INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

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SEMINAR ON COMBINATORIAL TOPOLOGY

by

Eric-Christopher ZEEMAN

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FASCICULE 1

(Exposés I à V inclus)

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Seminar on Combinatorial Topology

by E.C. ZEEMAN

INTRODUCTION

The purpose of these seminars is to provide an introduction to combinatorial topology. The topics to be covered are :

1. The combinatorial category and subdivision theorems,
2. The polyhedral category,
3. Regular neighbourhoods,
4. Unknotting of spheres,
5. General Position,
6. Engulfing lemmas,
7. Embedding and isotopy theorems.

At first sight the unattractive feature of combinatorial theory as applied to manifolds is the kinkiness and unhomogeneity of a complex as compared with the roundness and homogeneity of a manifold. However this is due to a confusion between the techniques and subject matter. We resolve this confusion by separating into two different categories the tools and objects of study. The tools in the combinatorial category we keep as special as possible, namely finite simplicial complexes embedded in Euclidean space.
These possess two crucial properties :

i) finiteness, and the use of induction
ii) tameness, and niceness of intersection.

Meanwhile objects of study we make as general as possible. Our definition of polyhedral category contains not only

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i) polyhedra;

ii) manifolds (bounded or not, compact or not),

but also the following spaces, which have not been given a combinatorial structure before :

iii) non-paracompact manifolds, for example the Long Line;

iv) infinite dimensional manifolds, for example the orthogonal group,

v) joins of non-compact spaces; for example the suspension of an open interval.

vi) function spaces; for example the space of all piecewise linear embeddings of compact manifold in another manifold.

As the examples show, a polyhedral space need not be triangulable, and if it is, it does not have a specific triangulation, but is a set with a structure. The structure is, roughly speaking, a maximal family of subpolyhedra, and the structure determines the topology.

Our theory is directed towards the study of manifolds, and in particular of embeddings and isotopies. Recently it has become apparent that combinatorial results differ substantially from differential results; a striking case is S^3 in S^6, which knots differentially, and unknots combinatorially. In fact combinatorial theory seems to behave well in, and to have techniques to handle, most situations with codimension ≥ 3. Just as differential theory behaves well and can handle most situations in the stable range.

We shall therefore concentrate on geometry in codimension ≥ 3. This means we shall neglect a number of interesting and allied topics that depend more on algebra, for example

i) codimension 2

ii) immersion theory

iii) relations with differential theory.

Chapter I : THE COMBINATORIAL CATEGORY

Simplexes

Let E^p denote Euclidean p-space. An n-simplex $(n \geqslant 0)$ A in E^p is the convex hull of $n+1$ linearly independent points.
We call the points _vertices_, and say that A _spans_ them. A is closed and compact; $\dot{A}$ denotes the boundary, A the interior. A simplex B spanned by a subset of the vertices is called a _face_ of A, written $B < A$. Simplexes A, B are _joinable_ if their vertices are linearly independent. If A, B are joinable we define the _join_ AB to be the simplex spanned by the vertices of both; otherwise the join is undefined.

Complexes

A _finite simplicial complex_, or _complex_, K in E^p is a finite collection of simplexes such that

(i) if $A \in K$, then all the faces of A are in K,

(ii) if $A, B \in K$, then $A \cap B$ is empty or a common face.

The _star_ and _link_ of a simplex A K are defined :

$$st(A,K) = \{B;\ A < B\}, \qquad lk(A,K) = \{B;\ AB \in K\}.$$

Two complexes K, L in E^p are _joinable_ provided :

(i) if $A \in K$, $B \in L$ then A, B joinable

(ii) if $A, A' \in K$ and $B, B' \in L$, then $AB \cap A'B'$ is empty or a common face.

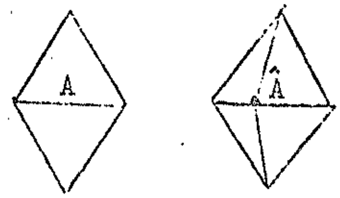
If K, L are joinable, we define the _join_ $KL = K \cup L \cup [AB; A \in K, B \in L]$; otherwise the join is undefined.

The underlying point set $|K|$ of K is called a _euclidean polyhedron_ :

L is called a <u>subdivision</u> of K if $|L| = |K|$, and every simplex of L is contained in some simplex of K.

<u>Examples</u>, 1) Choose a point $\hat{A} \in \mathring{A}$. Let

$$L = (K - st(A,K)) \cup \hat{A}\,\dot{A}\ \ lk(A,K)\ .$$



Then L is a subdivision of K, and we say L is obtained from K by starring A (at $\hat{A}$).

2) <u>A first derived</u> $K^{(1)}$ of K is obtained by starring all the simplexes of K in some order such that if $A > B$ then A preceedes B (for example in order of decreasing dimension).

Another way of defining $K^{(1)}$ is to define the subdivision of each simplex, inductively in order of increasing dimension, by the rule $A' = \hat{A}\dot{A}'$. Therefore a typical simplex of $K^{(1)}$ is $\hat{A}_0 \hat{A}_1 \dots \hat{A}_p$, where $A_0 < A_1 < \dots < A_p$ in K. An <u>r^{th} derived</u> $K^{(r)}$ is defined inductively as the first derived of an $(r-1)^{th}$ derived. <u>The barycentric first derived</u> is obtained by starring at the barycentres.

<u>Convex linear cells</u>

A <u>convex linear cell</u>, or <u>cell</u>, A in E^p is a non-empty compact subset given by

$$\begin{cases} \text{linear equations} & f_1 = 0, \dots, f_r = 0 \quad \text{and} \\ \text{linear inequalities} & g_1 \geq 0, \dots, g_s \geq 0\ . \end{cases}$$

A <u>face</u> B of A is a cell (i.e. non-empty) obtained by replacing some of the inequaties $g_i \geq 0$ by equations $g_i = 0$.

The 0-dimensional faces are called vertices. It is easy to deduce the following elementary properties :

1) A is the convex hull of its vertices
2) A is a closed compact topological n-cell, where $n + 1$ is the maximum number of linearly independent vertices,

3) A simplex is a cell.
4) The intersection or product of two cells is another.
5) Let x be a vertex of the cell A, and let B be the union of faces of A not containing x.
   Then $A =$ the cone xB.

A <u>convex linear cell complex</u>, or <u>cell complex</u>, K is a finite collection of cells such that

(i) if $A \in K$, then all the faces of A are in K,
(ii) if $A, B \in K$, then $A \cap B$ is empty or a common face.

<u>Lemma 1</u> : <u>A convex linear cell complex can be subdivided into a simplicial complex without introducing any more vertices.</u>

<u>Proof</u> : Order the vertices of the cell complex K. Write each cell A as a cone $A = xB$, where x is the first vertex. Subdivide the cells inductively, in order of increasing dimension. The induction begins trivially with the vertices. For the inductive step, we have already defined the subdivision A' of A, and so define A' to be the cone $A' = xB'$. The definition is compatible with subdivision C' of any face C of A containing x, because since x is the first vertex of A, it is also the first vertex of C. Therefore each cell, and hence K, is subdivided into a simplicial complex.

<u>Corollary 1</u> : <u>The underlying set of a cell complex is a euclidean polyhedron.</u>

<u>Corollary 2</u> : <u>The intersection or product of two euclidean polyhedra is another.</u>

For the intersection or product of simplicial complexes is a cell complex.

Maps.

Suppose K in E^p, L in E^q.

A map $f : K \to L$ is a continuous map $|K| \to |L|$.

Call f simplicial if it maps vertices to vertices and simplexes linearly to simplexes. Call f an isomorphism, written $f : K \cong L$, if it is a simplicial homeomorphism. The graph Γf of f is defined as usual

$$\Gamma f = \{ (x,fx) \; ; \; x \in |K| \} \subset |K| \times |L| \subset E^{p+q} .$$

Call f piecewise linear if either of the two definitions hold :

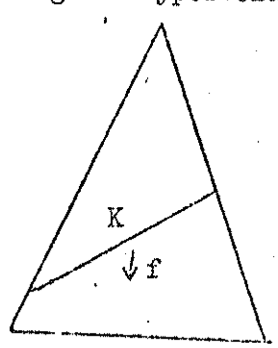
(1) The graph Γf of f is a euclidean polyhedron

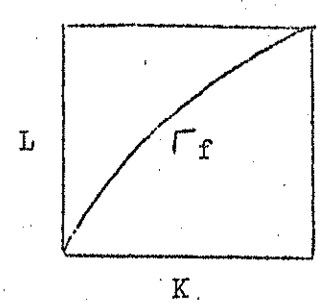
(2) There exist subdivision K', L', of K,L with respect to which f is simplicial .

Notice that condition (2) clearly implies condition (1), because the graph of a linear map from a simplex to a simplex is a simplex, and so the graph of a simplicial map $K \to L$ is a complex isomorphic to K. We shall prove the converse, and therefore the equivalence of the two definitions, in Lemma 7 . Definition (1) is the aesthetically simpler, while definition (2) is the one which is used continually in practice.

The reader is warned against the standard mistake of confusing projective maps with piecewise linear maps. For example the projection onto the base of a triangle from the opposite vertex of a line not parallel to the base is not piecewise linear .

Infact the graph Γf in the square $|K| \times |L|$ is part of a rectangular hyperbola .





