{\mathrm{Top}}_{k-1}(M)$ in terms of the nonzero elements in the subquotient groups $\theta_k([M, G/\mathrm{Top}])/\theta_{k-1}([M, G/\mathrm{Top}])$.

If M^{2n-1} is a fake lens space with fundamental group $\mathbb{Z}_p$, where p is an odd prime, and $n \geq 3$, then we shall see that the groups $\theta_k([M, G/\mathrm{Top}])$ are all cyclic p-groups (possibly trivial) and hence the same is true of the subquotients $\theta_k([M, G/\mathrm{Top}])/\theta_{k-1}([M, G/\mathrm{Top}])$. We shall prove that every such subquotient is either trivial or isomorphic to $\mathbb{Z}_p$. Furthermore we shall prove that $\theta_k([M, G/\mathrm{Top}]) = \theta([M, G/\mathrm{Top}])$ well below the range of the Stable Equivalence Theorem in part (i) of Theorem 6 (in particular, the equation holds when $k > \frac{n}{p-1}$), and in about half of the remaining cases the subquotient is isomorphic to $\mathbb{Z}_p$. We shall begin with the cases that are the simplest to describe:

Theorem 7. *Let p be an odd prime, let $n \geq 3$ and let M^{2n-1} be a fake lens space with fundamental group $\mathbb{Z}_p$. Assume further that $n \not\equiv 0 \bmod p-1$. Then the subquotients*

$$\theta_k([M, G/\mathrm{Top}])/\theta_{k-1}([M, G/\mathrm{Top}]), \quad \theta_{2j+2}([M, G/\mathrm{Top}])/\theta_{2j}([M, G/\mathrm{Top}])$$

are given as follows:

(i) $\theta_{k+1}([M, G/\mathrm{Top}])/\theta_k([M, G/\mathrm{Top}]) = 0$ *if* $k \geq 2\left[\frac{n}{p-1}\right] + 2$, *where* $[-]$ *denotes the greatest integer function.*

(ii) *If* $k = 2j$ *and* $1 \leq j \leq \left[\frac{n}{p-1}\right]$ *then* $\theta_{2j+2}([M, G/\mathrm{Top}])/\theta_{2j}([M, G/\mathrm{Top}]) \cong \mathbb{Z}_p$*; we set* $\theta_2([M, G/\mathrm{Top}]) = 0$ *by definition.*

(iii) *If* $2 \leq j \leq \left[\frac{n}{p-1}\right]$, *then either* $\theta_{2j+1}([M, G/\mathrm{Top}]) = \theta_{2j}([M, G/\mathrm{Top}])$ *or else* $\theta_{2j+1}([M, G/\mathrm{Top}]) = \theta_{2j+2}([M, G/\mathrm{Top}])$.

There is a similar but slightly weaker conclusion when $n \equiv 0 \bmod p-1$.

Theorem 8. *Suppose we are in the same setting as in Theorem 7, but $n \equiv 0 \bmod p-1$. Then (i) and (ii) remain valid. However, if $k = 2j$ and*

$$1 \leq j \leq \left[\frac{n}{p-1}\right]$$

then $\theta_{2j+2}([M, G/\mathrm{Top}])/\theta_{2j}([M, G/\mathrm{Top}]) \cong \mathbb{Z}_p$ except for precisely one value j_0 of j.

We shall say more about the exceptional value in Section 7; unfortunately, our methods only yield limited information about the exceptional value j_0, but we shall provide some evidence for conjecturing that $j_0 = 1$ in all cases.

Here is a more qualitative consequence of the preceding results:

Theorem 9. *Let L^{2n-1} be a lens space with $n \geq 3$.*

(i) *If $n \not\equiv 0 \bmod p-1$, then for each j such that $1 \leq j \leq \left[\frac{n}{p-1}\right]$, there exist manifolds L_j tangentially homotopy equivalent to L such that $L_j \times \mathbb{R}^{2j}$ and $L \times \mathbb{R}^{2j}$ are not homeomorphic but $L_j \times \mathbb{R}^{2j+2}$ and $L \times \mathbb{R}^{2j+2}$ are homeomorphic.*
(ii) *If $n \equiv 0 \bmod p-1$, then the same conclusion holds for all but one value of j such that $1 \leq j \leq \left[\frac{n}{p-1}\right]$.*
(iii) *If N is a fake lens space which is tangentially homotopy equivalent to L and $k \geq 2\left[\frac{n}{p-1}\right] + 2$, then $L \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are homeomorphic.*

The proofs of these results will appear in Sections 2–7 below. In Section 2 we shall use surgery-theoretic methods as in [37] to prove Theorems 3 and 4. Section 3 gives a surgery-theoretic criterion for two manifolds to have tangential thickness $\leq k$, where $k \geq 3$; most of this material is surely well known, but we include it since it is fundamental to our work and difficult to extract from literature. In the case of odd-dimensional $\mathbb{Z}[\frac{1}{2}]$ homology spheres, these results will be restated very simply in terms of desuspending classes in the stable cohomotopy groups of such manifolds (see Propostition 3.4). In Section 4 we shall use the ideas of Section 3 and results on the K-theory of lens spaces [34] to prove Theorems 1 and 2. We shall then specialize the general setting of Section 3 to fake lens spaces in Section 5; this uses a variety of results about the structure of the classifying spaces for surgery theory (mostly contained in [42]). In Section 6 we shall analyze the cohomotopy desuspension questions from Section 3 for the case of $\mathbb{Z}_p$ lens spaces using the work of F. Cohen, J.C. Moore and J. Neisendorfer (*e.g*, see [20], [21]) and [49]) on exponents of homotopy groups. We shall bring everything together in Section 7 to prove Theorems 5–9. Finally, Section 8 contains some comments and remarks concerning smooth and twisted tangential thickness.

2. Results in Low Codimensions

In this section and the next, we shall derive the basic surgery theoretic conditions for determining the tangential homotopy equivalences $h : M \to N$ such that $h \times \mathrm{Id}_{\mathbb{R}^k}$ is properly homotopic to a homeomorphism. As in many other situations within geometric topology, the cases with codimension $k \geq 3$ differ greatly from the cases where $k = 1$ or 2, and in this section we shall dispose of the latter cases.

Proof. (Theorem 3) Let M^{2n-1} $(n \geq 3)$ be a fake spherical space form, and let $f : N^{2n-1} \to M^{2n-1}$ a tangential homotopy equivalence. Suppose $N \times \mathbb{R}$ and $M \times \mathbb{R}$ are isomorphic. Then, it follows that N and M are h-cobordant. The action of the Whitehead group $Wh(\pi_1(M))$ on M is free by the main result of [39].

On the other hand, if $(W; N, M)$ is an h-cobordism between N and M, then $W \times S^1$ is an s-cobordism between $N \times S^1$ and $M \times S^1$. Thus, $M \times S^1$ is isomorphic to $N \times S^1$ and hence $M \times \mathbb{R}$ and $N \times \mathbb{R}$ are isomorphic as well. □

Proof. (Theorem 4) Let $\pi \cong \pi_1(M^{2n-1})$ be the fundamental group of M^{2n-1}. If $N^{2n-1} \in \mathcal{TT}_2(M)$, then N is a fake lens space and there exists a homeomorphism $h : N \times \mathbb{R}^2 \to M \times \mathbb{R}^2$. This yields an h-cobordism W between $N \times S^1$ and $M \times S^1$ (*cf.* [39]). By taking infinite cyclic coverings, one gets a proper h-cobordism $\widetilde{W}$ between $N \times \mathbb{R}$ and $M \times \mathbb{R}$.

Conversely, if there is a proper h-cobordism V between $N \times \mathbb{R}$ and $M \times \mathbb{R}$, then $V \times S^1$ is a product cobordism between $N \times \mathbb{R} \times S^1$ and $M \times \mathbb{R} \times S^1$. In particular, $N \times \mathbb{R} \times S^1 \approx M \times \mathbb{R} \times S^1$ and hence $N \times \mathbb{R} \times \mathbb{R} \approx M \times \mathbb{R} \times \mathbb{R}$ (*i.e.*, $N \times \mathbb{R}^2 \approx M \times \mathbb{R}^2$).

Now, let $\tau_0 \in Wh(\widetilde{W}, M \times \mathbb{R}) \cong \widetilde{K}_0(\mathbb{Z}[\pi])$ (*cf.* [65]) be a proper Whitehead torsion of this proper h-cobordism. Analogously as in the compact case (*cf.* [19]) there is an involution on $Wh(\widetilde{W})$ and duality between $\tau_0 \in Wh(\widetilde{W}, M \times \mathbb{R})$ and $\tau_1 \in Wh(\widetilde{W}, M \times \mathbb{R})$ given by $\tau_1 = (-1)^{\dim(M \times \mathbb{R})} \tau_0^*$. Hence $\tau_1 = \tau_0^*$.

Let $f : N \times \mathbb{R} \to M \times \mathbb{R}$ be a proper homotopy equivalence given by the composition of the inclusion i and retraction r:

$$N \times \mathbb{R} \xrightarrow{i} \widetilde{W} \xrightarrow{r} M \times \mathbb{R}$$

It follows that $\tau(f) = \tau_0^* - \tau_1$. However, f is properly homotopic to a map $f_0 \times \mathrm{Id}_{\mathbb{R}} : N \times \mathbb{R} \to M \times \mathbb{R}$ (*cf.* [72]), with $f_0 : N \to M$. In particular, as $f_0 \times \mathrm{Id}_{S^1} : N \times S^1 \to M \times S^1$ is a simple homotopy equivalence (*cf.* [19]), so must be $f_0 \times \mathrm{Id}_{\mathbb{R}}$. As a consequence, $\tau_1 = \tau_0^*$ and $f : N \times \mathbb{R} \to M \times \mathbb{R}$ is a proper simple homotopy equivalence.

The standard construction shows that elements in $Wh(\widetilde{W})$ of the form $\rho + \rho^*$ can be realized by an inertial proper h-cobordism. Consider

$$\widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) = \left\{ \frac{\tau = \tau^*}{\tau + \tau^*} \right\} .$$

Claim 2.1. *Realization of elements in $\widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$ via proper h-cobordisms starting with $M \times \mathbb{R}$ yields manifolds of the form $N \times \mathbb{R}$ on the other end.*

Proof. To see this, let $(\overline{W}; M \times \mathbb{R}, K)$ be a proper h-cobordism with $\tau_0 \in Wh(\overline{W}, M \times \mathbb{R})$, $\tau_0 \in \widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$. Then there is a proper homotopy equivalence

$$f : K \hookrightarrow \overline{W} \to M \times \mathbb{R}$$

which is simple. By the one-sided splitting theorem for proper maps and noncompact manifolds (*cf.* [70]), f is properly homotopic to a map g with $g^{-1}(M \times \{0\}) = N \subset K$ and $g|_N : N \to M \approx M \times \{0\}$ a homotopy equivalence. We have a splitting

where $g|_{K_0} : K_0 \to M \times [0, \infty)$ and $g|_{K_1} : K_1 \to (-\infty, 0] \times M$ are proper homotopy equivalences. Now, the Collaring theorem of Siebenmann (*cf.* [66]) implies $K_0 \approx N \times [0, \infty)$ and $K_1 \approx (-\infty, 0] \times N$, and hence $K \approx N \times \mathbb{R}$. □

It remains to show that the action of $\widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$ yields all manifold classes in $\mathcal{T}\mathcal{T}_2^{\mathrm{Top}}(M) \smallsetminus \mathcal{T}\mathcal{T}_1^{\mathrm{Top}}(M)$. To see this, we use a proper surgery theory of S. Maumary and L. Taylor [45], [46], [70], [52]. Consider the long Wall-Sullivan exact sequence for proper surgery theory:

$$\cdots \to L_{*+1}^{s,\mathrm{open}}(M \times \mathbb{R}) \to \mathbb{S}_{\mathrm{Top}}^s(M \times \mathbb{R}) \to [M \times \mathbb{R}; G/\mathrm{Top}] \to L_*^{s,\mathrm{open}}(M \times \mathbb{R}) \to \cdots$$

We have $L_{*+1}^{s,\mathrm{open}}(M \times \mathbb{R}) \cong L_{even}^h(\pi) \cong L_{even}^{p,s}(\pi) \oplus \widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[\pi]))$ (*cf.* []). By using the equivariant Hopf theorem (much as in [38]), one shows that the action of $L_{even}^h(\pi)$, and hence the action of $\widehat{H}^0(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$, is not through self-homotopy equivalences of $M \times \mathbb{R}$.

This, combined with the description of $\mathcal{TT}_1^{\mathrm{Top}}(M)$, shows the one-to-one correspondence between $\mathcal{TT}_2^{\mathrm{Top}}(M) \setminus \mathcal{TT}_1^{\mathrm{Top}}(M)$ and the nonzero elements of $\widehat{H}^0(\widetilde{K}_0(\mathbb{Z}[\pi]))$. □

3. Tangential Thickness and Normal Invariants

Suppose that M and N are closed n-manifolds and $k \geq 3$ is such that $n + k \geq 6$. Surgery theory then yields the following criteria for $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ to be homeomorphic:

Proposition 3.1. *If M, N and k are as above, the $M \times \mathbb{R}^k$ is homeomorphic to $N \times \mathbb{R}^k$ if and only if the compact bounded manifolds $M \times D^k$ and $N \times D^k$ are h-cobordant in the following sense: There is a compact manfold with boundary X^{n+k+1} and a compact manifold with boundary $W^{n+k} \subseteq \partial X$ such that the following hold:*

(i) *∂W^{n+k} is homeomorphic to a disjoint union of $M \times S^{k-1}$ and $N \times S^{k-1}$*
(ii) *$\partial X^{n+k+1} \cong M \times D^k \cup W \cup N \times D^k$, where $M \times D^k \cap W = M \times S^{k-1}$ and $N \times D^k \cap W = N \times S^{k-1}$*
(iii) *The inclusion of pairs $(M \times D^k, M \times S^{k-1}) \subseteq (\partial X, W) \subseteq (X, W)$ and $(N \times D^k, N \times S^{k-1}) \subseteq (\partial X, W) \subseteq (X, W)$ are homotopy equivalences of pairs.*

Proof. This is fairly standard. If $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are homeomorphic, then the homeomorphism maps $M \times D^k$ into some subset $N \times r\,D^k$, where $r\,D^k$ is the disk of radius r for some very large value of r. Let W be the bounded manifold $N \times r\,D^k \setminus \mathrm{Int}(M \times D^k)$, and take X to be $N \times D^k \times [0, 1]$. The decomposition of ∂X in (ii) is then given by identifying $M \times D^k$ with $M \times D^k \times \{0\}$, W with $W \times \{0\}$ and $N \times D^k$ with $N \times r\,D^k \times \{1\} \cup N \times \partial(r\,D^k) \times [0, 1]$. It is then fairly straightforward to check that the inclusions in (iii) are homotopy equivalences of pairs. Conversely, if we are given X as in the theorem, then it follows that $X \setminus \mathrm{Int}(W)$ is a proper h-cobordism from $M \times \mathrm{Int}(D^k) \cong M \times \mathbb{R}^k$ to $N \times \mathrm{Int}(D^k) \cong N \times \mathbb{R}^k$ in the sense of [65]. Then, by the proper h-cobordism theorem of [65], it follows that $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are homeomorphic. □

Complement 3.1. *Similar results are true in the categories of piecewise linear (PL) or smooth manifolds if we stipulate that all manifolds lie in the given category and the homeomorphisms are PL-homeomorphisms or diffeomorphisms, respectively.*

This is true because one has analogs of the proper h-cobordism theorem in the PL and smooth categories (in fact, they predate the topological version). In the smooth category there are some issues about rounding corners in a product of two bounded smooth manifolds, but there are standard ways of addressing such points. (*e.g.*, see Section I.3 of [22] or the appendix to [14]).

These results lead to the use of surgery theoretic structure sets; the latter are defined for closed manifolds in [55] and one can treat the bounded case using maps and homotopy equivalences of pairs as in Chapter 10 of Wall's book [74]. In order to translate Proposition 3.1 and Complement 3.1 into the language of structure sets, we need to work with certain function spaces. Following James [30], we shall denote the identity component of the continuous function space $\mathcal{F}(S^{k-1}, S^{k-1})$ by SG_k, and SF_{k-1} will denote the subspace of basepoint preserving maps (which is also arcwise connected). By the results of [30] and [67], there is a Serre fibration $SF_{k-1} \to SG_k \to S^{k-1}$ and a corresponding classifying space fibration $S^{k-1} \to BSF_{k-1} \to BSG_k$. The space of degree zero basepoint preserving self-maps is homeomorphic to the iterated loop space $\Omega^{k-1}S^{k-1}$ and the map $w : \Omega^{k-1}S^{k-1} \to SF_{k-1}$ sending $f : S^{k-1} \to S^{k-1}$ to the composite

$$S^{k-1} \xrightarrow{\text{pinch}} S^{k-1} \vee S^{k-1} \xrightarrow{f \vee \mathrm{Id}} S^{k-1} \vee S^{k-1} \xrightarrow{\text{fold}} S^{k-1}$$

is a homotopy equivalence. It is important to note that this homotopy equivalence does **not** send the loop sum on $\Omega^{k-1}S^{k-1}$ to the composition product SF_{k-1} (the precise relationship is described at the beginning of Section 6). The unreduced suspension functor defines a continuous homomorphisms $SG_k \to SF_{k+1}$, and if $\Omega^{k-1}S^{k-1} \to \Omega^k S^k$ is the suspension map induced by the suspension adjoint $\sigma : S^{k-1} \to \Omega S^k$, then we have the following homotopy commutative diagram:

$$\begin{tikzcd}
\Omega_0^{k-1}S^{k-1} \arrow[r, "\Omega^{k-1}\sigma"] \arrow[d, "w_{k-1}"'] & \Omega_0^k S^k \arrow[r, "="] \arrow[d] & \Omega_0^k S^k \arrow[d, "w_k"] \\
SF_{k-1} \arrow[r] & SG_k \arrow[r] & SF_k
\end{tikzcd}$$

The preceeding chain of maps can be extended by adjoining SG_{k+1} on the right, and if we take limits we obtain a topological monoid that is denoted SG or SF (it is the simultaneous limit of the sequences $\{SG_k\}$ and $\{SF_k\}$). With this preparation, we can restate Proposition 3.1 and Complement 3.2 in the piecewise linear and topological categories as follows:

Proposition 3.2. *Let M and N be closed connected PL (resp., topological) manifolds, $k \geq 3$, dim M = dim $N \geq 5$ and let $f : M \to N$ be a homotopy equivalence. Then, $M \times D^k$ is PL-homeomorphic (resp., homeomorphic) to $N \times D^k$ if and only if the normal invariant in $[N, G/\mathrm{PL}]$ (resp., $[N, G/\mathrm{Top}]$) lies in the image of $[N, SG_k]$ under the map induced by the composite $G_k \to G \to G/\mathrm{PL}$ (resp., $SG_k \to SG \to SG/\mathrm{Top}$).*

There is an analog of this result in the smooth category, but the proof is longer and we shall not need the smooth version of Proposition 3.2.

Proof. We begin with the case of the PL category since the argument is simpler but also contains the ideas to be employed in the topological category. Given a homotopy equivalence

$f : M \to N$, we want to consider the homotopy structure on $N \times D^k$ given by the product map $f \times \mathrm{Id}_{D^k}$. Standard properties of normal invariants imply that $\eta(f \times \mathrm{Id}_{D^k}) = p^*\eta(f)$, where $\eta(-)$ denotes the normal invariant and $p^* : [N, G/\mathrm{PL}] \to [N \times D^k, G/\mathrm{PL}]$ is induced by the coordinate map $p : N \times D^k \to N$; the map p^* is an isomorphism because D^k is contractible. Since $k \geq 3$, it follows that the maps $\pi_i(N \times S^{k-1}) \to \pi_i(N \times D^k)$ are isomorphisms for $i = 0$ or 1, and hence the $\pi - \pi$ theorem of [74] implies that the normal invariant map $\mathbb{S}(N \times D^k) \to [N \times D^k, G/\mathrm{PL}]$ is 1-1 and onto. By the embedding theorem of Browder, Casson, Haefliger, Sullivan and Wall (see [58], (8.10), p. 161), there is a k-dimensional block bundle over N — call it ξ — and a PL-homeomorphism φ from its total space $E(\xi)$ to $M \times D^k$ such that $[f \times \mathrm{Id}_{D^k}] \circ \varphi$ is homotopic to the identity (see [59], [60], [61] for background on block bundles). These data correspond to a unique class $\alpha \in [N, G_k/\widetilde{\mathrm{PL}}_k]$ with the following properties:

(i) The image of α in $[N, G/\mathrm{PL}]$ under a canonical stabilization map $G_k/\widetilde{\mathrm{PL}}_k \to G/\mathrm{PL}$ (which is a homotopy equivalence) is the normal invariant $\eta(f)$.
(ii) The image of α in $[N, \mathrm{B}\widetilde{\mathrm{PL}}_k]$ under a canonical map $G_k/\widetilde{\mathrm{PL}}_k \to \mathrm{B}\widetilde{\mathrm{PL}}_k$ classifies the block bundle ξ.

In the theory of block bundles, a block bundle ξ is trivial if and only if $E(\xi)$ is PL-homeomorphic to $N \times D^n$. Therefore, $M \times D^k$ is PL-homeomorphic to $N \times D^k$ if and only if the image of α in $[N, \mathrm{B}\widetilde{\mathrm{PL}}_k]$ is trivial. The latter is true if and only if α lies in the image of the map $[N, G_k] \to [N, G_k/\mathrm{PL}_k]$, and hence the result follows in the PL category. The proof in the topological category is similar, but one must replace the theory of PL block bundles with a corresponding theory of topological regular neighborhoods as in [62] and [24]. One crucial step in the PL proof uses the fact that the stabilization map $G_k/\widetilde{\mathrm{PL}}_k \to G/\mathrm{PL}$ is a homotopy equivalence if $k \geq 3$. The corresponding fact for the map $G_k/\widetilde{\mathrm{Top}}_k \to G/\mathrm{Top}$ is contained in [62]. □

If X is a connected finite complex, then diagram (3.3) yields an isomorphism of sets from the stable cohomotopy group $\{X, S^0\}$ to $[X, SG]$. Under this isomorphism, the image of the map $[X, SG_k] \to [X, SG]$ is trapped between the images of the iterated suspension homomorphisms $[S^{k-1}X, S^{k-1}] \to \{X, S^0\}$ and $[S^kX, S^k] \to \{X, S^0\}$. The results of [30] show that the images of $[X, SG_k] \to [X, SG]$ correspond to the image of $[S^kX, S^k] \to \{X, S^0\}$ if $\dim X \leq 2k - 2$. We shall also need the following criteria for determining whether a class in $[X, SG]$ lifts back to $[X, SG_k]$:

Proposition 3.3. *Let X be a connected finite complex and let $\alpha \in [X, SG]$ be a class such that α lifts to $[X, SG_3]$; take the group structures on these spaces induced by the composition products on the function spaces $\mathcal{F}(S^3, S^3)$ and $\lim\limits_{m\to\infty} \mathcal{F}(S^m, S^m)$. Then $\alpha = \alpha_1\alpha_2$ where α_2 lies* *in the image of* $[X, SO] \to [X, SG]$ *(where* SO *is the group* $\lim_{m\to\infty} SO_m$*) and* α_1 *corresponds to an element in the image of* $[S^2X, S^2] \to \{X, S^0\}$.

Proof. It will suffice to show that the images of $[X, SG_3]$ and $[X, SF_2]$ in $[X, G/O]$ are equal, for this implies that the image of $[X, SG_3]$ in $[X, SG]$ is generated by $[X, SF_2]$ and $[X, SO]$, and by (3.3) the image of $[X, SF_2]$ in $[X, SG]$ corresponds to the image of $[S^2X, S^2]$ in $\{X, S^0\}$.

We begin with the following commutative diagram whose rows are given by fibrations:

$$\begin{array}{ccccccccc}
SO_2 & \longrightarrow & SO_3 & \longrightarrow & S^2 & \longrightarrow & BSO_2 & \longrightarrow & BSO_3 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
SF_2 & \longrightarrow & SG_3 & \longrightarrow & S^2 & \longrightarrow & BSF_2 & \longrightarrow & BSG_3
\end{array}$$

It follows that the fibers of $BSO_2 \to BSF_2$ and $BSO_3 \to BSG_3$, which are SF_2/SO_2 and SG_3/SO_3, are homotopy equivalent. Since the map $SO_3 \to SO$ is well known to be 2-connected and $SG_3 \to SG$ is also 2-connected by [30], it follows that $SG_3/SO_3 \to G/O$ is 2-connected. Therefore, $\pi_1(SG_3/SO_3) \cong \pi_1(G/O) = 0$, so that SF_2/SO_2 is also simply connected. Furthermore, since SO_2 is aspherical it follows that the composite of the universal covering space projection $\widetilde{SF}_2 \to SF_2$ and the canonical map $SF_2 \to SF_2/SO_2$ is a homotopy equivalence. Thus, we have shown that the images of $[X, SG_3]$ and $[X, SF_2]$ in $[X, G/O]$ are equal. Finally, since the image of $[X, SF_2]$ lies between the images of $[X, \widetilde{SF}_2]$ and $[X, G_3]$, it follows that the images of all three of these groups in $[X, G/O]$ must coincide. $\square$

Proposition 3.4. *Let* p *be an odd prime, let* $k \geq 2$*, and let* $\alpha \in [X, SG]$ *be an element of order* p^r *for some* $r > 0$*. Then* α *lies in the image of* $[X, SG_{2k}] \to [X, SG]$ *if and only if* α *corresponds to an element in the image of* $[S^{2k-1}X, S^{2k-1}] \to \{X, S^0\}$.

Proof. We shall work with p-localization in the sense of Sullivan [69]. Since connected H-spaces all have good localization at p, it is meaningful to discuss the localized spaces

$$SG_{(p)}, \quad SG_{2k(p)}, \quad SF_{2k-1(p)}, \quad S^{2k-1}_{(p)}, \quad SO_{2k(p)}, \quad \Omega^\infty_0 S^\infty_{(p)} \quad \text{and} \quad \Omega^{2k-1}_0 S^{2k-1}_{(p)}$$

where $\Omega^m_0 Y$ denotes the path component of the constant map in the iterated loop space $\Omega^m Y$. Note that if W is an arcwise connected H-space whose homotopy groups are all finite, then W is naturally homotopy equivalent to the weak product of its localizations $W_{(q)}$ at all primes q; in particular, this applies to the H-spaces $SG \simeq \Omega^\infty_0 S^\infty$ and $SF_{2k-1} \simeq \Omega^{2k-1}_0 S^{2k-1}$.

Recall that we have a fibration $SO_{2k-1} \to SO_{2k} \to S^{2k-1}$ and that the tangent bundle $T(S^{2k})$ is classified by a map $S^{2k-1} \to SO_{2k}$ such that the composite $S^{2k-1} \to SO_{2k} \to S^{2k-1}$ has degree 2. If we compose the map $S^{2k-1} \to SO_{2k}$ with the inclusion of SO_{2k} in SG_{2k} and the fibration $SG_{2k} \to S^{2k-1}$, the resulting composite also have degree 2. Therefore the map

$$SF_{2k-1} \times S^{2k-1} \longrightarrow SG_{2k} \times SG_{2k} \xrightarrow{\text{mult.}} SG_{2k}$$

becomes a homotopy equivalence when localized at the odd prime p. Since the composite

$$S^{2k-1} \to SO_{2k} \to SO$$

is nullhomotopic, it follows that the image of $[X, SG_{2k(p)}]$ in $[X, SG_{(p)}] \cong [X, SG]_{(p)}$ is equal to the image of $[X, SF_{2k-1(p)}]$, which corresponds to the image of $[S^{2k-1}X, S^{2k-1}_{(p)}] \cong [S^{2k-1}X, S^{2k-1}]_{(p)}$ in $\{X, S^0\}_{(p)}$; note that the codomain is the Sylow p-subgroup of $\{X, S^0\}$ with respect to the loop sum, and likewise the domain is the Sylow p-subgroup of the finite group $[S^{2k-1}X, S^{2k-1}]$. These observations imply that if $\alpha \in [X, SG]$ is p-primary with respect to the composition product (which is homotopy abelian) and lifts to $[X, SG_{2k}]$, then α corresponds to an element of $\{X, S^0\}$ which desuspends to $[S^{2k-1}X, S^{2k-1}]$. □

We shall also need the following result:

Proposition 3.5. *If $\alpha \in [X, SG]$ has odd order and lies in the image of $[X, SG_3]$, then the image of α in $[X, G/O]$ is trivial.*

Proof. Since the finite abelian group $[X, SG]$ splits into a product of the groups $[X, SG]_{(q)}$, where q runs through all odd primes, it will suffice to prove the result when the order of α is a power of some odd prime p.

By the proof of Proposition 3.5 we may assume that α lies in the image of $[X, \widetilde{SF}_2]$, and since the homotopy groups of $\widetilde{SF}_2$ and $\widetilde{\Omega^2_0 S^2}$ are finite, we can say that α corresponds to a class in $\{X, S^0\}_{(p)}$ which lies in the image of $[S^2X, S^2]_{(p)}$.