Lemma 2 : The composition of two piecewise linear maps is another .

Proof : We use definition (1), Given $K \xrightarrow{f} L \xrightarrow{g} M$, let

$$\Gamma = (\Gamma f \times |M|) \cap (|K| \times \Gamma g) \subset E^p \times E^q \times E^r .$$

Then Γ consists of all points (x, fx, gfx), $x \in |K|$
Therefore the projection $\pi : E^p \times E^q \times E^r \longrightarrow E^p \times E^r$ maps Γ homeomorphically onto $\Gamma(g\,f)$.
Since f,g are piecewise linear, Γ is a euclidean polyhedron by Lemma 1 Corollary 2 . The image under the linear projection π of any complex triangulating Γ gives an isomorphic complex triangulating $\Gamma(gf)$. Hence $\Gamma(gf)$ is a euclidean polyhedron, and gf is piecewise linear .

Definition : Lemma 2 enables us to define the combinatorial category $\mathcal{C}$ with

$$\begin{cases} \text{objects : finite simplicial complexes} \\ \text{maps : piecewise linear maps.} \end{cases}$$

We shall also need the subcategory of embeddings $\mathcal{E}$ with

$$\begin{cases} \text{the same objects} \\ \text{maps : injective piecewise linear maps.} \end{cases}$$

We proceed to prove some useful subdivision theorems .

Lemma 3 : If $K \supset L$, then (i) any subdivision K' of K induces a subdivision L' of L, and (ii) any subdivision L' of L can be extended to a subdivision K' of K .

Proof : (i) is obvious

(ii) subdivide, inductively in order of increasing dimension, those simplexes of $K-L$ that meet L, by the rule $A' = \hat{A}\dot{A}'$ where $\hat{A}$ is an interior point .

Corollary : Given a simplicial embedding $f : K \to L$, and a subdivision K' of K, there exists a subdivision L' of L such that $f : K' \to L'$ is simplicial .

Lemma 4 : If $|K| \supset |L|$, then there exists an r^{th} derived $K^{(r)}$ of K and a subdivision L' of L such that L' is a subcomplex of $K^{(r)}$.

Proof : By induction on the number of simplexes of L. The induction starts trivially when $L = \emptyset$. If A is a principal simplex of L (principal means not the face of another), then by induction choose $K^{(r-1)}$ to contain a subdivision of $L-A$.

Choose a derived $K^{(r)}$, by starring each simplex $B \in K^{(r-1)}$ at a point in $\mathring{A} \cap \mathring{B}$ if $\mathring{A}$ meets $\mathring{B}$, and arbitrarily otherwise . Then $K^{(r)}$ contains subdivision of $L-A$, A and hence of L.

Corollary 1. If $|K| = |L|$, then K, L have a common subdivision .

Corollary 2. If $|K| \supset |L_i|$, $i = 1,\dots,r$, then there exist subdivision K', L'_i such that all the L'_i are subcomplexes of K' .

Corollary 3. The union of two euclidean polyhedra is another .
For subdivide a large simplex containing them both, so that each appears as a subcomplex . The union is also a subcomplex .

Lemma 5 : Given a simplicial map $f : K \to L$, and a subdivision L' of L, then there exists a subdivision K' of K such that $f : K' \to L'$ is simplicial .

Proof : Let $K_1 = f^{-1}L'$, which is a cell complex subdividing K.
By Lemma 1 we can choose a simplicial complex K' subdividing K_1, introducing no new vertices. Then each simplex of K' is

mapped linearly to a simplex of L', and so $f : K' \to L'$ is simplicial.

Définition : A map $f : K \to E^q$ is linear if each simplex is mapped linearly.

Lemma 6 : Let e be the inclusion $L \subset E^q$. Given a map $f : K \to L$, such that $ef : K \to E^q$ is linear, then there exist subdivisions K', L' of K, L with respect to which f is simplicial.

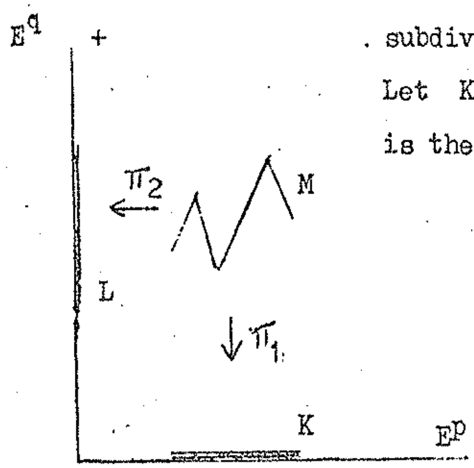
Proof : If $A_i \in K$, let $B_i = f A_i$.

By linearity B_i is a cell, possibly of lower dimension than A_i, and $|B_i| \subset |L|$. By Lemma 4 Corollary 2, choose simplicial subdivisions L', B_i' of L, B_i such that each B_i' is a subcomplex of L'. Then for each i, $f^{-1}B_i'$ is a cell complex subdividing A_i, and the union $f^{-1}L'$ is a cell complex subdividing K. By Lemma 1 choose a simplicial subdivision K' of $f^{-1}L'$, introducing no new vertices. Then $f : K' \to L'$ is simplicial.

Lemma 7 : The two definitions of piecewise linearity are equivalent.

Proof : We have observed $(2) \Longrightarrow (1)$ trivially. Therefore we shall prove $(1) \Longrightarrow (2)$. Suppose K in E^p, L in E^q and let $f : K \to L$ be a map whose graph Γf is a euclidean polyhedron. In other words, there exist a complex M in E^{p+q} such that $|M| = \Gamma f$.

The projection $E^p \times E^q \xrightarrow{\pi_1} E^p$ maps M homeomorphically onto K, and linearly into E^p; therefore by Lemma 6, there exist subdivisions M', K' with respect to which π_1 is simplicial. Hence $\pi_1 : M' \to K'$ is an isomorphism. Similarly



$\pi_2 : E^p \times E^q \to E^q$ maps M into L (not necessarily homeomorphically), and linearly into E^q; therefore there exist subdivisions M'', L' with respect to which π_2 is simplicial. Let K'' be the subdivision of K' isomorphic to M''. Then f is the composition of the simplicial maps

$$K'' \xrightarrow{\pi_1^{-1}} M'' \xrightarrow{\pi_2} L' .$$

Hence $f : K \to L$ is piecewise linear by definition (2)

Let T be a finite subset of $\mathcal{C}$, such that if a map is in T so is its range and domain. The <u>diagram</u> of T is the 1-complex obtained by replacing each complex by a vertex and each map by an edge. Call T a <u>tree in $\mathcal{C}$</u> if its diagram is simply-connected. Call T <u>one-way</u> if each complex is the domain of at most one map. Therefore in a one-way tree there is exactly one complex that is the domain of no map, and every other complex is the domain of exactly one map. Call T <u>simplicial</u> if every map of T is simplicial. Call T' a <u>subdivision</u> of T if it has the same diagram, and each complex of T' is a subdivision of the corresponding complex of T, and each map of T' (qua map between the underlying polyhedra) is the same as the corresponding map of T.

<u>Theorem 1</u> : <u>If T is a one-way tree in $\mathcal{C}$, or a tree in $\mathcal{J}$, then T has a simplicial subdivision</u>.

<u>Proof</u> by induction on the number of maps in T. Let T be a one-way tree in $\mathcal{C}$.

The induction begins trivially with no maps.

Suppose T has at least one map. Then there exist complex K and a map $f : K \dashrightarrow L$ in T, such that K is not the range or domain of any other map in T.

By Lemma 7, there exist subdivisions K', L' of K, L with respect to which f is simplicial. Let T_* be the one-way tree obtained from T by omitting K and f, and replacing L by L'. By induction there is a simplicial subdivision T_*' of T_*. In particular T_*' contains a subdivision L'' of L'. By Lemma 5 there exists a subdivision K'' of K' such that $f : K'' \to L''$ is simplicial. Let $T' = T_*'$ together with K'' ans f. Then T' is a simplicial subdivision of T.

Now suppose T is a tree in $\mathcal{E}$, not necessarily one-way. There is a complex K which is the range or domain of exactly one map. If K is the domain, proceed as before. If K is the range, let the map be $f : L \to K$. Proceed as before, except that we can use the Collorary to Lemma 3 instead of Lemma 5 to form K'', since f is an embedding. The proof of Theorem 1 is complete.

The following two examples show that the hypotheses of Theorem 1 are necessary as well as sufficient.

<u>Example 1</u>. It is necessary that a tree in $\mathcal{C}$ be one-way, otherwise it contains a subtree

$$\begin{tikzcd} & L \\ K \arrow[ur, dashed, "f"] \arrow[dr, "g"'] & \\ & M \end{tikzcd}$$

We can choose f, g so that there exists no simplicial subdivision as follows:

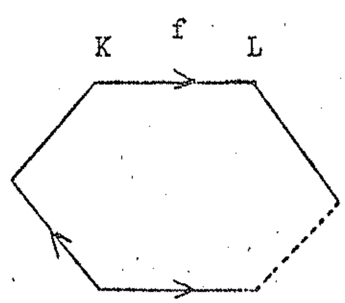
Let $K = L = M = I$, the unit interval, and let

$$f \text{ map } \begin{cases} 0 \longrightarrow 0 \\ 1/3 \longrightarrow 1 \\ 1 \longrightarrow 0 \\ [0,1/3] , [1/3,1] \qquad \text{linearly ,} \end{cases}$$

$$g \text{ map } \begin{cases} 0 \longrightarrow 0 \\ 2/3 \longrightarrow 1 \\ 1 \longrightarrow 0 \\ [0,2/3] , [2/3,1] \qquad \text{linearly .} \end{cases}$$

Suppose there is a simplicial subdivision, containing K' .
Let p,q,r be the numbers of vertices of K' between, respectively, 0 and 1/3, 1/3 and 2/3, 2/3 and 1 .
Since f is simplicial on K' , we have $p = q + 1 + r$.
From g similarly, $p + 1 + q = r$. Hence $q = -1$ a contradiction .
Therefore there is no simplicial subdivision .

<u>Example 2</u> . It is necessary that the diagram in $\mathcal{E}$ be a tree, otherwise it contains a circular subdiagram

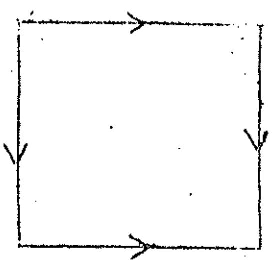


We can choose the maps so that there is no simplicial subdivision as follows :
Let all the complexes be I , and all the maps be the identity except f , and let

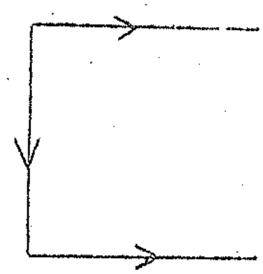
$$f \text{ map } \begin{cases} 0 \longrightarrow 0 \\ 1/3 \longrightarrow 2/3 \\ 1 \longrightarrow 1 \\ [0,1/3]\ \ [1/3\ ,\ 1] \quad \text{linearly} . \end{cases}$$

Suppose there is a simplicial subdivision containing $f : K' \to L'$. Going round all the other maps we have the identity map simplicial, and so $K' = L'$. Using the same notation as in Example 1, since f is simplicial, we deduce $p = p + 1 + q$. Hence $q = -1$, again a contradiction. Therefore T has no simplicial subdivision.

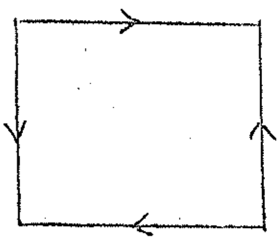
<u>Remark</u>. A "commutative diagram" in $\mathcal{E}$ has simplicial subdivision if the maps are determined by a maximal tree. For example :



is determined by



but



is not determined by a tree.

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Chapter 2 : THE POLYHEDRAL CATEGORY

In this chapter we give mainly definitions and examples to describe the category. We omit the proofs to most statements to make the reading easier, and because later chapters do not depend on them.

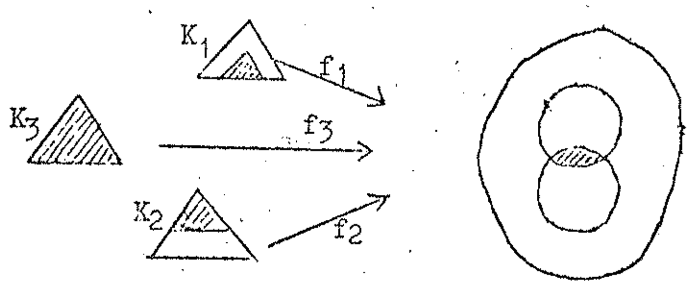
Let X be a set (without as yet any topology). A polyhedron in X is an injective function $f : K \to X$ where K is a finite simplicial complex. By a function we mean, as usual, a function from the set of points of the underlying euclidean polyhedron $|K|$ to the set X. We write

$$\text{dom } f = K \quad , \qquad \text{im } f = f K .$$

Two polyhedron $f_1 : K_1 \to X$ and $f_2 : K_2 \to X$ are related if there is a third $f_3 : K_3 \to X$, such that

i) $\text{im } f_3 = \text{im } f_1 \cap \text{im } f_2$

ii) $f_1^{-1} f_3 \, , \, f_2^{-1} f_3 \in \mathcal{E}$.



A *family* $\mathcal{F}$ of polyhedra in X is a set in which any two are related. Write $\operatorname{im} \mathcal{F} = \{ \operatorname{im} f \, ; \, f \in \mathcal{F} \}$.

A *polystructure*, (or more briefly a *structure*), $\mathcal{F}$ on X is a family such that

i) $\operatorname{im} \mathcal{F}$ covers X

ii) $\operatorname{im} \mathcal{F}$ is a lattice of subsets of X

iii) $\mathcal{F}\mathcal{E} = \mathcal{F}$

The last condition means that given $K \xrightarrow{g} L \xrightarrow{f} X$ with $f \in \mathcal{F}$ and g a piecewise linear embedding, then $fg \in \mathcal{F}$.

A *polyspace* $X = (X, \mathcal{F})$ is a set X together with a polystructure $\mathcal{F}$ on X.

T o p o l o g y

The *topology* $T(\mathcal{F})$ of the structure $\mathcal{F}$ is the identification topology

$$X = \operatorname{dom} \mathcal{F} / \mathcal{F} .$$

Here $\operatorname{dom} \mathcal{F}$ means the disjoint union of the euclidean polyhedra $\{ \operatorname{dom} f \, ; \, f \in \mathcal{F} \}$, and the identification is given by $\mathcal{F} : \operatorname{dom} \mathcal{F} \to X$. We can deduce (non-trivially) :

i) Each $f : K \to X$ is a homeomorphism into

ii) A set $U \subset X$ is open (or closed) if and only if $U \cap f K$ is open (or closed) in $f K$, each $f \in \mathcal{F}$.

If X is a topological space, then a polystructure on X is one with the same topology.

Example 1. The *discrete* structure on a set X is given by maps of points into X. This gives the discrete topology.

<u>Example 2</u> . The natural structure $\mathcal{F}(E^n)$ on Euclidean space E^n is the set of all piecewise linear embeddings $K \to E^n$. This gives the natural topology .

<u>Example 3</u> . The natural structure $\mathcal{F}(K)$ on a complex K is the set of all piecewise linear embeddings $L \to K$. The natural structure on the euclidean polyhedron $|K|$ is the same .

<u>Example 4</u> . Suppose $f : |K| \to X$ is a homeomorphism from a euclidean polyhedron onto a topological space X . Then $f\mathcal{F}(K)$ gives a polyhedral structure $\mathcal{F}(X)$ on X .

We call X , with this structure, a <u>polyhedron</u> .

Notice that $\mathcal{F}(X)$ contains the triangulation f , and all related triangulations . Conversely the structure is uniquely determined by any triangulation in it .

<u>Remark 1</u> . We have used the word polyhedron in three ways

i) euclidean-polyhedron
ii) polyhedron-in-a-set
iii) polyhedron .

The usage is coherent, because (i) with its natural structure is an example of (iii) , and the image of (ii) with its induced (see below) structure is an example of (iii) .

<u>Remark 2</u> . It is possible to have many structures on a set ; more examples are given below . However it can be shown (non-trivially) that

1) Any structure is maximal with respect to its topology : the topology of a strictly smaller structure is strictly finer (more open sets) .

2) The natural structures of E^n and of polyhedra are maximal .

Subspaces

Let $\mathfrak{X} = (X, \mathfrak{F})$ be a polyspace. If $Y \subset X$, we define the induced structure on Y to be

$$\mathfrak{F} \mid Y = \{ f \in \mathfrak{F} \ ; \ \operatorname{im} f \subset Y \} .$$

It is easy to verify that $\mathfrak{F} \mid Y$ is a polystructure on Y.

We call $Y = (Y, \mathfrak{F} \mid Y)$ a polysubspace if it has the induced topology :

$$T(\mathfrak{F} \mid Y) = T(\mathfrak{F}) \mid Y .$$

In general $T(\mathfrak{F} \mid Y)$ has a finer topology.

Example 1. E^n is a polysubspace of S^n.

This is a particularly satisfactory example, because combinatorially it is always a little embarrassing to regard the infinite triangulation of E^n as a satisfactory "substructure" of the finite triangulation of S^n.

We state elementary properties of polysubspaces, leaving the proofs to the reader :

i) Any open set of X is a polysubspace.

ii) Any polyhedron in X is a polysubspace.

iii) A polysubspace of a polysubspace is a polysubspace.

iv) The intersection of two polysubspaces is a polysubspace.

Therefore the notion of polysubspace substantially enlarges the concept of "tame" set to include both polyhedra and open sets.

Example 2. The union of two polysubspaces is not necessarily poly. For example let A = open disk in E^2

B = a boundary point.

Then $A \cup B$, with structure $\mathcal{F} \mid A \cup B$ is locally compact; a compact neighbourhood of B in $A \cup B$ is a closed disk D , having B on its boundary , and with $D - B \subset A$. But with the induced topology $A \cup B$ is <u>not</u> locally-compact, because B has no compact neighbourhood.

<u>Example 3</u> . A circle in E^2 is not a polysubspace , because the induced structure is discrete .