If $h : S^3 \to S^2$ is the Hopf whose fiber is S^1, then composition with h defines a homotopy equivalence from $\Omega^2_0 S^3$ to $\Omega^2_0 S^2$, where as before $\Omega^m_0 Y$ denotes the path component of the constant map in $\Omega^m Y$. Therefore, it follows that α corresponds to a class in $\{X, S^0\}_{(p)}$ which lies the in image of the composite

$$h : [S^2X, S^3]_{(p)} \to [S^2X, S^2]_{(p)}$$

and hence α factors homotopically as a composite $\overline{h} \circ \beta$, where β lies in $\{X, S^1\}_{(p)}$ and $\overline{h}$ denotes the image of h in the stable group $\{S^3, S^2\}_{(p)} \cong \{S^1, S^0\}_{(p)}$. Finally, since $\{S^1, S^0\} \cong \mathbb{Z}_2$ we have $\{S^1, S^0\}_{(p)} = 0$, and hence α corresponds to the trivial element of $\{X, S^0\}_{(p)}$, where we interpret the latter as a subgroup ot $\{X, S^0\}$. □

Corollary 3.1. *The same conclusion holds if we replace G/O by G/PL or G/Top.*

4. Tangential Equivalence of Lens Spaces

Throughout this section p will denote a fixed odd prime. We have already mentioned Folkman's result on tangentially homotopy equivalent lens spaces of sufficiently high dimension. In this section we shall derive a stronger version of Folkman's conclusion when the fundamental groups of the lens spaces are isomorphic to $\mathbb{Z}_p$:

Proposition 4.1. *Let M and N be $(2k-1)$-dimensional lens spaces that are (stably) tangentially equivalent. If $k \geq p-1$, then M and N are diffeomorphic (and in fact isometric).*

Proof. The results of Folkman yield that conclusion when $k \geq 2p-1$ so it is only necessary to prove the result when $k \leq 2p-1$, so that $p-1 \leq k \leq 2(p-1)$.

Let V and W be free representations of $\mathbb{Z}_p$ such that the associated lens spaces $L(V)$ and $L(W)$ are homotopy equivalent; assume that V and W are at least 4-dimensional. The free $\mathbb{Z}_p$-actions on the universal coverings $S(V)$ and $S(W)$ determine isomorphisms λ_V and λ_W from $\pi_1(L(V))$ and $\pi_1(LW))$ to $\mathbb{Z}_p$, respectively, and if h is a homotopy equivalence from $L(V)$ to $L(W)$ then we obtain an automorphism $\beta = \lambda_V h_* \lambda_W^{-1}$ of $\mathbb{Z}_p$. If $\widetilde{h} : S(V) \to S(W)$ is the associated map of universal covering spaces, then $\widetilde{h}$ satisfies the semi-equivariance identity $\widetilde{h}(g \cdot v) = \beta(g) \cdot \widetilde{h}(v)$. If we define a new representation V' with the same underlying vector space as V and a group action given by $g * v = \beta^{-1}(g) \cdot v$, then we may view $\widetilde{h}$ as a $\mathbb{Z}_p$-equivariant homotopy equivalence from $S(V')$, which equals $S(V)$ as a set, to $S(W)$. This means that we might as well assume the representations V and W are chosen so that β is the identity and $\widetilde{h}$ is equivariant. This is important for computational purposes because it yields the following commutative diagram, in which the map $RO(G)$ to $KO_G(X)$ sends a representation V to the class of the trivial $\mathbb{Z}_p$-vector bundle $[V] = X \times V$:

$$\begin{tikzcd}
RO(\mathbb{Z}_p) \arrow[r] \arrow[d, equal] & KO_{\mathbb{Z}_p}(S(W)) \arrow[r, "\cong"] \arrow[d, "\widetilde{h}^*"', "\cong"] & KO(L(W)) \arrow[d, "h^*"'] \\
RO(\mathbb{Z}_p) \arrow[r] & KO_{\mathbb{Z}_p}(S(V)) \arrow[r, "\cong"] & KO(L(V))
\end{tikzcd}$$

The horizontal arrows on the right are the standard isomorphisms $KO_G(X) \cong KO(X/G)$ for a free G-space X.

The KO-groups in the diagram are given by the results of T. Kambe [34]; in that paper the complex K-groups are computed, but one can extract computations for KO because the reduced KO groups of $L(V)$ and $L(W)$ are finite p-primary abelian groups, which implies that $\widetilde{KO}(L(V))$ and $\widetilde{KO}(L(W))$ are isomorphic to the self-conjugate elements in $\widetilde{K}(L(V))$ and $\widetilde{K}(L(W))$. For our purposes, the most important aspects of the computations are that the maps from $RO(\mathbb{Z}_p)$ to $KO(L(V))$ and $KO(L(W))$ are onto, and if $A \subseteq RO(\mathbb{Z}_p)$ denotes the commons kernel of these maps, then A is contained in $p \cdot RO(\mathbb{Z}_p) \subseteq RO(\mathbb{Z}_p)$. Hence we can adjoin the following commutative square to the preceding diagram, and in the expanded diagram the composites

$$RO(\mathbb{Z}_p) \to KO(L(V)) \to RO(\mathbb{Z}_p) \otimes \mathbb{Z}_p \quad \text{and} \quad RO(\mathbb{Z}_p) \to KO(L(W)) \to RO(\mathbb{Z}_p)$$

are the canonical maps induced by the mod p reduction map $\mathbb{Z} \to \mathbb{Z}_p$.

The preceding discussion yields the following consequences. Since the stable tangent bundles of $L(V)$ and $L(W)$ are the images of V and W under the maps from $RO(\mathbb{Z}_p)$ to $\widetilde{KO}(L(V))$ and $\widetilde{KO}(L(W))$, the tangential homotopy equivalence h yields the condition $V - W \in A$, and hence it follows that $V - W \in p \cdot RO(\mathbb{Z}_p)$.

On the other hand, since h is a homotopy equivalence the bundle $[V] - [W]$ is (stably) fiber homotopically trivial since $\widetilde{KO}(L(V))$ and $\widetilde{KO}(L(W))$ are p-primary; results of Adams ([1], [2] and [3]) imply that $[V] - [W]$ lies in the kernel of $\psi^r - 1$, where ψ^r is the r^{th} Adams operation. Since the nontrivial irreducible representations of $\mathbb{Z}_p$ are 2-dimensional and given by the 1-dimensional unitary representations ρ_k sending (g, z) to $g^k z$, where $a \leq k \leq \frac{1}{2}(p-1)$, it follows that ψ^r sends ρ_k to ρ_{kr} (where the subscript in ρ_{kr} is computed mod p and $\rho_j = \rho_{-j}$). Therefore the actions of ψ^r on $KO(L(V))$ and $KO(L(W))$ pass to the self-map of $RO(\mathbb{Z}) \otimes \mathbb{Z}_p$ sending ρ_k to ρ_{kr} for each k. Since r is a primitive root of unity mod p^2, it follows that the images of the stably fiber homotopically trivial elements in $RO(\mathbb{Z}_p)/RO(1)$ are precisely the linear combinations of the form $m \cdot \sum \rho_j$, where the summation is over all j such that $1 \leq j \leq \frac{1}{2}(p-1)$.

Express V and W as direct sums of irreducible representations $\sum a_j \rho_j$ and $\sum b_j \rho_j$, where $a_j, b_j \geq 0$. Since dim $V=$ dim $W \leq 4p - 4$, it follows that $\sum a_j = \sum b_j \leq 2p - 2$. By the preceding discussions we know that $a_j - b_j \equiv 0 \pmod{p}$ for all j and that $a_j \equiv a_1 \equiv b_1 \equiv b_j \pmod{p}$ for all j. The final step is to show that there are very few sequences a_j, b_j satisfying all these conditions unless $a_j = b_j$ for all j, and in all such cases the lens spaces $L(V)$ and $L(W)$ are diffeomorphic.

Suppose that $\{a_j\}$ and $\{b_j\}$ satisfy all the conditions given above. Since $\sum a_j = \sum b_j \leq 2p - 2$, there is at most one j_1 such that $a_{j_1} \geq p$ and at most one j_2 such that $b_{j_2} \geq p$. Since $a_j \equiv b_j \pmod{p}$, this implies that either $a_j = b_j$ for all j or else we have $a_{j_1} - b_{j_1} = p$, $b_{j_2} - a_{j_2} = p$ and $a_j = b_j$ otherwise. Furthermore, since $a_j \equiv a_{j_1} \pmod{p}$ and $b_j = b_{j_2} \pmod{p}$ for all j, it follows that there is some $c \geq 0$ such that $c < p$, $c = b_{j_1} = a_{j_2}$, and $a_j = b_j = c$ for $j \neq j_1, j_2$. It follows that

$$\sum a_j \ = \ \sum b_j \ = \ \frac{c(p-1)}{2} + p \ .$$

Twice this number is equal to $\dim V = \dim W$, and therefore the inequality $\dim V = \dim W \leq 4p - 4$ implies that $c = 0$ or $c = 1$. Also, since the right hand side is $\geq p$, it follows that we have eliminated the case where $\dim V = \dim W = 2(p-1)$; in other words, $a_j = b_j$ in this case, so that $L(V)$ is diffeomorphic to $L(W)$.

We shall extend the definitions of a_j and b_j to all nonzero elements of $\mathbb{Z}_p$ by setting $a_{-j} = a_j$ and $b_{-j} = b_j$. With these conventions the complexifications of V and W are given by $V \otimes \mathbb{C} = \sum a_j t^j$ and $W \otimes \mathbb{C} = \sum b_j t^j$ where t^j is the 1-dimensional unitary representation sending (z, v) to $z^j v$. Note that V and W are equivalent real representations if and only if $V \otimes \mathbb{C}$ and $W \otimes \mathbb{C}$ are equivalent complex representations. The standard diffeomorphism criteria for lens spaces (see [19]) implies that $L(V)$ and $L(W)$ are diffeomorphic if there is some $s \neq 0$ in $\mathbb{Z}_p$ such that $a_j = b_{sj}$ for all j.

Let $c = 0$ or 1, and assume that V and W satisfy the previous conditions on the coefficients a_j and b_j. Then $b_{j_2} = a_{j_1} = p + c$ implies that $b_{-j_2} = a_{-j_1} = p + c$; if we choose s to be the unique element of $\mathbb{Z}_p$ such that $j_2 = sj_1$, then it follows that

$$a_{j_1} \;=\; b_{sj_1} \;=\; p + c \;=\; b_{-sj_1} \;=\; a_{-j_1}$$

and also $a_j = b_{sj} = c$ where $j \neq \pm j_1$. Therefore the diffeomorphism criteria in the previous paragraph implies that $L(V)$ is diffeomorphic $L(W)$. □

Proof. (Theorem 2) All we need to do is check that the lens space diffeomorphisms are Riemannian isometries. This fact is contained in Folkman's work if $n > 2(p-1)$. In the remaining cases, the discussion in the second paragraph of the proof for 4.1 shows that it suffices to consider cases where the map $L(V) \to L(W)$ lifts to a $\mathbb{Z}_p$-equivariant map $S(V) \to S(W)$, and by the diffeomorphism criterion from [19], the sufficient condition in fact implies that the lens spaces are isometric (see [19], Section 31). □

Proof. (Theorem 1) If $\dim M = \dim N \geq 2p - 1$, then M and N are diffeomorphic by Theorem 2, so it is only necessary to consider cases where the dimension $2n - 1$ is $\leq 2p - 3$. Given f as in the theorem, the normal invariant $\eta(f)$ of f lies in $[N, G/O]$, and by the $\pi - \pi$ Theorem the map $f \times \mathrm{Id}_{D^3}$ is h-cobordant to a diffeomorphism if and only if $\eta(f)$ is trivial. Therefore the analog of Proposition 3.1 in the smooth category will imply that $f \times \mathrm{Id}_{\mathbb{R}^3}$ is properly homotopic to a diffeomorphism if and only if $\eta(f)$ is trivial, and accordingly it will suffice to prove the latter.

Since f is tangential, the normal invariant lifts back to $[N, SG] \cong \{N, S^0\}$. Furthermore, since the universal coverings of M and N are both diffeomorphic to S^{2n-1}, it follows that the pullback of $\eta(f)$ under the universal covering map $S^{2n-1} \to N$ is trivial.

Consider the Atiyah-Hirzebruch spectral sequence for $\{N, S^0\}$. Its E_2 terms have the form $\widetilde{H}^i(N; \pi_i)$ where π_i is the i^{th} stable stem $\pi_i(SG)$. These groups contain no p-torsion if $i < 2p-3$ (*e.g.*, see [71]), and since $\widetilde{H}_i(N; \mathbb{Z}) \cong \mathbb{Z}_p$ for i odd or 0 for i even when $i \leq 2n-2$, it follows that $\widetilde{H}^i(N; \pi_i) = 0$ if $i \leq 2n-2$ and $\widetilde{H}^{2n-1}(N; \pi_{2n-1}) \cong \pi_{2n-1}$. This means that the degree 1 collapsing map from N to S^{2n-1} induces an isomorphism from π_{2n-1} to $\{N, S^0\}$. But the composite $S^{2n-1} \to N \to S^{2n-1}$ has degree p, so that the composite $\pi_{2n-1} \to \{N, S^0\} \to \pi_{2n-1}$ is also multiplication by p.

If $n \leq p-2$, then π_{2n-1} has order prime to p and hence the map $\{N, S^0\} \to \pi_{2n-1}$ is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_{2n-1} & \longrightarrow & \{N, S^0\} & \longrightarrow & \pi_{2n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{2n-1}(G/O) & \longrightarrow & [N, G/O] & \longrightarrow & \pi_{2n-1}(G/O)
\end{array}$$

We already know that $\eta(f)$ lifts to a class x_0 in $\{N, S^0\}$ and hence lifts to a class x in π_{2n}. If y denotes the image of x in $\pi_{2n-1}(G/O)$, then y maps to $\eta(f)$. We also know that $\eta(f)$ maps to zero in $\pi_{2n-1}(G/O)$, and from this we may conclude that $py = 0$ in $\pi_{2n-1}(G/O)$. Now, $\pi_{2n-1}(G/O)$ has no nontrivial elements of order p, and therefore $y = 0$ so that $\eta(f) = 0$ and we are done if $n \leq p-2$.

We are left with the case $n = p-1$, in which $\pi_{2n-1} = \pi_{2p-3}$ is the direct sum of $\mathbb{Z}_p$ and a group of order prime to p []; we also know that the p-torsion maps to zero in $\pi_{2p-3}(G/O)$. Therefore, if we consider the same diagram as before we again obtain a class $y \in \pi_{2p-3}(G/O)$ which maps to $\eta(f)$, but in this case we can conclude that the order of y is prime to p, regardless of whether or not p divides the order of x. One can now reason as before to conclude that y and $\eta(f)$ must be trivial. $\square$

Note. A closer examination of results due to J. Ewing, S. Moolgavkar, R. Stong and L. Smith [25] shows that for each $n \geq 2$ there are infinitely many primes p for which one has homotopy equivalent but not diffeomorphic lens spaces that are stably parallelizable. So for each $n \geq 2$ there are many examples of nonhomeomorphic lens spaces M^{2n-1} and N^{2n-1} such that $M \times \mathbb{R}^3$ and $N \times \mathbb{R}^3$ are diffeomorphic.