<u>Example 4</u> . A closed disk in E^2 is not a polysubspace . With the induced structure it is non-compact ; any subset of the boundary being closed . It is like the Prüfer manifold with each attached disk shrunk to a point .

<u>Maps</u>

A function $f : X \to Y$ between two polyspaces is called a <u>polymap</u> if $f\mathcal{F}(X) \subset \mathcal{F}(Y)\mathcal{C}$.

In other words , given $g \in \mathcal{F}(X)$, then fg can be factored through the structure of Y , $fg = g'f'$ for some $g' \in \mathcal{F}(Y)$, where f' is piecewise linear ,

$$\begin{tikzcd} K \arrow[r, "f'"] \arrow[d, "g"] & L \arrow[d, "g'"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

It is easy to deduce

1) A polymap is continuous with respect to the structure topologies , and is therefore a map between the underlying topological spaces .

2) $f : K \to L$ is a polymap if and only if it is piecewise linear .

3) Identities and compositions of polymaps are polymaps.

Therefore we can define the polyhedral category $\mathcal{P}$ to consist of polyspaces and polymaps .

Call a polymap a polyhomeomorphism written $f : X \cong Y$, if $f\,\mathcal{F}(X) = \mathcal{F}(Y)$.

We deduce 1) it is a homeomorphism, and

2) f^{-1} is also a polyhomeomorphism .

Call a polymap a polyembedding , written $f : X \subset Y$, if it is an embedding , (i.e. a polyhomeomorphism only a polysubspace of Y) .

We deduce 3) $f : X \to Y$ is a polymap if and only if its graph

$1 \times f : X \to X \times Y$ is a polyembedding .

R e m a r k

It would be natural to call a polymap $f : X \to Y$ injective if $f\,\mathcal{F}(X) = \mathcal{F}(Y) \mid fX$. This definition is weaker than polyembedding , because it does not require the image fX to be a polysubspace of Y . But it is of interest for the following reason . Consider the categories :

(1) space and embeddings

(2) polyspaces and injective polymaps

(3) polyspaces and polyembeddings.

Then $(1) \cap (2) = (3)$. Now some constructions such as join and mapping cylinder are functorial in (2) but not in (1) , and therefore not in (3) . For these constructions the polystructure is more natural than its accompanying topology .

Bases.

A base $\mathcal{B}$ for a polystructure on a set X is a family of polyhedra such that im $\mathcal{B}$ covers X (i.e. only the first structure axiom). As with structures, the topology $T(\mathcal{B})$ is the identification topology $X = \mathrm{dom}\,\mathcal{B}/\mathcal{B}$. We say $\mathcal{B}$ is a base for $\mathcal{F}$ if

i) $\mathcal{B} \subset \mathcal{F}$

ii) every set of im $\mathcal{F}$ is contained in a finite union of sets of im $\mathcal{B}$.

We can deduce

1) Every structure has a base (trivially).

2) Every base is the base for a unique structure; and the base and structure have the same topology.

Example 1. Any polyspace has a base of simplexes, $\mathcal{B} = \{ f \in \mathcal{F} ; \mathrm{dom}\, f = \text{simplex} \}$.

Example 2. E^n has a base of all n-simplexes.

Example 3. A polyhedron X has a base of one element, namely a triangulation $f : K \to X$.

Example 4. The Woven Square. Let X be the square I^2. Let $\mathcal{B}$ be the base consisting of all horizontal and vertical intervals, or, more precisely, all horizontal and vertical linear embeddings of I. The resulting structure is smaller than the natural structure, because it contains no 2-dimensional polyhedra. The resulting topology is finer than the natural topology, and is therefore Hausdorff, but is not locally compact, nor simply-connected. A typical open neighbourhood of a point looks like a maltese cross. Any subset of the diagonal is a closed set.

Example 5. (The pathological Woven Square). We enlarge the structure of the Woven Square by weaving in one more thread so badly, that it produces a non-Hausdorff topology . Let $d: I \to I^2$ be the diagonal map ; and let $e: I \to I$ be the function that is the identity on the irrationals, but reflects tho rationals about the mid-point . We add to the base of the structure of the Woven Square one more element , the polyhedron $f = de : I \to I^2$. The topology of the Woven Square is thereby coarsened , so that the ends of the diagonal cannot be separated by disjoint open sets . (The proof uses measure theory, and depends upon the non-countability of the base) .

Definition, We call a base $\mathcal{B}$ topological if it is also a base for the topology $T(\mathcal{B})$ in the following sense : given $x \in X$, there exists $f \in \mathcal{B}$, such that im f is a (closed) neighbourhood of x in X in the topology $T(\mathcal{B})$. For instance , in Example 2 above , the set of all n-simplexes in E^n is a topological base , But in Example 4 , the base for the woven square is not topological . The structures for infinite manifolds and function spaces that we give below will not be topological .

Triangulable Spaces

The pathological examples 4 and 5 above indicate some of the consequences of the definitionsof polyspace . However since our interest lies towards manifolds, we do not stress the pathology , but rather use it to obtain insight into the structure of important polyspaces such as function spaces . One of the advantages of polyspace is that it is more general than the triangulable space , even if we use infinite triangulations . In fact we avoid infinite triangulations , because we regard them as alien to the subject , being too diffuse a tool , and defining too restrictive a space . The algebraic elegance of infinite complexes should not be confused with their geometric limitations . However it is worth mentioning the relationship between polyspaces and triangulable spaces .

Given a polyspace X , then there are six possibilities :

i) X is a polyhedron , i.e. its structure contains finite triangulations .

ii) X is not a polyhedron, but we can enlarge the structure of X to be a polyhedron - for example the woven square.

iii) There is a locally-finite infinite triangulation $f : K \to X$, whose restriction to any finite subcomplex is in the structure - as for example in E^n.
If X is connected, then a necessary and sufficient condition for this is that the structure have a countable topological base. A consequence is that the topology is paracompact, Hausdorff, and locally compact.

iv) The structure can be enlarged to give (iii).

v) The structure is maximal, but (i) and (iii) are not true; for example the Long Line (see below).

vi) The structure is not maximal, but (ii) and (iv) are not true; for example the pathological woven square, or ∞-dimensional manifolds, or function spaces (see below).

Compactness

Question 1. Is a compact polyspace a polyhedron? The answer is yes if it has a countable base, or if it has a topological base, but is unsolved otherwise.

Question 2. Does the lattice of compact subsets of a polyspace refine the lattice of polyhedra?
The question is important for studying the homotopy structure of function spaces.

Manifolds

An n-polyball is a polyhedron triangulated by an n-simplex.
An n-polysphere is a polyhedron triangulated by the boundary of an (n+1)-simplex.

<u>Definition</u> : an <u>n-polymanifold</u> M is a polyspace , each point of which has an n-polyball neighbourhood .

More precisely , each point has a closed neighbourhood (with respect to the structure topology) which is a polysubspace , and which, with the induced structure , is an n-polyball . The <u>boundary</u> $\dot{M}$ is the closed polysubspace of those points which lie on the boundary of their neighbourhoods , and is an (n-1)-polymanifold . The <u>interior</u> $\mathring{M} = M - \dot{M}$ is the complementary open polysubspace .

We call M <u>closed</u> if compact and $\dot{M} = \emptyset$.
<u>bounded</u> if compact and $\dot{M} \neq \emptyset$.
<u>open</u> if non-compact and $\dot{M} = \emptyset$.

If M compact then any triangulation in the structure is a combinatorial manifold (i.e. the link of every vertex is an (n-1)-sphere or ball according as to whether the vertex is in the interior or boundary ; the proof is by verifying that the property is invariant under subdivision, and true in an n-simplex) .

<u>Example 1</u> . The <u>Long Line</u> is obtained by filling in (with unit intervals) all the ordinals up to the first non-countable , and is given the order topology . Then it can be shown (non trivially) that the Long Line has a 1 - polymanifold structure, although it is non-paracompact, and therefore non-triangulable .

<u>Example 2</u> . The Prüfer-manifolds are non-triangulable n-manifolds , $n \geq 2$.

<u>Direct Limits</u>

Suppose X_n , $n = 0, 1, 2, \ldots$, is a sequence of polyspaces , such that, for each n , X_n is a polysubspace of X_{n+1} . Define the <u>limit structure</u>

on $X = UX_n$ to be

$$\mathcal{F}(X) = U\,\mathcal{F}(X_n) \ .$$

The topology of $\mathcal{F}$ is the same as the limit topology .

Example 1

Let $X_n = E^n$. Assume $E^n \subset E^{n+1}$ linearly .

Then $E^\infty = U\,E^n$ is Euclidean ∞-space . This is not to be confused with , nor homeomorphic (in either topology) to , R^∞ , Hilbert space , which is the product of countable copies of the reals .

Example 2

Let B^0 = point ; $B^n = S\,B^{n-1}$, the suspension . Then B^n is an n-polyball , and $B^\infty = U\,B^n$ the ∞-polyball . This is not to be confused with , nor is homeomorphic to , I^∞ , the Hilbert cube .

Example 3

Let S^0 = two points ; $S^n = SS^{n-1}$, the suspension . Then S^n is an n-polysphere, and $S^\infty = US^n$ the ∞-polysphere . It is true that B^∞ has S^∞ as a closed subpolyspace , with complementary open subspace $B^\infty - S^\infty \cong E^\infty$. Nevertheless we do not call these boundary and interior because it is fairly easy to show polyhomeomorphisms

$$E^\infty \cong B^\infty \cong S^\infty \ .$$

Therefore B^∞ is homogeneous without boundary , because S^∞ is .