4.1. **Addendum—Tangential Self-Equivalences of Lens Spaces.** At the beginning of this paper we stated the following result without proof:

Proposition 4.2. *Let M^{2n-1} be a lens space with fundamental group $\mathbb{Z}_p$. Then every tangential homotopy self-equivalence of M^{2n-1} is homotopic to a diffeomorphism (in fact, an isometry).*

This turns out to be a straightforward consequence of Folkman's theorem [26].

Proof. Suppose that M^{2n-1} is given by a $\mathbb{Z}_p$ representation $\rho_{a_1} + \cdots + \rho_{a_n}$, where $1 \leq a_j \leq \frac{1}{2}(p-1)$, and write $M = L(a_1, \ldots, a_n)$.

If $f : M^{2n-1} \to M^{2n-1}$ is a tangential homotopy self-equivalence, then there is some unit $v \in \mathbb{Z}_p$ such that f_* is multiplication by v. As usual, it follows that $L(a_1, \ldots, a_n)$ is isometric to $L(va_1, \ldots, va_n)$, where we now define $\rho_a = \rho_{p-a}$ if $\frac{1}{2}(p+1) \leq a \leq p-1$. On the universal covering space level this yields a $\mathbb{Z}_p$-equivariant map of spheres

$$\widetilde{g} : S(va_1, \ldots, va_n) \to S(a_1, \ldots, a_n)$$

covering a tangential homotopy equivalence

$$g : L(va_1, \ldots, va_n) \to L(a_1, \ldots, a_n).$$

If we take an equivariant join $\widetilde{g_0}$ of $\widetilde{g}$ with the identity and pass to orbit spaces, we obtain a new tangential homotopy equivalence

$$g_0 : L(va_1, \ldots, va_n, 1, \ldots, p-1) \to L(a_1, \ldots, a_n, 1, \ldots, p-1)$$

Given a sequence $(b_1, \ldots, b_n)$ of nonzero integers mod p, and some fixed nonzero integer k mod p, let $\mu[k](b_1, \ldots, b_n)$ be the number of b_i's such that $b_i \equiv \pm k \bmod p$. In our preceding examples, we clearly have

$$\mu[k](va_1, \ldots, va_n, 1, \ldots, p-1) \;=\; \mu[k](va_1, \ldots, va_n) + 2$$

$$\mu[k](a_1, \ldots, a_n, 1, \ldots, p-1) \;=\; \mu[k](a_1, \ldots, a_n) + 2$$

for all k. We can now apply Folkman's theorem to conclude that the two quantities on the left sides of the displayed equations are equal. Therefore, we also have

$$\mu[k](va_1, \ldots, va_n) \;=\; \mu[k](a_1, \ldots, a_n)$$

for all k. Therefore, there is an isometry φ from $L(va_1, \ldots, va_n)$ to $L(a_1, \ldots a_n)$ which induces the identity on fundamental groups. Since g is a tangential homotopy equivalence and induces the identity on fundamental groups, the maps φ and g agree on fundamental groups and have equal degrees, and hence they must be homotopic. $\square$

5. Normal Invariants for Tangential Homotopy Lens Spaces

Throughout this section p will denote a fixed odd prime. If $f : M \to N$ is a homotopy equivalence of compact topological manifolds (possibly with boundary) and $\eta(f) \in [N, G/\mathrm{Top}]$ is its normal invariant, then f is a tangential homotopy equivalence if and only if a canonical map from $[N, G/\mathrm{Top}]$ to $[N, B\,\mathrm{STop}]$ sends $\eta(f)$ to zero (*e.g.* see [42]), and by the exactness of the fibration sequence $SG \to G/\mathrm{Top} \to B\,\mathrm{STop}$ the image vanishes if and only if $\eta(f)$ lies in the image of the associated map from $[N, SG]$ to $[N, G/\mathrm{Top}]$. In this section we shall describe this image when N is a $\mathbb{Z}_p$ lens space. Our analysis is based upon fundamental results on the structure of the localized spaces $SG_{(p)}$, $G/O_{(p)}$, $G/\mathrm{Top}_{(p)}$, $BSO_{(p)}$, $B\,\mathrm{STop}_{(p)}$ and similar objects; some basic references are [42], chapter V of [48] and lecture 4 of [4].

We are particularly interested in the structure of $SG_{(p)}$ and $G/\mathrm{Top}_{(p)}$. Results of Sullivan (compare [42]) imply that the localized spaces $BSO_{(p)}$ and $G/\mathrm{Top}_{(p)}$ are homotopy equivalent and that $G/O_{(p)}$ is homotopy equivalent to $BSO_{(p)} \times \mathrm{Cok}\ J_{(p)}$ for some space $\mathrm{Cok}\ J_{(p)}$ (see [42] for the definition of the latter). Furthermore, if J_p is defined as the fiber of the map $\psi^r - 1 : BSO_{(p)} \to BSO_{(p)}$, where r is a primitive root of unity mod p^2 and ψ^r is the Adams operation in K-theory, then there is a homotopy equivalence from $SG_{(p)}$ to $J_p \times \mathrm{Cok}\ J_{(p)}$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccccc} SG_{(p)} & \longrightarrow & G/O_{(p)} & \longrightarrow & G/\mathrm{Top}_{(p)} \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ J_p \times \mathrm{Cok}\ J_{(p)} & \xrightarrow{\beta \times 1} & BSO_{(p)} \times \mathrm{Cok}\ J_{(p)} & \xrightarrow{\varphi} & BSO_{(p)} \end{array}$$

In this diagram $\beta : J_p \to BSO_{(p)}$ is the homotopy fiber of $\psi^r - 1$ and φ factors as follows:

$$BSO_{(p)} \times \mathrm{Cok}\ J_{(p)} \xrightarrow{\text{proj.}} BSO_{(p)} \xrightarrow{\varphi'} BSO_{(p)}$$

Since $[N, J_p]$ can often be computed fairly directly but $[N, \mathrm{Cok}\ J_{(p)}]$ generally cannot, these splittings are very helpful for describing the image of $[N, SG_{(p)}]$ in $[N, G/\mathrm{Top}_{(p)}]$.

In addition to the splittings described above, there are also splittings of $BSO_{(p)}$ that will be useful in this section. The results of [42] and [4] imply that the localized complex K-theory spectrum $K_{(p)}$ splits into a sum of $(p-1)$ periodic spectra $E_\alpha K_{(p)}$, where α runs through the elements of $\mathbb{Z}_{p-1}$. Each of these spectra is periodic of period $2p-2$, and the coefficient groups $E_\alpha K_{(p)}(S^n)$ are given by $\mathbb{Z}_{(p)}$ if $n \equiv 2\alpha \mod 2p-2$ and zero otherwise.

If we view the localized real K-theory spectrum $KO_{(p)}$ as the direct summand given by the self-conjugate part of $K_{(p)}$, then $KO_{(p)}$ corresponds to the sum of the spectra $E_\beta K_{(p)}$, where β runs through all the even elements of $\mathbb{Z}_{p-1}$. We shall be particularly interested in $E_0 K_{(p)}$; which is just the first (p-local) Morava K-theory $K(1)$, with $K(1)(S^n) = \mathbb{Z}_{(p)}$ if $n \equiv 0 \mod 2p-2$ and 0 otherwise (see [76] for background on Morava K-theories).

If L is a $\mathbb{Z}_p$ lens space, then there is a canonical map k_L from L to the classifying space $B\mathbb{Z}_p$, and our analysis of the image of $[L, SG] \to [L, G/\mathrm{Top}]$ begins with a study of the analogous problem with $B\mathbb{Z}_p$ replacing L. Both $[B\mathbb{Z}_p, SG] \cong \{B\mathbb{Z}_p, S^0\}$ and $[B\mathbb{Z}_p, BSO] \cong \widetilde{KO}(B\mathbb{Z}_p)$ are well understood; results of D.W. Anderson (summarized in [7], with full details in [8]) imply that the latter is isomorphic to the completion of the ideal $IO(\mathbb{Z}_p)$ in the real representation ring $RO(\mathbb{Z}_p)$ given by all 0-dimensional virtual representations (this also follows directly from [9]), while the proof of the Segal Conjecture for $\mathbb{Z}_p$ (see [57] and [5]) implies that $\{B\mathbb{Z}_p, S^0\}$ is isomorphic to the completion of the ideal $IA(\mathbb{Z}_p)$ in the Burnside ring $A(\mathbb{Z}_p)$ given by all virtual finite $\mathbb{Z}_p$-sets with virtual cardinality 0. Although the set theoretic isomorphism from $\{B\mathbb{Z}_p, S^0\}$ to $[B\mathbb{Z}_p, SG]$ is not additive, one can prove that the latter is also isomorphic to the completion of $IA(\mathbb{Z}_p)$ using the methods of [9], and this is explained in [41]. The ideal $IA(\mathbb{Z}_p)$ is infinite cyclic and it turns out that the image of one generator in the completed ideal $\widehat{IA(\mathbb{Z}_p)} \cong \{B\mathbb{Z}_p, S^0\}$ corresponds to the reduced stable homotopy-theoretic transfer $B\mathbb{Z}_p \to S^0$ associated to the standard p-fold covering $E\mathbb{Z}_p \to B\mathbb{Z}_p$ (see [32] and [33]), whose total space is contractible.

It is fairly straightforward to prove that the completion $\widehat{IA(\mathbb{Z}_p)}$ is topologically and additively isomorphic to the additive p-adic integers $\widehat{\mathbb{Z}_{(p)}}$ and $\widehat{IO(Z_p)}$ is similarly isomorphic to a sum of $\frac{1}{2}(p-1)$ copies of the $\widehat{\mathbb{Z}_{(p)}}$. One can describe these groups and their interrelationships more precisely as follows:

Let $I(\mathbb{Z}_p)$ be the ideal in the complex representation ring $R(\mathbb{Z}_p)$ given by the kernel of the virtual dimension map from $R(\mathbb{Z}_p)$ to $\mathbb{Z}$. Then $\widetilde{K}(B\mathbb{Z}_p)$ is isomorphic to the completion $\widehat{I(\mathbb{Z}_p)}$ by [9], and hence it is a free $\widehat{Z_p}$-module on $(p-1)$ generators. We can choose these free generators to have the form $\mathbf{e}_a$, where a runs through the nonzero elements of $\mathbb{Z}_p$, and if r is a primitive root of unity mod p^2 then the Adams operation ψ^r on $\widetilde{K}(B\mathbb{Z}_p)$ sends $\mathbf{e}_a$ to $\mathbf{e}_{ra}$; furthermore, if θ is the additive automorphism of $\mathbb{Z}_p$ sending $a \in \mathbb{Z}_p$ to ra and $B\theta$ is the induced self-map of $B\mathbb{Z}_p$ which induces θ on the fundamental group level (so $B\theta$ is unique up to homotopy) then the induced automorphism $B\theta^*$ in K-theory also sends e_a to e_{ra}. Furthermore, the complexification map from $\widetilde{KO}(B\mathbb{Z}_p)$ to $\widetilde{K}(B\mathbb{Z}_p)$ is split injective, and its image is the free submodule whose generators have the form $e_a + e_{-a}$, where a runs through all nonzero elements of $\mathbb{Z}_p$ (note that there are $\frac{1}{2}(p-1)$ elements of this form). With this background, we can describe a canonical homomorphism from $[B\mathbb{Z}_p, SG]$ to $[B\mathbb{Z}_p, BSO]$ as follows:

Proposition 5.1. *Let $F : SG_{(p)} \to BSO_{(p)}$ be the composite*

$$SG_{(p)} \longrightarrow G/O_{(p)} \simeq BSO_{(p)} \times \text{Cok } J_p \longrightarrow BSO_{(p)}$$

where the final arrow is coordinate projection. Then the image of $[B\mathbb{Z}_p, SG] \cong [B\mathbb{Z}_p, SG_{(p)}]$ in $[B\mathbb{Z}_p, BSO] \cong [B\mathbb{Z}_p, BSO_{(p)}]$ corresponds to the split free submodule of $\widetilde{K}(B\mathbb{Z}_p) \cong \bigoplus^{p-1} \widehat{\mathbb{Z}_{(p)}}$ generated by the sum of the basis elements $\sum_a \mathbf{e}_a$, and the image corresponding to the direct summand $E_0K_{(p)}$ in $KO(B\mathbb{Z}_p) \cong KO_{(p)}(B\mathbb{Z}_p)$.

Proof. If $V : G/O \to BSO$ is the homotopy fiber of $BSO \to BSG$, then by construction the composite

$$BSO_{(p)} \xrightarrow{\text{slice}} BSO_{(p)} \times \text{Cok } J_p \simeq G/O_{(p)} \xrightarrow{V_{(p)}} BSO_{(p)}$$

is given by $\psi^r - 1$. Now $V_{(p)}$ is trivial on Cok J_p, and therefore for all connected CW complexes the image of $[X, SG_{(p)}]$ in $[X, G/O_{(p)}]$ will be the kernel of the map $V_{(p)*} : [X, G/O_{(p)}] \to [X, BSO_{(p)}]$. If we combine these we see that the kernel of $V_{(p)*}$ is generated by $[X, \text{Cok } J_p]$ and the kernel of $\psi^r - 1$ on $\widetilde{KO}_{(p)}(X)$. If we let $X = B\mathbb{Z}_p$, then the localized and unlocalized groups are isomorphic, and if we expand an element ξ of $\widetilde{KO}_{(p)}(B\mathbb{Z}_p)$ as $\sum c_a \mathbf{e}_a$ for suitable corefficients c_a (note that $c_a = c_{-a}$), then ξ lies in the kernel of $\psi^r - 1$ if and only if $c_{ra} = c_a$ for all a. We claim this happens if and only if the coefficients c_a are all equal. Sufficiency is obvious; on the other hand, it follows by induction that $c_{r^k a} = c_a$ for all k and a, and since the powers r^k exhaust the nonzero elements of $\mathbb{Z}_p$ we must have $c_a = c_b$ for all $a, b \neq 0$. If we now denote the image of $[B\mathbb{Z}_p, SG]$ in $\widetilde{KO}(B\mathbb{Z}_p)$ as M, the preceding discussion shows that M is a direct summand of $\widetilde{KO}(B\mathbb{Z}_p)$ which is isomorphic to $\widehat{\mathbb{Z}_{(p)}}$ and the complementary summand M' is a free $\widehat{\mathbb{Z}_{(p)}}$-module on $(p-2)$ generators. In particular, $M' \cong \widetilde{KO}(B\mathbb{Z}_p)/M$ is torsion free.