Example 4

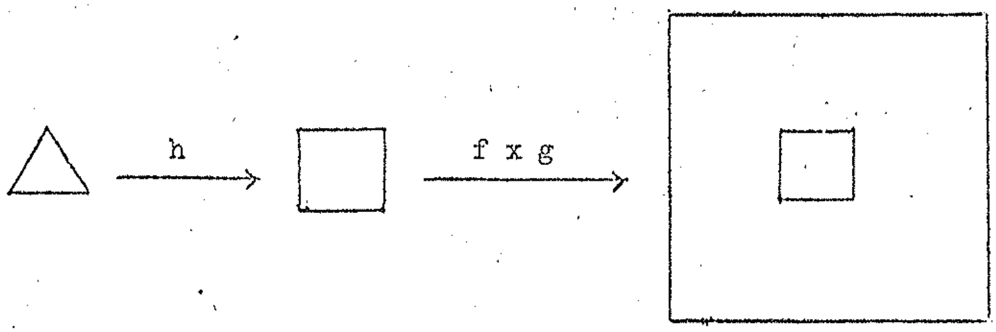
Let $O = U\,O_n$ be the infinite orthogonal group . Any triangulation of O_n can be extended to a triangulation of O_{n+1} . The resulting structures define a polystructure on O .

Infinite manifolds . The above definition is good for $n = \infty$. The above examples are all infinite manifolds. Similarly other classical groups , and the infinite Grassman and Stiefel manifolds . We observe that an ∞ -manifold has no boundary because an ∞ -ball has no boundary .

Products.

Let K,L be complexes in E^p , E^q . Then $K \times L$ is a cell complex in E^{p+q} , and so the natural structure $\mathcal{J}(K \times L)$ is uniquely defined . Given now two polyspaces X, Y , define the product structure .

$$\mathcal{J}(X \times Y) = \{(f \times g)\, h \,;\; f \in \mathcal{J}(X) , g \in \mathcal{J}(Y), h \in \mathcal{J}(\mathrm{dom} f \times \mathrm{dom} g) .$$



We can deduce :

1) The product is functorial on $\mathcal{P}$.

2) A function $f : X \to Y$ is a polymap if and only if the graph $1 \times f : X \to X \times Y$ is a polyembedding .

Joins

The topological join $X * Y$ of two spaces X , Y is obtained from $X \cup (X \times I \times Y) \cup Y$ by identifying $x = (x, 0, y)$, $y = (x, 1, y)$ all $x \in X$, $y \in Y$, and giving the identification topology .

If X , Y are polyspaces we define the join structure $\mathcal{F}(X * Y)$ as follows .

Given K in E^p , L in E^q , we identify E^p , E^q with $E^p \times 0 \times 0$, $0 \times E^q \times 1$ in $E^p \times E^q \times I$, $\subset E^{p+q+1}$.

The images of K, L are joinable in E^{p+q+1}, and we define $K * L$ to be their join . The complex $K * L$ has a natural structure $\mathcal{F}(K * L)$. Define

$$\mathcal{F}(X * Y) = \{ (f * g)h \; ; \; f \in \mathcal{F}(X) , g \in \mathcal{F}(Y) , h \in \mathcal{F}(\text{domf} * \text{domg}) .$$

We can deduce that the topology of $\mathcal{F}(X * Y)$ is the same as the topology of the join $X * Y$ above .

Remark

The join $*$ is functorial on the category of maps, but not on the subcategory of embeddings . Give $X' \subset X$, $Y' \subset Y$ then $X' * Y'$ does not always have the topology induced from the inclusion $X' * Y' \subset X * Y$. For example let $X = I$, $X' = \mathring{I}$, and $Y = Y' =$ a point . Then the cone on $\mathring{I}$ is not a subspace of the cone on I ; the cone on $\mathring{I}$ has a finer topology than the induced topology , and is locally compact at the vertex , whereas the induced topology is not (cf. the polysubspace Example 2) .

On the other hand the join is functorial in the category of polysubspaces and injective polymaps; the naturality in the category dictates the topology to be chosen on the join , which is then not functorial in the subcategory of polyembeddings . The explanation is that the concept join is essentially a combinatorial idea, and so as we should expect, in this context the polystructure is more basic than the topology .

Function Spaces

Let X be a polyhedron , and Y a polyspace . Let Y^X be the set of polymaps $X \to Y$. We define the function space structure $\mathcal{F}(Y^X)$ on Y^X

as follows . If $f : K \to Y^X$ is an injective function , let $f' : X \times K \to Y$ be the associated function given by $f'(x , k) = (fk)\,x$. Define

$$\mathcal{F}(Y^X) = \{ f \,;\, f' \text{ is a polymap} \} .$$

<u>Lemma</u> (Hudson) <u>Any two such f'_s are related</u> .

Therefore $\mathcal{F}(Y^X)$ is a family of polyhedra in Y^X and the three axioms for a polystructure are easy to verify. We can deduce the following properties

1) The structure of Y^X is functorial on X,Y in $\mathcal{P}$. In other words if $f: X_1 \to X$ and $g : Y \to Y_1$ are polymaps, then the induced function

$$g^f : Y^X \longrightarrow {Y_1}^{X_1}$$

is also a polymap .

2) If X,Y are polyhedra and Z a polyspace then there is a natural polyhomeomorphism

$$(Z^Y)^X \cong Z^{Y \times X} .$$

<u>Remark 1</u> . If X not a polyhedron (not compact) then the above definition does not give a polystructure . For example if $X = E'$, $Y = E^2$ then Hudson's theorem fails; there exist two $f's$ that are not related .

<u>Remark 2</u> . The topology of the structure is strictly finer than the compact open topology , and is therefore Hausdorff . If Y is a polyhedron or a manifold then both topologies give the same homotopy structure on Y^X . (Question : is this true for general Y ?)

<u>I s o t o p y</u>

Let $(X \subset Y)$ denote the polyspace of polyembeddings of X in Y , with strunture induced from Y^X . (Question : is it a polysubspace of Y^X ?)

One of the main reasons for the way we have developed the theory is that the following four definitions of isotopy are now trivially equivalent .

A <u>polyisotopy</u> of X in Y is

i) a point of $(X \subset Y)^I$

ii) a polymap $I \twoheadrightarrow (X \subset Y)$

iii) a polymap $X \times I \twoheadrightarrow Y$, which is a polyembedding at each level ,

iv) a level-preserving polyembedding $X \times I \rightarrow Y \times I$.

If $f, g : X \twoheadrightarrow Y$ are the beginning and end points of the isotopy , we say the isotopy <u>moves fX onto gX</u> , and that f , g are <u>isotopic</u> .

Let $H(Y)$ denote the polyspace of polyhomeomorphisms of Y onto itself, with structure inducec from Y^Y . An <u>ambient polyisotopy</u> of Y is a polyarc in $H(Y)$ starting at the identity, and finishing at e , say . If X is a polysubspace of Y we say the ambient isotopy <u>moves X onto eX</u> . If $f: X \rightarrow Y$ is a polyembedding (or polymap) we say f, ef are <u>ambient isotopic</u> . Later we shall prove a theorem of Hudson , which says that the notions of isotopy and ambient isotopy coincide for manifolds of codimension $\geqslant 3$. In codimension 2 they are essentially different, because ordinary knots in E^3 can be untied by isotopy , but not by ambient isotopy .

<u>R e m a r k</u>

If $Y = E^n$, there is another definition of isotopy favoured by some writers , which we call <u>linear isotopy</u>, and it is worthwhile analysing the difference . A <u>linear homotopy</u> of X in E^n is constructed as follows : choose a fixed triangulation K of X , and for each vertex $v \in K$, a polymap $f_v : I \longrightarrow E^n$. For each t , let $g_t : K \longrightarrow E^n$ be the linear map determined by the vertex map $v \longrightarrow f_v(t)$. Then $\{g_t\}$ or $g : K \times I \longrightarrow E^n$ is the linear homotopy . If g is an embedding at each level we call g a linear isotopy . We make the following observations :

i) Not every linear isotopy is poly, because in general the track $g(K \times I)$ left by the linear isotopy is a curvilinear ruled surface rather than a euclidean polyhedron .

ii) Not every polyisotopy is linear, as shown by the example below .

iii) If two polymaps are linearly isotopic then they are polyisotopic . The converse is also true (non-trivially) if X is a manifold of codimension $\geqslant 3$.

iv) Linear isotopy is not functorial. We justify this last statement by defining a polystructure on $(X \subset E^n)$ that exactly captures linear isotopy ; more precisely we shall construct a polystructure, $\mathcal{F}_L$ say , on $(E^n)^X$ such that linear homotopies are the polymaps $I \longrightarrow (E^n)^X$ with respect to $\mathcal{F}_L$, and linear isotopies are the polymaps $I \rightarrow (X \subset E^n)$ with respect to the induced structure .

Define $\mathcal{F}_L$ as follows: if K is a triangulation of X with k vertices , then the set M_K of linear maps $K \longrightarrow E^n$ can be given a polystructure $M_K \cong E^{kn}$. If K' is a subdivision of K , then $M_K \subset M_{K'}$ is a polyembedding .

Therefore $(E^n)^X = \bigcup_K M_K$, the union taken over all triangulations in the structure of X , and $\mathcal{F}_L$ is defined by the limit polystructure . We shall show that if $E^n \rightarrow E^n$ is a polyhomeomorphism, then the induced function $(E^n)^X \longrightarrow (E^n)^X$, which is a polyhomeomorphism with respect to the function space structure, is not even continuous with respect to $\mathcal{F}_L$.

<u>Example 1</u> . Let $X = I$ and $Y = E^2$, and consider the isotopies $I \rightarrow (I \subset E^2)$ performed by a caterpillar crawling firstly along a straight twig , and secondly along a bent twig . The first isotopy , f say , is linear , and therefore also poly . The second isotopy , g say , is poly but <u>not</u> linear , because we cannot describe it in terms of a fixed triangulation of $X = L$. This shows $\mathcal{F} \not\subset \mathcal{F}_L$. Now suppose the caterpillar

performs g by starting with his nose, and finishing with his tail, at the bend in the twig. Then $g\mathring{I}$ is a closed set in the topology of $\mathcal{F}_L$, whereas $f\mathring{I}$ is not. There is an obvious polyhomeomorphism η of E^2 bending a straight twig into a bent twig, and the induced map of $(I \subset E^2)$ into itself maps $f\mathring{I}$ into $g\mathring{I}$. Therefore it cannot be continuous with respect to the topology of $\mathcal{F}_L$.

The explanation is that $\mathcal{F}$ is functorial on $X, Y \in \mathcal{P}$, whereas $\mathcal{F}_L$ is functorial only on $X \in \mathcal{P}$ and Y in the subcategory of euclidean spaces and linear maps. Since our theory is directed towards isotopies of manifolds in manifolds, we favour $\mathcal{F}$ and reject $\mathcal{F}_L$.

Example 2. Let H^n denote the set of polyhomeomorphisms of E^n onto itself having compact support. The hypothesis of compact support enables us to define on H^n, as above, both a function space polystructure $\mathcal{F}_1$, and a linear polystructure $\mathcal{F}_2$. Let H_1^n, H_2^n be the resulting topological spaces, both having H^n as underlying set. Then it appears that H_1^n, H_2^n have different homotopy structures. By Alexander's Lemma on isotopy, it is easy to show that H_1^n is contractible. However Kuiper has used the queer differential structures on S^7 to show that either $\pi_0(H_2^6) \neq 0$ or $\pi_1(H_2^5) \neq 0$. This is essentially a phenomen on of codimension zero.

Degeneracy

Let $f: X \to Y$ be a polymap. Define the non-degenerate structure $\eta(f)$ of f by

$$\eta(f) = \{ g \in \mathcal{F}(X) \; ; \; fg \in \mathcal{F}(Y) \} .$$

Note that in general $fg \notin \mathcal{F}(Y)$ because it is not injective. Call f non-degenerate if $\eta(f)$ is a base for $\mathcal{F}(X)$. Otherwise f is degenerate.

Example 1. A polyembedding is non-degenerate.

Example 2 . A polyimmersion (local embedding) is non-degenerate .

Example 3 . A simplicial map is non-degenerate if and only if it maps each simplex non-degenerately .

Example 4 . We shall show that any map of a polyhedron of dimension $\leqslant n$ to an n-manifold can be put into "general position" where it is non-degenerate .

The mapping cylinder problem

The problem is to define a natural structure on the mapping cylinder $\mathcal{C}$ of a map $f : X \to Y$. We explain why this problem is, in a sense, insoluble .

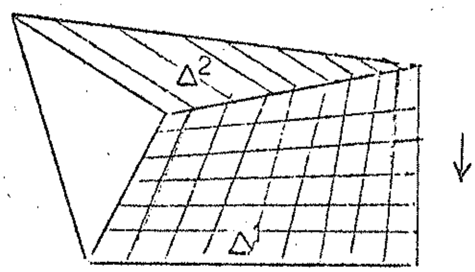
1. Topological . The topological mapping cylinder $\mathcal{C}$ is obtained from $X \times I \cup Y$ by identifying $(x,1) = fx$, all $x \in X$, and is given the identification topology . Then $\mathcal{C}$ is functorial on the category of maps .

2. Combinatorial . Suppose $f : K \to L$ is a simplicial map . Whitehead gave a rule for defining the simplicial mapping cylinder , G say , of f , which is a triangulation $g : G \to C$ of the topological mapping cylinder . This rule is functorial on the category of simplicial maps , but not on the category of piecewise linear maps . For suppose K', L' are subdivisions of K , L , giving rise to the simplicial cylinder $g' : G' \to C$. Then , although G, G' are piecewise linearly homeomorphic , g , g' are not in general related . Therefore the identity maps $K' \to K$, $L' \to K$ induce the identity $C \to C$, but only a piecewise projective map $G' \to G$.

3. Polyhedral . The inclusion $C \subset X * Y$ of the mapping cylinder in the join induces a natural polystructure $\mathcal{F}(C)$ on C that is functorial on the category of polymaps . However $\mathcal{F}(C)$ gives the wrong topology (too fine a one) .

Example 1 . The identity on I has mapping cylinder a square , and polystructure the Woven Square (of example 4 above) .

Example 2 . The mapping cylinder of a simplicial map of a 2-simplex onto a 1-simplex epitomises the problem , because when embedded in E^3 it looks like the prow of a ship .



The structure $\mathcal{F}(C)$ has a base consisting of all horizontal sections , and all vertical sections going athwartships , but no 3-dimensional stuff .

Example 3 . If f is simplicial , then the simplicial mapping cylinder $G \to C$ is related to $\mathcal{F}(C)$. In other words , $\mathcal{F}(C)$ can be enlarged (non-naturally) to contain any simplicial cylinder, and is the intersection of all the structures determined by the simplicial cylinders .

Example 4 . On the subcategory of non-degenerate polymaps the natural structure $\mathcal{F}(C)$ can be enlarged to a structure $\mathcal{F}_1(C)$ that (i) is functorial on this subcategory (ii) contains all simplicial cylinders , and (iii) gives the correct topology . A base for $\mathcal{F}_1(C)$ is

$$\mathcal{B} = \mathcal{F}(Y) \quad \cup \left\{ (g \times 1)\, h \; ; \; g \in \eta(f) \; , \; h \in \mathcal{F}(\mathrm{dom}\, g \times I) \right\} .$$

Concluding Remarks

We can enlarge or change $\mathcal{P}$ by enlarging or changing the tool $\mathcal{C}$.

Example 1 . Enlarge $\mathcal{C}$ to contain piecewise projective maps .

Example 2 . Further enlarge $\mathcal{C}$ to contain piecewise algebraic complexes and piecewise algebraic maps. Then algebraic varieties in E^n

would become polysubspaces .

Example 3 . Replace $\mathcal{C}$ by the category of open subsets of E^n and differential maps . Then $\mathcal{P}$ would be the category of differential manifolds and differential maps.

Gabrielle has pointed out that a polyspace is equivalent to a contravariant fuctor from $\mathcal{C}$ to the category of sets and functions , obeying two axioms of intersection and union; a polymap is a natural transformation between two such functors .

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

1 9 6 3

Seminar on Combinatorial Topology

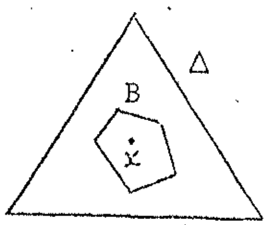
by E.C. ZEEMAN

Chapter 3 : REGULAR NEIGHBOURHOODS

From now on we shall omit the prefix "poly", and whenever we say space, map, manifold, etc., we mean polyspace, polymap, polymanifold, etc...

Lemma 8. A convex linear cell is a ball

Proof : Given a convex linear cell B we have to exhibit a specific piecewise linear homeomorphism from a simplex Δ onto B. Since B is in some Euclidean space, we can choose $\Delta \supset B$. Let x be a point in $\mathring{B}$. Then radial projection from x gives a homeomorphism $\dot{\Delta} \longrightarrow \dot{B}$, but this is not piecewise linear by the Standard Mistake. We get round this difficulty by defining a pseudo radial projection as follows. Let $\dot{\Delta}'$ be the cell subdivision of $\dot{\Delta}$ consisting of all cells



$\Delta_1 \cap x B_1$, $\Delta_1 \in \dot{\Delta}$, $B_1 \in \dot{B}$. Let $\dot{\Delta}''$ be a simplicial subdivision of $\dot{\Delta}'$. Radial projection of $\dot{\Delta}''$ determines an isomorphic subdivision $\dot{B}''$ of $\dot{B}$, and radial projection of the vertices determines the simplicial isomorphism, which is of course piecewise linear. Joining to x gives the required homeomorphism $A \longrightarrow B$.

Corollary. Joins of spheres and balls obey the rules :

i) $B^p B^q \cong B^{p+q+1}$

......

ii) $B^pS^q \cong B^{p+q+1}$

iii) $S^pS^q \cong S^{p+q+1}$.