Claim 5.1. *M is contained in the summand $E_0K_{(p)}(B\mathbb{Z}_p)$*

Proof. To see this, let $E_0^{\perp}K_{(p)}$ denote the sum of the other cohomology theories $E_iK_{(p)}$, and let $\overline{M}$ denote the projection of M onto $E_0^{\perp}K_{(p)}$ with respect to the splitting $\widetilde{KO}(B\mathbb{Z}_p) \cong E_0K_{(p)}(B\mathbb{Z}_p) \oplus E_0^{\perp}K_{(p)}(B\mathbb{Z}_p)$. We know that $\psi^r - 1$ restricted to M is trivial, but we also know that $\psi^r - 1$ restricted to $E_0^{\perp}K_{(p)}(B\mathbb{Z}_p)$ is injective (compare [42]), and these combine to imply that $\overline{M}$ is trivial, so that M must be contained in $E_0K_{(p)}(B\mathbb{Z}_p)$. □

The results of [34] imply that the summand $E_0K_{(p)}(B\mathbb{Z}_p)$ of $\widetilde{K}(B\mathbb{Z}_p)$ must also be isomorphic to $\widehat{\mathbb{Z}_{(p)}}$, and the complementary summand $E_0^{\perp}K_{(p)}(B\mathbb{Z}_p)$ must be torsion free. Therefore the quotient $\widetilde{KO}(B\mathbb{Z}_p)/M$ is isomorphic to the direct sum of $E_0^{\perp}K_{(p)}(B\mathbb{Z}_p)$ and a quotient

$M_1 \cong \widehat{\mathbb{Z}_{(p)}}/M$, where M is also isomorphic to $\widehat{\mathbb{Z}_{(p)}}$. If M is a proper subgroup of $\widehat{\mathbb{Z}_{(p)}}$, then the quotient M_1 must be a nontrivial finite cyclic p-group, and therefore the quotient M_1 has nontrivial elements of finte order. Since M is a direct summand of $\widetilde{KO}(B\widehat{\mathbb{Z}_{(p)}})$ and the latter is a direct sum of $\frac{1}{2}(p-1)$ copies of $\widehat{\mathbb{Z}_{(p)}}$, this cannot happen and hence we must have $M = E_0K_{(p)}(B\mathbb{Z}_p)$. $\square$

We also have a similar conclusion regarding the image of $[B\mathbb{Z}_p, SG]$ in $[B\mathbb{Z}_p, G/\mathrm{Top}] \cong [B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}]$; as noted before, the results of Sullivan show the codomain is isomorphic to $\widetilde{KO}(B\mathbb{Z}_p)$.

Proposition 5.2. *The image of the composite*

$$[B\mathbb{Z}_p, SG] \longrightarrow [B\mathbb{Z}_p, G/\mathrm{Top}] \cong \widetilde{KO}(B\mathbb{Z}_p)$$

is the image of $E_0K_{(p)}(B\mathbb{Z}_p)$, and the kernel of this map is trivial.

Proof. To prove this first statement, we need to analyze the image of the map

$$E_0K_{(p)}(B\mathbb{Z}_p) \to \widetilde{KO}(B\mathbb{Z}_p) \subseteq [B\mathbb{Z}_p, G/O_{(p)}] \to [B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}] \cong \widetilde{KO}(B\mathbb{Z}_p)$$

and the composite of this map with the splitting retraction $\widetilde{KO}(B\mathbb{Z}_p) \to E_0K_{(p)}(B\mathbb{Z}_p)$. We claim this composite is an isomorphism. Since the composite is given by a natural transformation of cohomology theories, it suffices to show that this transformation induces an isomorphism of cohomology theories, and the latter in turn reduces to showing that the induced self-maps of the localized homotopy groups $\pi_{2k(p-1)}(BSO)_{(p)}$ are isomorphisms.

We have the following commutative diagram, in which each group except $\pi_{2k(p-1)}(BJ_p)$ is a direct sum of $\mathbb{Z}_{(p)}$ and a finite abelian p-group:

$$\begin{tikzcd}
\pi_{2k(p-1)}(BSO)_{(p)} \arrow[r, "\text{splitting}", "\text{injection}"'] \arrow[d] & \pi_{2k(p-1)}(G/O)_{(p)} \arrow[r, "\beta_*^O"] \arrow[d] & \pi_{2k(p-1)}(BSO)_{(p)} \arrow[r, "e_*"] \arrow[d] & \pi_{2k(p-1)}(J_p) \arrow[d, "="] \\
\pi_{2k(p-1)}(BSO)_{(p)} \arrow[r, "\cong", "\text{Sullivan}"'] & \pi_{2k(p-1)}(G/\mathrm{Top})_{(p)} \arrow[r, "\beta_*^{\mathrm{Top}}"] & \pi_{2k(p-1)}(B\,\mathrm{STop})_{(p)} \arrow[r] & \pi_{2k(p-1)}(J_p)
\end{tikzcd}$$

As noted in [42] (see p. 117), modulo torsion the two vertical arrows on the left are multiplication by the number explicitly given on that page. The maps β_*^O and β_*^{Top} give the underlying bundles, and the first line is exact β_*^O and e_*; furthermore, up to units in $\mathbb{Z}_{(p)}$, the map β_*^O is multiplication by the order of the inverse of the J-homomorphism, and this order is divisible by p. On the other hand, up to torsion and units in $\mathbb{Z}_{(p)}$, the maps $\pi_{2k(p-1)}(G/O)_{(p)} \to \pi_{2k(p-1)}(G/\mathrm{Top})_{(p)}$ and $\pi_{2k(p-1)}(BSO)_{(p)} \to \pi_{2k(p-1)}(B\,\mathrm{STop})_{(p)}$ are multiplication by

$$c = (2^{k(p-1)-1} - 1) \cdot \mathrm{NUM}\left(\frac{B_{k(p-1)}}{2kp - 2k}\right)$$

where "NUM(...)" denotes the numerator of a fraction reduced to least terms and $B_{k(p-1)}$ is the appropriate Bernoulli number (see [42], p. 117, for the first map and [16] for the second). Therefore, modulo torsion, the right hand square is given by

$$\begin{array}{ccc} \mathbb{Z}_{(p)} & \longrightarrow & \mathbb{Z}_{p^m} \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ \mathbb{Z}_{(p)} & \longrightarrow & \mathbb{Z}_{p^m} \end{array}$$

where the horizontal arrows are epimorphisms and the top arrows the standard quotient projection. Such a diagram can exist only if c is relatively prime to p, and therefore the two vertical arrows at the left are isomorphisms, which is what we wanted to prove.

Finally, we need to check that the image of $[B\mathbb{Z}_p, SG]$ in $[B\mathbb{Z}_p, G/\mathrm{Top}]$ corresponds to $E_0K_{(p)}(B\mathbb{Z}_p)$ and that $[B\mathbb{Z}_p, SG]$ is mapped isomorphically onto its image. By the preceding discussion we know that this image is a direct summand of $[B\mathbb{Z}_p, G/\mathrm{Top}] \cong \widetilde{KO}(B\mathbb{Z}_p)$ and is isomorphic to $\widehat{\mathbb{Z}_{(p)}}$. Thus the map from $[B\mathbb{Z}_p, SG]$ to its image is given by a surjective homomorphism from $\widehat{\mathbb{Z}_{(p)}}$ to itself. Since every such surjection is an isomorphism, we see that $[B\mathbb{Z}_p, SG]$ must be mapped isomorphically to its image. To prove that the image in $[B\mathbb{Z}_p, G/\mathrm{Top}] \cong \widetilde{KO}(B\mathbb{Z}_p)$ is $E_0K_{(p)}(B\mathbb{Z}_p)$ we can use the reasoning in the proof of Proposition 5.1 to reduce the question to checking that the image of $[B\mathbb{Z}_p, SG]$ in $[B\mathbb{Z}_p, G/\mathrm{Top}] \cong \widetilde{KO}(B\mathbb{Z}_p)$ is contained in the kernel of $\psi^r - 1$. We have already noted that on $\widetilde{KO}(B\mathbb{Z}_p)$ one has $\psi^r = B\theta^*$ for some automorphism θ of $\mathbb{Z}_p$, so everything reduces to showing that the map

$$\widetilde{KO}(B\mathbb{Z}_p) \subseteq [B\mathbb{Z}_p, G/\mathrm{Top}] \longrightarrow [B\mathbb{Z}_p, G/\mathrm{Top}] \cong \widetilde{KO}(B\mathbb{Z}_p)$$

sends the kernel of $B\theta^* - 1$ to itself. Since the displayed map arises from some self-map of $BSO_{(p)}$, it follows immediately that this mapping does send the kernel to itself, proving the remaining assertions in the proposition. □

Now let L be a $(2n-1)$-dimensional lens space, and let $\eta_L : L \to B\mathbb{Z}_p$ be its classifying map. We may assume that $B\mathbb{Z}_p$ is constructed so that its $(2n-1)$-skeleton is L, and we shall do so henceforth. Our next objective it to derive analogs of Propositions 5.1 and 5.2 in which $B\mathbb{Z}_p$ is replaced by L.

More precisely, we need to extend our observations about the map

$$[B\mathbb{Z}_p, SG] \to [B\mathbb{Z}_p, G/\mathrm{Top}]$$

into an effective analysis of all the objects and morphisms in the following commutative diagram:

$$\begin{array}{ccccc}
[B\mathbb{Z}_p, SG] & \longrightarrow & [B\mathbb{Z}_p, G/\mathrm{Top}] & \xrightarrow{\cong} & \widetilde{KO}(B\mathbb{Z}_p) \\
{\scriptstyle \eta^*}\downarrow & & {\scriptstyle \eta^*}\downarrow & & {\scriptstyle \eta^*}\downarrow \\
[L, SG] & \longrightarrow & [L, G/\mathrm{Top}] & \xrightarrow{\cong} & \widetilde{KO}(L)
\end{array}$$

The results of [34] show that the induced map in reduced KO-theory is surjective and yield an explicit description of its kernel; the following result on the summand $E_0K_{(p)}$ is a straightforward consequence of the methods and conclusion of [4]:

Proposition 5.3. *If L is a $(2n-1)$-dimensional $\mathbb{Z}_p$ lens space, then the Atiyah-Hirzebruch spectral sequence of $E_0K_{(p)}(L)$ collapses, and this group is cyclic of order p^m where $m = \left[\frac{n-1}{p-1}\right]$ (and $[-]$ denotes the greatest integer function).*

Proof. (*Sketch*) The spectral sequence collapses because the analogous spectral sequence for $\widetilde{KO}_{(p)}(L)$ collapses (see [9] or [34]), and the cyclic nature of the group follows because $\widetilde{KO}(B\mathbb{Z}_p) \to KO_{(p)}(L)$ is onto (see [34]) and $E_0K_{(p)}$ is a direct summand of KO. ☐

In contrast to this result, the map from $[B\mathbb{Z}_p, SG]$ to $[L, SG_{(p)}]$ is not necessarily onto, but we shall show that the image is a natural direct summand which maps, with an easily described kernel, onto $E_0K_{(p)}(L) \subseteq KO_{(p)}(L) \cong [L, G/\mathrm{Top}_{(p)}]$ and the complementary summand of $[L, SG_{(p)}]$ maps to zero in $[L, G/\mathrm{Top}_{(p)}]$. The summands of $SG_{(p)}$ are given by the splitting $SG_{(p)} \simeq J_p \times \mathrm{Cok}\ J_p$; results of [48] imply this decomposition comes from a splitting of infinite loop spaces.

Most of what we need to know about $[B\mathbb{Z}_p, J_p] \to [L, J_p]$ is contained in the following results.

Proposition 5.4. *We have $[B\mathbb{Z}_p, \mathrm{Cok}\ J_p] = 0$, and $[B\mathbb{Z}_p, SG] \cong [B\mathbb{Z}_p, SG_{(p)}]$ is isomorphic to $[B\mathbb{Z}_p, J_p]$*

Proof. (*Sketch*) Propositions 5.1 and 5.2 imply that the composite

$$\begin{array}{ccccc}
[B\mathbb{Z}_p, J_p] \times [B\mathbb{Z}_p, \mathrm{Cok}\ J_p] & \longrightarrow & [B\mathbb{Z}_p, SG_{(p)}] \times [B\mathbb{Z}_p, SG_{(p)}] & & \\
\| & & \downarrow & & \\
[B\mathbb{Z}_p, SG_{(p)}] & & [B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}] \times [B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}] & \xrightarrow{\oplus} & [B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}]
\end{array}$$

in which the first horizontal arrow is inclusion, is split injective, and since the composite $\mathrm{Cok}\ J_p \to SG_{(p)} \to G/\mathrm{Top}_{(p)}$ is nullhomotopic the displayed composite can be rewritten more simply as

$$[B\mathbb{Z}_p, SG_{(p)}] \xrightarrow{\text{proj}} [B\mathbb{Z}_p, J_p] \longrightarrow [B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}]$$

It follows that the projection map induces a split injection from $[B\mathbb{Z}_p, SG_{(p)}]$ to $[B\mathbb{Z}_p, J_p]$. Since the projection is onto by construction, it follows that the map $[B\mathbb{Z}_p, SG_{(p)}] \to [B\mathbb{Z}_p, J_p]$ is an isomorphism, proving one assertion in the proposition. To see that $[B\mathbb{Z}_p, \mathrm{Cok}\ J_p] = 0$, notice that if this group were nonzero then the projection $[B\mathbb{Z}_p, SG_{(p)}] \to [B\mathbb{Z}_p, J_p]$ would not be injective. □

We are now ready to analyze objects like $[L, SG_{(p)}]$ and its summands where L is a lens space as above.

Proposition 5.5. *Let L be a $(2n-1)$-dimensional $\mathbb{Z}_p$ lens space, let $\eta : L \to B\mathbb{Z}_p$ denote its classifying map and let $q : L \to S^{2n-1}$ be a map of degree 1 (which is unique up to homotopy). Then $[L, J_p]$ is the sum of the image of $\eta^* : [B\mathbb{Z}_p, J_p] \to [L, J_p]$ and $q^* : \pi_{2n-1}(J_p) \to [L, J_p]$. The image of η^* is cyclic of order p^m, where $m = \left[\frac{n}{p-1}\right]$, the map q^* is injective and the structures of $[L, J_p]$ and the map $[L, J_p] \to [L, G/\mathrm{Top}] \cong \widetilde{KO}(L)$ are given as follows:*

- (i) *Suppose that $n \not\equiv 0 \mod p-1$, so that $\pi_{2n-1}(J_p) = 0$. Then $[B\mathbb{Z}_p, J_p] \to [L, J_p]$ is onto and $[L, J_p] \to [L, G/\mathrm{Top}_{(p)}] \cong \widetilde{KO}(L)$ is split injective with image corresponding to $E_0K_{(p)}(L)$. Furthermore, the latter also equals the image of $[L, SG_{(p)}] \to [L, G/\mathrm{Top}_{(p)}]$, and this map is a split injection.*
- (ii) *Suppose that $n = p^\nu(p-1)r$ where r is prime to p, so that $\pi_{2n-1}(J_p) \cong \mathbb{Z}_{p^\nu}$. Then the images of η^* and q^* intersect in a subgroup of order p, the image of q^* is the kernel of the map $[L, SG_{(p)}] \to [L, G/\mathrm{Top}_{(p)}]$ and the image of the latter is given by $E_0K_{(p)}(L)$. Furthermore, the latter is also equal to the images ot $[L, J_p] \to [L, G/\mathrm{Top}_{(p)}]$ and the kernel of this map has order p.*

Note: Since the group $[L, \mathrm{Cok}\ J_p]$ is usually nontrivial, the map $[B\mathbb{Z}_p, SG_{(p)}] \to [L, SG_{(p)}]$ is usually not onto; furthermore, since the homotopy groups of Cok J_p are given by largely unknown factors in the stable homotopy groups of spheres, the groups $[L, \mathrm{Cok}\ J_p]$ are usually not easy to describe explicitly. This leads to major complications in studying the notion of smooth tangential thickness for lens spaces.

Notation: Given a CW complex X and an arcwise connected space Y, the *skeletal filtration* of the set of homotopy classes $[X, Y]$ is the family of subsets

$$\mathbf{XF}_k([X, Y]) = \{u \in [X, Y] \ : \ u \,|\, X_k \text{ is homotopically trivial}\} .$$

Clearly if $f : X' \to X$ is a cellular map, then the map $f^* : [X, Y] \to [X', Y]$ is also filtration preserving. Similarly, if $g : Y \to Y'$ is continuous, then $g_* : [X, Y] \to [X, Y']$ is filtration preserving. If Y is a double loop space, set

$$\mathbf{FF}_k([X,Y]) = \mathbf{XF}_k([X,Y]) \setminus \mathbf{XF}_{k-1}([X,Y])$$

and note that (i) this has a natural abelian group structure and (ii) $\mathbf{FF}_k([X,\Omega^2 W])$ is functorial in the second variable W.

Proof. The first step is an anlysis of the Atiyah-Hirzebruch spectral sequence for $[B\mathbb{Z}_p, J_p]$ and $[L, J_p]$.

Claim: *The Atiyah-Hirzebruch spectral sequence for $[B\mathbb{Z}_p, J_p]$ collapses.*

Proof. The relevant E_2 terms are given by $\widetilde{H}^i(B\mathbb{Z}_p; \pi_i(J_p))$; these groups are isomorphic to $\mathbb{Z}_p$ if $i = 2k(p-1)-1$ for some integer k and zero otherwise. We also know that $[B\mathbb{Z}_p, J_p]$ maps isomorphically to $E_0K_{(p)}(B\mathbb{Z}_p)$, and the collapsing Atiyah-Hirzebruch spectral sequence for the latter has E_2 terms given by $\widetilde{H}^i(B\mathbb{Z}_p; E_0K_{(p)}(S^i))$, which are isomorphic to $\mathbb{Z}_p$ if $i \equiv 0 \bmod 2(p-1)$ and zero otherwise. It is a fairly straightforward exercise to check that the bijectivity of $[B\mathbb{Z}_p, J_p] \to E_0K_{(p)}(B\mathbb{Z}_p)$ implies that the spectral sequence for $[B\mathbb{Z}_p, J_p]$ must also collapse. □

Next, we shall use the naturality properties of the Atiyah-Hirzebruch spectral sequence to analyze $[L, J_p]$ and related objects. Let $\lambda : S^{2n-1} \to L$ be the universal covering projection; then the mapping cone $\widehat{L}$ of λ can be viewed as the $2n$-skeleton of $B\mathbb{Z}_p$ and the restriction $H^*(B\mathbb{Z}_p) \to H^*(\widehat{L})$ is an isomorphism in dimensions $\leq 2n$ for all coefficients. Therefore a naturality argument implies that the Atiyah-Hirzebruch spectral sequence for $[\widehat{L}, J_p]$ collapses and the restriction $\widehat{\mathbb{Z}}_{(p)} \cong [B\mathbb{Z}_p, J_p] \to [\widehat{L}, J_p]$ is onto with image $\mathbb{Z}_{p^m}$ where $m = \left[\frac{n}{p-1}\right]$. Since $\pi_{2n}(J_p) = 0$, the Barratt-Puppe exact sequence associated to

$$S^{2n-1} \xrightarrow{\lambda} L \longrightarrow \widehat{L} \longrightarrow S^{2n} \longrightarrow \cdots$$

implies that the restriction map $[\widehat{L}, J_p] \to [L, J_p]$ is injective, and hence the image of $[B\mathbb{Z}_p, J_p]$ in $[L, J_p]$ is also cyclic of order p^m (where m is as given above). To describe the entire group $[L, J_p]$, let $L_0 = L \setminus \mathrm{Int}\ D$ where D is a smoothly embedded closed $(2n-1)$-disk; then L_0 may be viewed as a $(2n-2)$-skeleton for $B\mathbb{Z}_p$ and $H^*(B\mathbb{Z}_p) \to H^*(L_0)$ is an isomorphism in dimensions $\leq 2n-2$. As before, it follows that $[B\mathbb{Z}_p, J_p] \to [L_0, J_p]$ is onto. If we consider the exact sequence for the Barratt-Puppe sequence

$$S^{2n-2} \subseteq L_0 \subseteq L \xrightarrow{q} S^{2n-1}$$

we see that if $\overline{y} \in [L, J_p]$, the restriction of y to $[L_0, J_p]$, is the image of some class $z \in [B\mathbb{Z}_p, J_p]$ and therefore $y - \eta^* z \in [L, J_p]$ must lie in the image of q^*. Thus the images of q^* and η^* generate $[L, J_p]$. To see that q^* is injective, we begin by noting that $\pi_{2n-1}(J_p) = 0$ unless

$n \equiv 0 \bmod p-1$, and if $n = p^\nu(p-1)r$, where $\nu \geq 1$ and r is prime to p, then $\pi_{2n-1}(J_p)$ is cyclic of order p^ν (see [1]). If $n \equiv 0 \bmod p-1$, then our computations of $[L, J_p]$ and $[\widehat{L}, J_p]$ show that q^* maps a generator for the p-torsion in $\pi_{2n-1}(J_p) \cong \mathbb{Z}_{p^\nu}$ to a class of order p in the image of $\eta^* : [B\mathbb{Z}_p, J_p] \to [L, J_p]$. In particular, q^* maps the p-torsion injectively, and hence it must map the entire cyclic p-group $\pi_{2n-1}(J_p)$ injectively. Observe that if $n \equiv 0 \bmod p-1$, the preceding argument and skeletal filtration considerations show that the intersection of $\mathrm{Im}\ \eta^*$ and $\mathrm{Im}\ q^*$ is a cyclic subgroup of order p.

We must now describe the image of $[L, J_p]$ in $[L, G/\mathrm{Top}_{(p)}]$. First of all, we claim that the map $[L, J_p] \to [L, G/\mathrm{Top}_{(p)}]$ is trivial on the image of q^*. More or less by construction, the composite maps of homotopy groups $\pi_*(SO) \to \pi_*(SG) \to \pi_*(J_p)$ are onto in all dimensions, and since the composite in the commutative diagram

$$\begin{array}{ccccc} SO & \longrightarrow & SG_{(p)} & \longrightarrow & G/O_{(p)} \\ & & \downarrow & & \downarrow \\ & & J_p & \longrightarrow & G/\mathrm{Top}_{(p)} \end{array}$$

is homotopically trivial, and therefore the composite

$$\begin{array}{ccccc} \pi_{2n-1}(SO)_{(p)} & \xrightarrow{\text{onto}} & \pi_{2n-1}(J_p) & \longrightarrow & \pi_{2n-1}(G/\mathrm{Top}_{(p)}) \\ & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\ & & [L, J_p] & \longrightarrow & [L, G/\mathrm{Top}_{(p)}] \end{array}$$

must be zero.

The final step is to check that the morphism from the cyclic p-group $[L, J_p]$ to the cyclic p-group $[L, G/\mathrm{Top}_{(p)}]$ maps onto the summand $E_0K_{(p)}(L)$ and the kernel is precisely the image of q^*. Is is convenient to split the discussion into two cases, depending on whether or not $n \equiv 0 \bmod p-1$. In both cases the argument uses the commutative diagram

$$\begin{array}{ccc} [L, J_p] & \longrightarrow & [L, G/\mathrm{Top}_{(p)}] \\ \downarrow & & \downarrow \\ [L_0, J_p] & \longrightarrow & [L_0, G/\mathrm{Top}_{(p)}] \end{array}$$

in which the vertical arrow on the right is an isomorphism by standard results on $\widetilde{KO}(L)$ and $\widetilde{KO}(L_0)$ which follow from the collapsing of their Atiyah-Hirzebruch spectral sequence.

Case (i): If $\not\equiv 0 \bmod p-1$, then the restriction map from $[L, J_p]$ to $[L_0, J_p]$ is an isomorphism, and the restrictions $[B\mathbb{Z}_p, J_p] \to [L_0, J_p]$ and $[B\mathbb{Z}_p, G/\mathrm{Top}_{(p)}] \to [L_0, G/\mathrm{Top}_{(p)}]$ are onto with isomorphic images. A diagram chase now shows that $[L_0, J_p] \to [L_0, G/\mathrm{Top}_{(p)}]$ is a split injection whose image is $E_0K_{(p)}(L)$, which is what we wanted to prove.

Case (ii): If $n \equiv 0 \bmod p-1$, then the kernel of the restricion map from $[L, J_p]$ to $[L_0, J_p]$ is the image of q^*, and the kernel of the map from Im β^* to $[L_0, J_p]$ has order p. As in the preceding case, the map $[L_0, J_p] \to [L_0, G/\mathrm{Top}_{(p)}]$ is an isomorphism, so the conclusion in this case also follows from a diagram chase.

To see the statements about the images of $[L, SG_{(p)}]$ and $[L, J_p]$ in $[L, G/\mathrm{Top}_{(p)}]$, note that these images are equal by the splitting $SG_{(p)} \simeq J_p \times \mathrm{Cok}\ J_p$ and the homotopic triviality of Cok $J_p \to G/\mathrm{Top}_{(p)}$. □

6. Desuspension Results

If X is a connected finite complex, it is well known that the standard "loop sum with identity" bijection from $\{X, S^0\}$ to $[X, SG]$ is not necessarily a homomorphism with respect to the loop sum structure on the domain and the composition/direct sum structure on the codomain; specifically, if we view $\{X, S^0\}$ as a ring using the standard smash product ring spectrum structure on the spectrum for S^0, then the composition/direct sum structure is given by:

$$a \circ b = a + b + ab$$

(*i.e.*, the Perlis circle operation; see [35], p. 81, line 4, or [53], Section 9.4, p. 298). Fortunately, one can often show that these two algebraic structures are similar in key respects (for example, they are equal if X is a suspension [75], pp. 124-125). In particular, we have the following:

Proposition 6.1. *Let r be an arbitrary positive integer, and let $\sigma : \{B\mathbb{Z}_p, S^0\} \to [B\mathbb{Z}_p, SG]$ be the standard set-theoretic isomorphism such that $\sigma^{-1}(u \oplus v) = u + v + uv$. Then $y \in \{B\mathbb{Z}_p, S^0\}$ is divisible by p^r with respect to the loop sum operation if and only if σy is divisible by p^r with respect to the composition or direct sum (or circle) operation.*

Proof. By construction the standard map from $\Omega_0^\infty S^\infty$ to SG induces a set-theoretic bijection from $\{B\mathbb{Z}_p, S^0\}$ to $[B\mathbb{Z}_p, SG]$ which is skeletal filtration preserving. The sets in these skeletal filtrations are subgroups with respect to the standard binary operations on the respective sets. Therefore the sets $\mathbf{XF}_k(\{B\mathbb{Z}_p, S^0\})$ are subgroups with respect to both the loop sum and the circle operation corresponding to the operation on $[B\mathbb{Z}_p, SG]$. Furthermore, it follows that each subquotient $\mathbf{FF}_k(\{B\mathbb{Z}_p, S^0\})$ has group structures given by each binary operation. These subquotients have order equal to 1 or p; since $\{B\mathbb{Z}_p, S^0\}$ and $[B\mathbb{Z}_p, SG]$ are both isomorphic to $\widehat{\mathbb{Z}_{(p)}}$, this means that the classes in $\mathbf{FF}_k(\{B\mathbb{Z}_p, S^0\})$ are precisely those which are divisible by the same prime power p^t with respect to each operation. □

We shall need the following dualization of Propostion 6.1 for lens spaces:

Proposition 6.2. *Let $T(L) \subseteq \{L, S^0\}$ denote the image of $\{B\mathbb{Z}_p, S^0\}$ in $\{L, S^0\}$, so that $T(L)$ corresponds to a cyclic subgroup $[L, J_p]$ of order p^m in $[L, SG]$, where*

$$m = \left[\frac{n}{p-1}\right].$$

Then $T(L)$ is a cyclic subgroup of $\{L, S^0\}$ with respect to the loop sum, and for all positive integers t, a class $x \in T(L)$ has order p^t with respect to the loop sum if and only if it has order p^t with respect to the circle operation.

Proof. The assertion that $T(L)$ is a finite cyclic group follows because the image of $\{B\mathbb{Z}_p, S^0\}$ is a subgroup with respect to the loop sum, the group $\{L, S^0\}$ is finite, and a finite quotient of $\widehat{Z_{(p)}} \cong \{B\mathbb{Z}_p, S^0\}$ must be cyclic. As before the sets in the skeletal filtration are subgroups with respect to both binary operations, and the subquotients either have order 1 or p. Since the set of all elements of exponent p^t in $\mathbb{Z}_{p^m}$ is cyclic of order p^t, it follows that there is some k such that $\mathbf{XF}_k(\{B\mathbb{Z}_p, S^0\})$ has order p^t and the latter contains all elements of exponent p^t with respect to both operations. Likewise, there is some $k' > k$ such that $\mathbf{XF}_{k'}(\{B\mathbb{Z}_p, S^0\})$ has order p^{t-1} with respect to either operation. Therefore, $\mathbf{XF}_k - \mathbf{XF}_{k'}$ is the set of all elements with order p for each operation. $\square$

In view of the results from Section 3, we are interested in determining how far one can desuspend the classes in $T(L)$, and here is the main result:

Proposition 6.3. *Let k be an integer such that $1 \leq k \leq m-1$, where L, n and m are given as above, Then a class in $T(L)$ desuspends to $[S^{2k-1}L, S^{2k-1}]$ if and only if its order divides p^k.*

Proof. Fundamental results of F. Cohen, J.C. Moore and J. Neisendorfer [20], [21] imply that if a p-primary element α in the stable homotopy groups of spheres desuspends to $\pi_{m+2k+1}(S^{2k+1})$, then the orders of the element α and its preimage have orders dividing p^k. In fact these methods immediately yield a far more general conclusion:

Lemma 6.1. *Let X be a finite complex, and let α be a p-primary element of the stable cohomotopy group $\{X, S^0\}_{(p)}$ which desuspends to $[S^{2k+1}X, S^{2k+1}]_{(p)}$. Then α and its preimage have orders dividing p^k.*

The "only if" part of Proposition 6.3 is an immediate consequence of this result.