Proof. Since the structure of a join is functorial, it suffices to prove one example .

i) The join of two simplexes is a simplex

ii) In E^{p+q+1} choose B^p, B^{q+1} to be simplexes crossing at their barycentres. Then $B^p\dot{B}^{q+1}$ is a convex linear cell.

iii) Take the boundary of ii) .

We call a complex J a combinatorial n-manifold if the link of each vertex is an (n-1)-sphere or an (n-1)-ball.

Lemma 9 . Suppose $|J| = M$. Then J is a combinatorial manifold if and only if M is a manifold .

Proof . One way is trivial; for if J is a combinatorial manifold , then the closed vertex stars of J give a covering of M by balls, such that each point of M has some ball as a neighbourhood .

Conversely suppose M is an n-manifold , and let x be a vertex of J in $\mathring{M}$. By the definition of manifold (polymanifold), there is a piecewise linear embedding $f : \Delta \longrightarrow J$ covering a neighbourhood of x , where Δ is an n-simplex , such that $f^{-1}x \in \mathring{\Delta}$. Subdivide so that $f : \Delta' \longrightarrow J'$ is simplicial; we have piecewise linear homeomorphisms

$$\dot{\Delta} \longrightarrow \mathrm{lk}(f x , \Delta') \longrightarrow \mathrm{lk}(x, J') \longrightarrow \mathrm{lk}(x, J)$$

Where the middle arrow is an isomorphism and the other two arrows are pseudo radial projections . Hence $\mathrm{lk}(x,J)$ is an (n-1)-sphere .

If x is a vertex of J in $\dot{M}$, there is a similar situation except that $f^{-1}_x \in \dot{\Delta}$, and so it follows that $\mathrm{lk}(x,J)$ is a ball.

Corollary 1 . Let $|J| = M$ be an n-manifold . If A is a p-simplex of J , then

$$\text{either } lk(A,J) = (n-p-1)\text{-sphere and } \mathring{A} \subset \mathring{M}$$

$$\text{or } lk(A,J) = (n-p-1)\text{-ball and } A \subset \dot{M} .$$

Proof . We show the link is a sphere or ball by induction on p , the induction starting at $p = 0$ by the Lemma . If $p > 0$, write $A = xB$, and then $lk(A,J) = lk(x, lk(B,J))$, which is the link of a vertex in an (n-p)-sphere or ball, by induction, and is therefore an (n-p-1)-sphere or ball by the Lemma .

Any point of $\mathring{A}$ has $A\, lk(A;J)$ as a closed neighbourhood , and so lies in $\mathring{M}$ or $\dot{M}$ according as to whether it lies in the interior or boundary of this neighbourhood, i.e. according as to whether $lk(A,J)$ is a sphere or ball. Therefore if the link is a sphere then $\mathring{A} \subset \mathring{M}$, and if the link is a ball then $A \subset \dot{M}$, since $\dot{M}$ is closed .

Corollary 1 justifies the following definition : if J is a combinatorial manifold, define the boundary $\dot{J}$ to be the subcomplex

$$\dot{J} = \{ A \in J ;\ lk(A,J) = \text{ball} \}$$

and the interior to be the open subcomplex $\mathring{J} = J - \dot{J}$.

We deduce at once :

Corollary 2. If $|J| = M$ = manifold, then $|\dot{J}| = \dot{M}$.

Definition . If B^{n-1} is an (n-1)-ball contained in the boundary $\dot{M}^n$ of an n-manifold M^n , we call B^{n-1} a face of M^n and write $B^{n-1} < M^n$. We are particularly interested when $M^n = B^n$ a ball also. Let Δ^n denote an n-simplex .

Theorem 2 . If $B^{n-1} < B^n$ and $\Delta^{n-1} < \Delta^n$ then any homeomorphism $B^{n-1} \longrightarrow \Delta^{n-1}$ can be extended to a homeomorphism

$B^n \longrightarrow \Delta^n$.

Corollary . If two balls meet in a common face , then their union is a ball . (For by Theorem 2 the union is homeomorphic to the suspension of a simplex) .

Theorem 3 . If $B^n \subset S^n$ then $\overline{S^n - B^n}$ is a ball

Remark 1 .

The original proofs of Theorem 2 and 3 were given by Newman and Alexander in the 1920's and 30's and used "stellar theory" instead of combinatorial theory . The essential notion of the proof is to replace the finite simplicial structure of a ball by some ordered finite structure, and then use induction on the number of steps in the ordering (the induction starting trivially with a simplex) . Newman and Alexander used an ordering by stellar subdivisions; we give a new proof here, based on ordering by collapsing . The collapsing technique was invented by Whithehead in 1939, and is more powerful than stellar theory because it includes the theories of regular neighbourhoods and simple homotopy type . Notice that some concept of an ordered structure seems vital, because without it we cannot prove :

Schönflies Conjecture : If $S^{n-1} \subset S^n$ then the closures of each component of the complement is a ball .

The conjecture is true for $n \leqslant 3$, but unsolved for $n > 3$. It is known by Morton Brown's result that they are triangulated topological balls, but not known whether they are polyballs . Our ignorance of whether they are polyballs when $n = 4$ implies our ignorance of whether they are even polymanifolds when $n = 5$ (the links of boundary vertices may go haywire) .

Remark 2 .

The proof of Theorem 2 and 3 is done together by induction on n . The induction starts trivially with $n = 0$. We shall show first that

Theorem 2_n is equivalent to Theorem 3_{n-1}. The inductive step is achieved by showing that

$$\left.\begin{array}{l}\text{Theorem } 2_r\ ,\ r \leq n \\ \\ \text{Theorem } 3_r\ ,\ r < n\end{array}\right\} \Longrightarrow \text{Theorem } 3_n$$

The inductive step is long, involving Lemmas 10 - 17 and Theorems 4 - 8, during which we shall often have to make inductive use of Theorems 2 and 3. However we can avoid going round in a circle by

i) assuming everything to be of dimension $\leq n$

ii) avoiding the use of Theorem 3_n

until Theorem 3_n is proved. To emphasise which statements are involved in the induction, and at the same time avoid repetition, we put a star against all those lemmas or theorems which depend upon Theorem 2_r and its Corollary, $r \leq n$, and Theorem 3_r, $r < n$.

<u>Lemma 10</u>. <u>Any homeomorphism between the boundaries of two balls can be extended to the interiors</u>.

<u>Proof</u>. We are given $f : \dot{B}_1 \to \dot{B}_2$.

Choose triangulations $g_i : \Delta \longrightarrow B_i$. Define $h : \dot{\Delta} \longrightarrow \dot{\Delta}$ by the commutative diagram

$$\begin{array}{ccc} \dot{\Delta} & \xrightarrow{h} & \dot{\Delta} \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ \dot{B}_1 & \xrightarrow{f} & \dot{B}_2 \end{array}$$

Extend h conewise to a homeomorphism $h' : \Delta \longrightarrow \Delta$.

Then the required homeomorphism $f' : B_1 \to B_2$ is given by the commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{h'} & \Delta \\ \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_2} \\ B_1 & \xrightarrow{f'} & B_2 \end{array}$$

<u>Theorem 2_n is equivalent to Theorem 3_{n-1}</u> .

<u>Proof</u> . Assume Theorem 2_n . Given $B^{n-1} \subset S^{n-1}$, then joining to a point x , we have $B^{n-1} < xS^{n-1}$. Let Δ^n be a simplex with face Δ^{n-1} and opposite vertex y . Choose a homeomorphism $B^{n-1} \longrightarrow \Delta^{n-1}$, and extend it to $xS^{n-1} \longrightarrow \Delta^n$. Therefore $\overline{S^{n-1} \quad B^{n-1}}$ is homeomorphic to the ball $y\,\dot{\Delta}^{n-1}$.

Conversely assume Theorem 3_{n-1} . Given $B^{n-1} < B^n$, then we know $\overline{\dot{B}^n - B^{n-1}}$ is a ball . Therefore given a homeomorphism $B^{n-1} \longrightarrow \Delta^{n-1}$, we can extend $\dot{B}^{n-1} \to \dot{\Delta}^{n-1}$ to a homeomorphism $\overline{\dot{B}^n - B^{n-1}} \longrightarrow y\,\dot{\Delta}^{n-1}$, by Lemma 10 . Therefore we have defined $\dot{B}^n \longrightarrow \dot{\Delta}^n$, and can extend to $B^n \longrightarrow \Delta^n$, again by Lemma 10 .

<u>S t e l l a r   s u b d i v i s i o n</u>

Recall from Chapter 1 that an <u>elementary stellar subdivision</u> of K is given by

$$K' = (K - \mathrm{st}(A,K)) \cup a\,\dot{A}\,\mathrm{lk}(A,K) \qquad \text{where} \quad a \in \overset{o}{A} , \quad A \in K .$$

A <u>stellar subdivision</u> of K , written σK , is the result of a finite number of elementary ones .

<u>Examples</u> i) An r^{th} derived is a stellar .

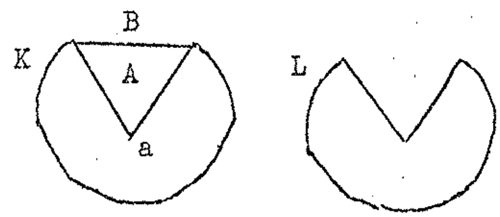
ii) If $K \supset L$, then any stellar subdivision of K determines a unique stellar subdivision of L , and conversely .

iii)  is <u>not a</u> stellar subdivision of a triangle .