Proof. (Lemma 6.1) As noted in [49], Cor. 11.8.2, p. 461, if $\Psi_p : \Omega^2 S^{2r+1} \to \Omega^2 S^{2r+1}$ is the double looping of the degree p self-map for S^{2r+1}, then $\Psi_p = \sigma \circ \pi$, where $\sigma : S^{2r-1} \to \Omega^2 S^{2r+1}$ is adjoint to the identity. Therefore, if Y is a connected finite complex, then the H-space structure on S^{2r+1} and the square lemma (see the previous citation from [75]) imply that if $\beta \in [S^{2r+1}Y, S^{2r+1}]_{(p)}$, then $p \cdot \beta$ desuspends to $[S^{2r-1}Y, S^{2r-1}]_{(p)}$. One can now proceed by induction as in [20] and [49] to conclude that $p^r \cdot \beta = 0$ (*e.g.*, see the proof of [49], Cor. 11.8.3, p. 462). $\square$

Proof of necessity in Proposition 6.3. By Lemma 6.1 and Proposition 6.2, it will suffice to show that a generator τ of $T(L)$ desuspends to S^{2t+1}, where $t = \left[\frac{n}{p-1}\right]$. Since τ has order p^t by Proposition 6.2, we can use Lemma 6.4 to conclude that τ cannot double desuspend any further. Similarly, if $r < t$, then it will follow that $p^r\tau$ must desuspend to $S^{2(t-r)+1}$ but cannot double desuspend any further. The conclusion in Proposition 6.3 follows because a multiple $a\tau$ of τ satisfies $p^k(a\tau) = 0$ if and only if a is divisible by p^{t-k}.

It is well known that the localized stabilization maps

$$S^{2m+1}_{(p)} \longrightarrow Q_0(S^{2m+1}) \ = \ \lim \ \Omega_0^{2m+1} S^{2m+1}_{(p)}$$

are very highly connected. In fact, using the fibration sequences in (1.5.3) and (1.5.5) of [56], and an inductive argument, one can prove that the localized stabilization map is $(2(m+1)(p-1)-3)$-connected (a related statement appears in [56]). Therefore, if $2n-1 \leq 2(m+1)(p-1)-3$, then τ (and its loop sum multiples) will automatically desuspend to S^{2m+1}. In particular, if the preceding inequality holds when $m = \left[\frac{n}{p-1}\right]$, then τ will desuspend to S^{2m+1}, and hence the conclusion of Proposition 6.3 will follow. $\square$

Write $n = j(p-1) + s$, where $0 \leq s \leq p-1$, so that $j = \left[\frac{n}{p-1}\right]$. With this notation the dimension versus connectivity inequality reduces to

$$m \ \geq \ \frac{n+1}{p-1} - 1 \ = \ j + \frac{s+1}{p-1} - 1$$

and, as indicated in the preceding paragraph, we want to verify that this holds when $m = j$. To see this, note that $0 \leq s \leq p-2$ implies

$$-1 \ < \ \frac{s+1}{p-1} - 1 \ \leq \ \frac{p-1}{p-1} - 1 \ = \ 0$$

and therefore we do have

$$j \ \geq \ j + \frac{s+1}{p-1} - 1$$

which is what we wanted to verify.

As at the beginning of this section, let X be a connected finite complex. The remarks in the first paragraph of this section show that, if we take the loop sum operation on $\{X, S^0\}$

and the direct sum operation on $[X, G/\mathrm{Top}]$, then the composite

$$\{X, S^0\} \longrightarrow [X, SG] \longrightarrow [X, G/\mathrm{Top}]$$

is not usually additive. However, we have the following useful result:

Proposition 6.4. *In the setting above, there is an infinite loop space structure on* G/Top *such that the displayed composite is a homomorphism.*

In fact, this structure is given by suitable versions of D. Sullivan's Characteristic Variety Theorem (compare [68], [31] or [50]).

Proof. If X is a closed oriented manifold, then the infinite loop space structure on G/Top has the following description on the the set $[X, G/\mathrm{Top}]$: (i) Take Sullivan's family of morphisms $\varphi_i : V_i \to X$, where each V_i is either a closed manifold or a near-manifold with explicitly specified singularities. (ii) For each $\alpha \in [X, G/\mathrm{Top}]$ construct surgery problems associated to the various classes $\varphi_i^*\alpha \in [X, G/\mathrm{Top}]$, and take their Kervaire invariant or (possibly reduced) signature invariants which live in suitable cyclic abelian groups Λ_i. These yield an embedding of $[X, G/\mathrm{Top}]$ into $\prod_i \Lambda_i$, and the abelian group operation on $[X, G/\mathrm{Top}]$ given by this embedding corresponds to the Characteristic Variety infinite loop space structure on G/Top (the associated spectrum is frequently denoted by symbols like $\mathbb{L}(1)$).

Suppose now that we are given classes u and v in $\{X, S^0\}$, and let $\chi(u), \chi(v) \in \prod_i \Lambda_i$ by given by the Characteristic Variety construction. We need to show that $\chi(u + v) = \chi(u) + \chi(v)$. One way of constructing tangential surgery problems associated to u and v is to begin by taking their S-duals, which lie in the stable homotopy groups $\pi^{\mathbf{S}}_{\dim X}(X^\nu)$, where as usual X^ν denotes the Thom complex of the (formally) 0-dimensional stable normal bundle ν of X. If we make these dual maps "transverse to the zero section" (stably of course), we obtain degree zero tangential normal maps (f_i, b_i) for suitable $f_i : Y_i \to X$ $(i = u, v)$. The surgery problems associated to u, v and $u + v$ are then given by

$$(f_u, b_u) \amalg \mathrm{Id}_X, \qquad (f_v, b_v) \amalg \mathrm{Id}_X, \qquad (f_u, b_u) \amalg (f_v, b_v) \amalg \mathrm{Id}_X$$

respectively. It is now straightforward to check that if $\chi(u)$ and $\chi(v)$ are the characteristic variety surgery obstructions for u and v respectively, then $\chi(u) + \chi(v)$ will give the characterstic variety surgery obstructions for $u + v$ (see [54] for a more detailed analysis of such problems). □

The preceding result yields a useful complement to Proposition 6.3.

Proposition 6.5. *Let* p *be an odd prime, let* X *be a closed oriented manifold, and let* $a \in [X, G/\mathrm{Top}]_{(p)}$ *be a class which lies in the image of*

$$[S^{2k+1}X, S^{2k+1}]_{(p)} \longrightarrow \{X, S^0\}_{(p)} \cong [X, SG]_{(p)} \longrightarrow [X, G/\mathrm{Top}]_{(p)}$$

where $k \geq 1$. If $*$ denotes the binary operation on the codomain given by the Characteristic Variety Theorem and $*^p y$ denotes $y * y * \cdots * y$ (p factors), then $*^p a$ lies in the image of $[S^{2k-1}X, S^{2k-1}]_{(p)}$.

Remark. The results of [6] imply that the direct sum and Characteristic Variety structures determine isomorphic group structures on $[X, G/\mathrm{Top}]_{(p)}$, but the self-map inducing this isomorphism is not necessarily the identity map.

Proof. Let $a' \in [S^{2k+1}X, S^{2k+1}]_{(p)}$ be a preimage of a. Then, if $*$ denotes the loop sum in $[S^{2k+1}X, S^{2k+1}]_{(p)}$, the results of Cohen, Moore and Neisendorfer imply that $*^p a'$ lies in the image of $[S^{2k-1}X, S^{2k-1}]_{(p)}$. Since the displayed composite is additive if we take the loop space sum on the domain and the Characteristic Variety sum on the codomains, it follows that $*^p a$ lifts in the described manner. □

In the next section we shall prove a similar result if the Characteristic Variety operation is replaced by the direct sum and X is a mod p lens space (see Proposition 7.2).

7. Proofs of Theorems 5–8

As in Sections 3–6, unless stated otherwise, we take p to be a fixed odd prime.

All that remains is to combine the results of Sections 3–6 into proofs of the results on $\mathfrak{T}\mathfrak{T}_k^{\mathrm{Top}}(L)$, where $k \geq 3$ and L is a $\mathbb{Z}_p$ lens space. In fact, since the orbit space of an arbitrary free $\mathbb{Z}_p$ action on a sphere is homotopy equivalent to a lens space, one can extend the entire discussion to cases where L is a fake lens space. We begin with the result (Theorem 6) characterizing the normal invariants of homotopy structures in $\mathfrak{T}\mathfrak{T}_k^{\mathrm{Top}}(L)$ for $k \geq 3$.

Proof. (Theorem 6) By Proposition 3.2 the set $\theta_k([M, G/\mathrm{Top}])$ consists of all classes which are in the image of the normal invariant map η and in the image of the map

$$[M, SG_k] \longrightarrow [M, G/\mathrm{Top}]$$

We are assuming that the image of η is a subgroup, so it suffices to check that the images of $[M, SG_k]$ in $[M, G/\mathrm{Top}]$ is a subgroup. The composition product defines a group structure on $[M, SG_k]$, and the stabilization map from the latter to $[M, SG]$ is a homomorphism with respect to the composition operation of SG. But the composition and direct sum operations are identical on the set of homotopy classes $[M, SG]$, and since $[M, SG] \to [M, G/\mathrm{Top}]$ is a homomorphism with respect to connected sum, it follows that the image of $[M, SG_k]$ in $[M, G/\mathrm{Top}]$ is a subgroup. □

We can now prove Theorem 5 very easily.

Proof. (Theorem 5) Suppose that L is homotopy equivalent to a $\mathbb{Z}_p$ lens space. Then $[L, G/\mathrm{Top}]$ is a cyclic p-group (hence of odd order), and Proposition 3.5 implies that the image of $[L, SG_3]$ in $[L, G/O]$, and hence also in $[L, G/\mathrm{Top}]$, must also be trivial. But this means that $\theta_3([L, G/\mathrm{Top}]) = 0$ and therefore a homotopy structure in $\mathcal{TT}_3^{\mathrm{Top}}(L)$ must be normally cobordant to the identity. □

Our next result implies the conclusions of Theorems 7–9 for $\theta_{2k+1}([L, G/\mathrm{Top}])$ where $k \geq 2$.

Proposition 7.1. *For all $k \geq 2$ the subquotients $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ are either trivial or cyclic of order p. Furthermore, either $\theta_{2k-1}([L, G/\mathrm{Top}]) = \theta_{2k-2}([L, G/\mathrm{Top}])$ or $\theta_{2k-1}([L, G/\mathrm{Top}]) = \theta_{2k}([L, G/\mathrm{Top}])$*

Since $\theta_3([L, G/\mathrm{Top}]) = 0$ by Theorem 5, we set $\theta_2([L, G/\mathrm{Top}]) = 0$ by definition.

Proof. Suppose that $x \in [L, G/\mathrm{Top}]_{(p)}$ lies in the image of $[L, SG_{2k}]_{(p)}$. Then by Proposition 3.4 we know that x also lies in the image of $[L, SF_{2k-1}]_{(p)} \cong [S^{2k-1}L, S^{2k-1}]_{(p)}$. Therefore, if we let $\star(k, w)$ denote the k-fold loop or Characteristic Variety sum on $[L, SG]_{(p)}$ or $[L, G/\mathrm{Top}]_{(p)}$, then by Proposition 6.5 we know that $\star(p, x)$ lies in the image of $[S^{2k-3}L, S^{2k-3}]_{(p)}$. We need to show this implies that px lies in the image of $[L, SG_{2k-2}]_{(p)}$.

If τ is the generator of the cyclic p-group $T(L)$ described in Section 6, then there are unique integers $r \geq 0$ and b such that b is prime to p and $\star(bp^r, \tau) \in \{L, S^0\}_{(p)}$ maps to x. By Proposition 6.4 we know that $\star(bp^{r+1}, \tau)$ maps to $\star(p, x)$.

Proposition 6.1 now implies that $\star(bp^r, \tau) = b'p^r\tau$ and $\star(bp^{r+1}, x) = b''p^{r+1}\tau$ for some integers b' and b'' prime to p. By construction and our previous observations, it follows that $b'p^r\tau$ maps to $x \in \theta_{2k}([L, G/\mathrm{Top}])$ and $b''p^{r+1}\tau$ maps to some element of $\theta_{2k-2}([L, G/\mathrm{Top}])$. Choose an integer c such that $cb'' \equiv b'$ modulo a sufficiently large power of p (say at least p^n). Then we can also conclude that

$$p\,x = p \cdot \mathrm{Image}(b'p^r\tau) = \mathrm{Image}(b'p^{r+1}\tau) = \mathrm{Image}(cb''p^{r+1}\tau) = c \cdot \mathrm{Image}(b''p^{r+1}\tau)$$

lies in $\theta_{2k-2}([L, G/\mathrm{Top}])$. Since x is arbitrary, this means that

$$p\;\theta_{2k}([L, G/\mathrm{Top}]) \;\subseteq\; \theta_{2k-2}([L, G/\mathrm{Top}])$$

and since the image of $[L, SG]_{(p)}$ in $[L, G/\mathrm{Top}]$ is a finite cyclic p-group this means that $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ is either trivial or cyclic of order p □

Note. If X is a finite complex, it is well known that an element of $\{X, S^0\}_{(p)}$ desuspends to $[S^{2k}X, S^{2k}]_{(p)}$ if and only if it desuspends to $[S^{2k-1}X, S^{2k-1}]_{(p)}$ (*e.g.*, see [56] or [71]) but apparently very little is known about classes in $[X, SG]_{(p)} \cong \{X, S^0\}_{(p)}$ which lift to $[X, SG_{2k+1}]_{(p)}$ outside the stable range where $[S^{2k-1}X, S^{2k-1}]_{(p)} \to \{X, S^0\}_{(p)}$ is an isomorphism (as in Section 6, this is roughly the range in which $(p-1)k \geq \dim X$.

Proof. (Theorem 7) The stable range results of Section 6 imply that $\theta_{2k}([L, G/\mathrm{Top}]) = \theta_N([L, G/\mathrm{Top}])$ for all $N \geq 2k$ if we take $k = \left[\frac{n}{p-1}\right]$. Therefore, by the preceding result, it is only necessary to prove that

$$\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}]) \cong \mathbb{Z}_p$$

if $2 \leq k \leq \left[\frac{n}{p-1}\right] + 1$. Since $n \not\equiv 0 \bmod p-1$, we know that $[L, J_p]$ maps bijectively onto the image of $[L, SG_{(p)}]$ in $[L, G/\mathrm{Top}]$ and that $[L, J_p]$ is cyclic of order $p^{\left[\frac{n}{p-1}\right]}$. By Proposition 7.1 we know that the subquotients $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ have order equal to 1 or p, where $2 \leq k \leq \left[\frac{n}{p-1}\right] + 1$. Since the product of their orders equals the order of $[L, J_p]$, it follows that each factor $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ must have order p. $\square$

Proof. (Theorem 8) The main difference between this case and the previous ones is that the map from $[L, J_p]$ to $[L, G/\mathrm{Top}]$ has a kernel isomorphic to the nonzero group $\pi_{2n-1}(J_p)$. Similarly, if $T'(L)$ is the image of $T(L)$ in $[L, G/\mathrm{Top}]$ (with $T(L)$ as in Section 6), then the map $T(L) \to T'(L)$ has a kernel of order p. We now have $\left[\frac{n}{p-1}\right]$ factors of the form $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ where $2 \leq k \leq \left[\frac{n}{p-1}\right] + 1$, but the order of $T'(L)$ is $p^{\left[\frac{n}{p-1}\right]-1}$. Since the orders of the factors are again either 1 or p and their product is the order of $T'(L)$, it follows that all but one factor $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ must have order p and the remaining factor will necessarily have order 1. $\square$

In general, the determination of the exceptional factor $\theta_{2k}([L, G/\mathrm{Top}])/\theta_{2k-2}([L, G/\mathrm{Top}])$ seems to be a very difficult problem in homotopy theory. However, one can obtain strong restrictions on k for a smooth version of the tangential thickness problem, and these lead to partial results in other cases. We shall only illustrate the latter with a few examples; their statement requires the following observation.

Proposition 7.2. *In the usual setting as above, assume that* $\dim L = 2n-1$ *where* $n = p^\nu(p-1)r$ *for some* $\nu \geq 1$ *and* r *is prime to* p. *Let* $J\theta_n(L)$ *denote all classes in* $[L, G/O]_{(p)}$ *which lie in the images of* $[L, J_p]$ *and* $[L, SG_n]_{(p)}$. *Then for all* k *the quotients* $J\theta_{2k}(L)/J\theta_{2k-2}(L)$ *are either* 0 *or* $\mathbb{Z}_p$. *The quotients vanish if* $k > \left[\frac{n}{p-1}\right] + 1$, *and there is also a unique value* k_0 *of* k *such that* $2 \leq k_0 \leq \nu + 1 \leq \left[\frac{n}{p-1}\right] + 1$ *and the quotient vanishes.*

Proof. Let $y \in [X, SG_{(p)}]$, where X is a connected complex. Then the image of y in $[X, G/O]_{(p)}$ lifts to $[X, SG_{2k(p)}]$ if and only if $y = y_1 + y_2$ where y_1 lies in the latter group and y_2 lies in the image of $[X, SO]_{(p)} \to [X, SG]_{(p)}$. If X is the lens space L^{2n-1}, then by Proposition 5.5 we know that $[L, J_p]$ is generated by the image of $\pi_{2n-1}(J_p)$ under the degree 1 normal map from L^{2n-1} to S^{2n-1}. Furthermore, the latter map also induces an isomorphism from $\pi_{2n-1}(SO)$ to $[L, SO]$, and hence the image of $[L, SO]$ in $[L, J_p]$ equals the image of $\pi_{2n-1}(J_p)$ in $[L, J_p]$. By naturality, the images of $[B\mathbb{Z}_p, J_p] \cong [B\mathbb{Z}_p, SG]$ and $\pi_{2n-1}(J_p) =$ (Image of the J-homomorphism in $\pi_{2n-1}(SG)$) are also subgroups with respect to the loop sum operation on $SG_{(p)}$ that we have denoted by $*$ or $\star$, and therefore it turns out that $[L, J_p] \subseteq [L, SG]_{(p)}$ is also a subgroup with respect to this loop space operation (of course, usually one cannot expect such a conclusion).

Since $[L, J_p]$ contains the image of J, a class $y \in [L, J_p]$ maps to $J\theta_{2k}(L) \subseteq [L, G/O]_{(p)}$ if and only if it has the form $y_1 * y_2$ (with respect to the loop sum), where y_2 comes from $\pi_{2n-1}(J_p)$ and y_1 lies in the image of the stabilization map from $[S^{2k-1}L, S^{2k-1}]_{(p)}$ to $\{L, S^0\}_{(p)}$. As noted before, the group $\pi_{2n-1}(J_p)$ is cyclic of order p^{ν}. Therefore, if a class $w \in T(L) = \text{Image } [B\mathbb{Z}_p, J_p]$ has order p^j for $j \geq \nu + 1$, then w plus anything coming from $\pi_{2n-1}(J_p)$ will also have order p^j. In particular, this means that no such sum can desuspend to $[S^{2j-1}L, S^{2j-1}]_{(p)}$. On the other hand, it is known that the generator of $\pi_{2n-1}(J_p)$ does desuspend to $\pi_{(2n-1)+(2\nu+1)}(S^{2\nu+1})$ (for example, see [23]), and if we combine this with proposition 6.3 we conclude that if $k \geq \nu + 1$, then a class lies in $J\theta_{2k}(L)$ if and only if it has order dividing p^k. The nontriviality assertion about the quotients $J\theta_{2k+2}(L)/J\theta_{2k}(L)$ is an immediate consequence of this. □

When $\nu = 1$ the proposition states that the factors $J\theta_{2k+2}(L)/J\theta_{2k}(L)$ are nontrivial for all $k \geq 2 = \nu + 1$, and by the arguments employed in the proofs of Theorems 7 and 8 it follows that $J\theta_4(L)/J\theta_2(L)$ must be trivial. This suggests the following:

Conjecture. *In Theorem* 8*, the unique trivial quotient*

$$\theta_{2k}([L, G/\text{Top}]_{(p)})/\theta_{2k-2}([L, G/\text{Top}]_{(p)})$$

is given by $\theta_4([L, G/\text{Top}]_{(p)})/\theta_2([L, G/\text{Top}]_{(p)})$ *and accordingly the remaining quotients* $\theta_{2k}([L, G/\text{Top}]_{(p)})/\theta_{2k-2}([L, G/\text{Top}]_{(p)})$ *are nontrivial for all* $k \geq 3$.

Finally, we shall use Proposition 7.2 to verify the conjecture on exceptional dimensions when $n = j(p-1)$ for $j = 1, \ldots, p-1$.

Proposition 7.3. *Assume the setting of Theorem* 8 *and Proposition* 7.2*, and also let* $n = j(p-1)$ *where* $1 \leq j \leq p-1$*. Then the quotients* $\theta_{2k+2}([L, G/\text{Top}])/\theta_{2k}([L, G/\text{Top}])$ *are isomorphic to* $\mathbb{Z}_p$ *if* $k \geq 2$*, and* $\theta_4([L, G/\text{Top}]) = 0$.

Proof. The space $\mathrm{Cok}\ J_p$ is $(2p(p-1)-3)$-connected (see [71]), and therefore $[L, J_p] \cong [L, SG_{(p)}]$ if $2n-1 < 2p(p-1)-3$. As in the proof of Proposition 7.2, a class $y \in [L, J_p]$ maps into $\theta_{2j}([L, G/\mathrm{Top}])$ if and only if $y = y_1 + y_2$ where y_1 comes from $[L, \mathrm{STop}]_{(p)}$ and y_2 desuspends to $[S^{2k-1}L, S^{2k-1}]_{(p)}$.

In order to proceed further, we need to examine the image of $[L, \mathrm{STop}]_{(p)}$ in $[L, SG]_{(p)}$ using the Sullivan splittings:

$$\mathrm{STop}_{(p)} \simeq SO_{(p)} \times \mathrm{Cok}\ J_p\ , \qquad SG_{(p)} \simeq J_p \times \mathrm{Cok}\ J_p$$

It follows that the image of $[L, \mathrm{STop}]_{(p)}$ in $[L, SG]_{(p)}$ is the sum of $[L, \mathrm{Cok}\ J_p]$ with the image of $[L, SO]_{(p)}$ in $[L, J_p]$. If we are in the connectivity range of $\mathrm{Cok}\ J_p$, this means that the images of $[L, SO]_{(p)}$ and $[L, \mathrm{STop}]_{(p)}$ in $[L, SG]_{(p)} \cong [L, J_p]$ are equal. Therefore, if n satisfies the constraint in the proposition then we have $J\theta_{2k}(L) = \theta_{2k}([L, G/\mathrm{Top}])$. Since the quotients $J\theta_{2k+2}(L)/J\theta_{2k}(L)$ satisfy the conditions in the proposition, it follows that the quotients $\theta_{2k+2}([L, G/\mathrm{Top}])/\theta_{2k}([L, G/\mathrm{Top}])$ also satisfy these if $n = j(p-1)$ where $1 \leq j \leq p-1$. □

It should be possible to extend the range of dimensions for which similar conclusions hold if one uses known results on $\pi_m(\mathrm{Cok}\ J_p)$ for m roughly less than $2j(p-1)$ where $j < p^3 - (\textsf{some constant})$ due to Toda [71] and the splitting of the suspension of $B\mathbb{Z}_p$ in [29], but the calculations needed to do this would be considerably more complicated than the ones in this paper.

8. Comments on the Smooth Case and Twisted Tangential Thickness

In this section we shall discuss two variants of topological tangential thickness which arise naturally in other contexts.

8.1. **Smooth Tangential Thickness and Lens Spaces.** It is clearly possible to introduce a corresponding notion of tangential thickness in the smooth category, and in fact this goes all the way back to [47]. We shall state one such result for fake lens spaces without proof:

Proposition 8.1. *Let L^{2n-1} be a lens space, where $n \geq 2$.*

(i) *If $n \not\equiv 0 \bmod p-1$, then for each k such that $1 \leq k \leq \left[\frac{n}{p-1}\right]$ there is a manifold L_k (which is tangentially homotopy equivalent to L) such that $L_k \times \mathbb{R}^{2k}$ and $L \times \mathbb{R}^{2k}$ are not homeomorphic but $L_k \times \mathbb{R}^{2k+2}$ and $L \times \mathbb{R}^{2k+2}$ are diffeomorphic.*

(ii) *If $n = p^\nu(p-1)r$ where $\nu \geq 0$ and r is prime to p, then the same conclusion holds for all but one value of k, and the exceptional value is less than or equal to $\nu + 1$.*

One easy way of seeing the relative complexity of smooth tangential thickness is to consider this question for products of the form $(L\#\Sigma^{2n-1})\times\mathbb{R}^k$, where Σ^{2n-1} is an exotic sphere. If the order of Σ^{2n-1} in the Kervaire-Milnor group Θ_{2n-1} is prime to p and $k \geq 3$, then $(L\#\Sigma^{2n-1})\times\mathbb{R}^k$ is diffeomorphic to $L\times\mathbb{R}^k$ if and only if $\Sigma^{2n-1}\times\mathbb{R}^k$ and $S^{2n-1}\times\mathbb{R}^k$ are diffeomorphic. Smooth tangential thickness for exotic spheres has been fairly well understood for more than four decades (compare [63]; several individuals discovered these results independently). In particular, if we combine these results with some homotopy-theoretic input, we have the following:

Proposition 8.2.

(i) *Suppose that $2n-1 = 2^j+1 \geq 17$. Then these is a homotopy sphere Σ^{2n-1} such that $\Sigma^{2n-1}\times\mathbb{R}^{2n-7}$ is not diffeomorphic to $S^{2n-1}\times\mathbb{R}^{2n-7}$ but their products with $\mathbb{R}$ are diffeomorphic.*

(ii) *Suppose $2n-1 = 8k+1 \geq 9$, and let Σ^{8k+1} be a homotopy sphere not bounding a spin manifold. Then $\Sigma^{8k+1}\times\mathbb{R}^3$ and $S^{2k+1}\times\mathbb{R}^3$ are not diffeomorphic, but one can choose Σ^{8k+1} such that their products with $\mathbb{R}$ are diffeomorphic.*

The first statement follows by choosing Σ so that its Pontrjagin-Thom invariant of Σ in the group $\pi_{2n-1}/\text{Image } J$ is the $\eta_j\eta$, where $\eta_j \in \pi_{2^j}$ is the Mahowald element (see [56], Thm. 1.5.27(a), p. 38) and $\eta \in \pi_1$ is the stabiilization of the Hopf map from S^3 to S^2. The proof that the smaller products are not diffeomorphic is based upon results from [44], which show that $\eta_j\eta$ desuspends to S^{2^j-4} but does not desuspend to S^{2^j-5}(*e.g.*, see Table 1 on pp. 74–75 and Table 4.4 on p. 11; in the setting of the previous citation from [56], the class $\eta_j\eta$ corresponds to

$$\nu^2 \in E_1^{2^j+1,2^j-5} \approx \pi_6 \approx \mathbb{Z}_2$$

in the 2-primary EHP spectral sequence described in [56]). The second statement follows from the fact that the elements $\mu_k \in \pi_{8k+1}$ desuspend to S^3 but not S^2 (see [37] for desuspension to S^3; as in Section 3, if μ_k desuspended to S^2 it would be divisible by η in π_*). As noted above, these yield diffeomorphism and nondiffeomorphism results for smooth and fake lens spaces.

8.2. **Twisted Tangential Thickness.** The previously mentioned theorem of B. Mazur (see [47]) has a natural generalization to vector bundles (*i.e.,* Theorem 2 in [47]).

Theorem 10. (Mazur) *Let E and F be the total spaces of $\mathbb{R}^k$ bundles over smooth closed manifolds M^n and N^n with $k \geq n+2$. Then E and F are tangentially homotopy equivalent if and only if E and F are diffeomorphic.*

This leads to an obvious notion of **twisted tangential thickness** which in turn has nontrivial application to the geometry of nonnegatively curved manifolds (*cf.* [12]). Namely, let $\Theta_7 \cong \mathbb{Z}_{28}$ be the group of homotopy 7-spheres $\Sigma^7(d)$, $d = 0, \ldots, 27$. It is proved in [12] that although $\Sigma^7(d) \times \mathbb{CP}^2$ falls into one tangential homotopy type, there are four oriented (three unoriented) diffeomorphism classes of these manifolds, each admitting a nonnegatively curved metric by the main result of [27]. Similar results hold for manifolds of the form $\Sigma^7 \times \mathbb{CP}^{2n}$ for all n such that $n \not\equiv 0 \bmod 3$ [64].

It turns out that these four different manifolds have twisted tangential thickness 2; *i.e.*, the corresponding total spaces of the nontrivial $\mathbb{R}^2$ bundle are all diffeomorphic. This result gives the first examples of manifolds with complete enumeration of the different souls for Riemannian metrics which admit metrics of nonnegative sectional curvature:

Theorem. (Thm. 6 in [12]) *The total space N of any nontrivial complex line bundle over $S^7 \times \mathbb{CP}^2$ admits three complete nonnegatively curved metrics with pairwise nondiffeomorphic souls, S_0, S_1 and S_2, such that for any complete nonnegatively curved metric on N with soul S, there exists a self-diffeomorphism of N taking S to some S_i.*

Remarks. 1. Previously M. Özaydın and G. Walschap described examples of vector bundle total spaces which support no complete metrics with nonnegative sectional curvature [51].

2. It is worthwhile to note that the twisted tangential thickness of the non-diffeomorphic manifolds $\Sigma^7(d) \times \mathbb{CP}^2$ is equal to 2, but the standard (untwisted) tangential thickness is equal to 3.

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