Collapsing

If $K \supset L$, we say there is an <u>elementary simplicial collapse</u> from K to L if $K - L$ consists of a principal simplex A of K together with a free face . Therefore if $A = aB$, then

$$K = L \cup A$$
$$a\dot{B} = L \cap A$$

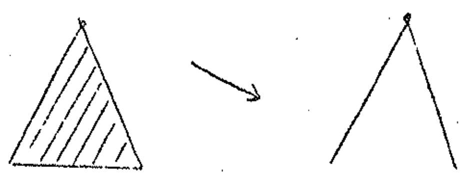


We describe the elementary simplicial collapse by saying <u>collapse A onto $a\dot{B}$</u> , or <u>collapse A from B</u> .

We say K <u>simplicially collapses</u> to L , written $K \overset{s}{\searrow} L$, if there is a sequence of elementary simplicial collapses going from K to L . If L is a point we call K <u>simplicially collapsible</u>, and write $K \overset{s}{\searrow} 0$.

<u>Examples</u> .

i) A cone simplicially collapses onto any subcone . For just collapse all the other simplexes in towards the vertex .



More precisely let aK be the cone on K , and aL be the subcone on L , where $L \subset K$. Then order the simplexes $B_1, \ldots, B_r$ of $K - L$ in order of decreasing dimension , and collapse aB_i from B_i , $i = 1, \ldots, r$.

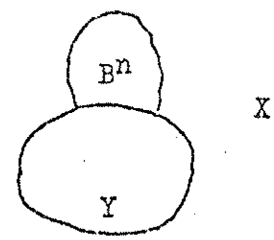
ii) A cone is simplicially collapsible .

iii) A simplex is simplicially collapsible . Both these are special cases of i) .

We now repeat the definition for polyhedra. If $X \supset Y$ are polyhedra, we say there is an <u>elementary collapse</u> from X to Y if there exists $B^n \succ B^{n-1}$ such that

$$X = Y \cup B^n$$

$$B^{n-1} = Y \cap B^n$$



We describe the elementary collapse by saying <u>collapse B^n onto B^{n-1}</u>, or <u>collapse B^n from $\overline{\dot{B}^n - B^{n-1}}$</u>.

We say <u>X collapses to Y</u>, written $X \searrow Y$, if there is a sequence of elementary collapses going from X to Y. If Y is a point we call X <u>collapsible</u>, and write $X \searrow 0$. For example a ball is collapsible.

We now investigate the relationship between simplicial collapsing and collapsing. We write $K \searrow L$ if $|K| \searrow |L|$. The significance of this last definition is that the balls across which the collapse takes place may not be subcomplexes of K. It is trivially true that

$$K \overset{s}{\searrow} L \implies K \searrow L ,$$

but the converse is unknown. What we can prove is:

<u>* Theorem 4</u>. <u>If $K \searrow L$, then there exists a subdivision K', L' of K, L such that $K' \overset{s}{\searrow} L'$</u>.

<u>* Corollary 1</u>. <u>If $X \searrow Y$ then there exists a triangulation such that $K \overset{s}{\searrow} L$</u>.

<u>* Corollary 2</u>. <u>If X is collapsible, then there exists a triangulation that is simplicially collapsible</u>.

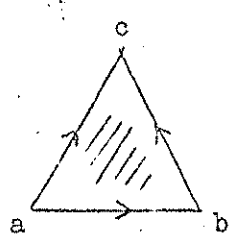
Before proving Theorem 4 we digress a little to indicate the consequences of the definition of collapsing.

Simple homotopy type

The relation $X \searrow Y$ between X and Y is ordered. If we forget the ordering, then we generate an equivalence relation between polyhedra called simple homotopy type. Since a collapse is a homotopy equivalence, this is a finer equivalence relation than homotopy type. It is strictly finer, because, for example, the lens spaces $L(7,1$, $L(7,2)$ are of the same homotopy type, but not of the same simple homotopy type. But for simply-connected spaces homotopy type = simply homotopy type, and there are simply-connected non-homeomorphic manifolds of the same homotopy type.

The Dunce Hat.

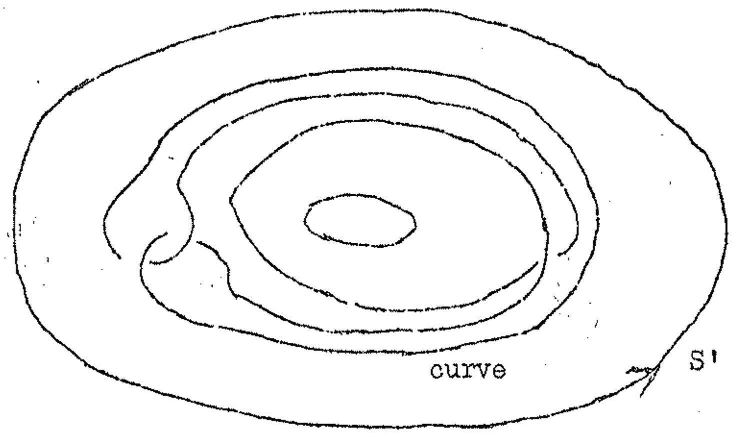
If we preserve the order $X \searrow Y$ then the relation between X, Y is much sharper. Trivially if X is collapsible then X is contractible (homotopywise). But the converse is not true. For example consider the Dunce Hat D which is defined to be a triangle with its sides identified $ab = ac = bc$. Then D is contractible (although the contraction is hard to visualise), and so D is the same simple homotopy type as a point; but D is not collapsible because there is nowhere to start. Although $D \not\searrow 0$, it can be shown that $D \times I \searrow 0$.



Conjecture. If K^2 is a contractible 2-complex then $K \times I \searrow 0$

This conjecture is interesting because it implies the 3-dimensional Poincaré Conjecture, as follows. Let M^3 be a compact contractible 3-manifold; it is sufficient to show that M^3 is a ball. Call X is a spine of M if $M \searrow X$. Now M^3 has a contractible spine K^2. By the conjecture $M^3 \times I \searrow K^2 \times I \searrow 0$, and we shall show in Theorem 8 Corollary 1 that this implies $M^3 \times I = B^4$. Hence $M^3 \subset \dot{B}^4 = S^3$, and by the Schönflies Theorem $M^3 = B^3$, a ball.

In particular the conjecture is true for the Dunce Hat, and so any M^n, $n = 3$, having D as a spine is a ball. This is also true for $n \geq 5$ because D unknots in $\geqslant 5$ dimensions. However it is <u>not</u> true for $n = 4$ because there is an $M^4 \neq B^4$ (in fact $\pi_1(\dot{M}^4) \neq 0$) having D as a spine. The construction of M^4 is due to Mazur, and defined by attaching a 2-handle to $S' \times B^3$, by a curve in the boundary that is homotopic, but not isotopic, to the first factor :



<u>Lemma 11. If $K \overset{s}{\searrow} L$, then we can reorder the elementary collapses so that they are in order of decreasing dimension</u>.

<u>Proof</u>. Suppose $K_1 \searrow K_2 \searrow K_3$ are consecutive elementary collapses, the first being across A^p from B^{p-1}, and the second across C^q from D^{q-1}. We shall show that if $p < q$ then we can interchange the order of the collapses (which is not true if $p \geqslant q$). The lemma follows by performing a finite number of such interchanges.

Since $p < q$, C^q is not a face of A^p or B^{p-1}. Therefore C^q, which is principal in K_2, remains principal in K_1. Also $D^{q-1} \neq A$ or B, because A, B do not lie in K_2, and so D^{q-1} cannot be a face of A or

B (again since $p \leqslant q-1$) . Therefore D remains a free face of C in K_2 . Therefore , if $K_2^* = K_1 - (C \cup D)$, then there is an elementary collapse $K_1 \searrow K_2^*$ across C from D . Meanwhile A remains principal in K_2^* , and B remains a free face . Therefore there is an elementary collapse $K_2^* \searrow K_1$ across A from B . The lemma is proved .

<u>Remark</u> . Although Lemma 11 indicates a certain freedom to rearrange the order of collapses, we cannot rearrange arbitrarily . For example if B^3 is a simplicially collapsible 3-ball , if we start collapsing B^3 carelessly we may get stuck before reaching a point - for instance the dunce hat is a spine of B^3 , so that by mistake we might get stuck at the dunce hat . This problem is the reason why the methods which classified 2-manifolds failed to classify 3-manifolds .

Again , if $K \overset{s}{\searrow} L$ and K' is an arbitrary subdivision of K , then trivially $K' \searrow L'$ but we do not know if $K' \overset{s}{\searrow} L'$. However we can prove a more limited result :

<u>Lemma 12</u> . <u>If $K \overset{s}{\searrow} L$ then $\sigma K \overset{s}{\searrow} \sigma L$ for any stellar subdivision σK of K</u> .

<u>Proof</u> . By induction we may assume both the simplicial collapse and stellar subdivision to be elementary . Suppose

$$K = L \cup A$$

$$a\dot{B} = L \cap A \text{ , and suppose}$$

σK is obtained by starring C at c . There are three cases

(i) If $C \not< A$, then the lemma is trivial

(ii) If $C \not< B$, then the cone $a(\sigma B)$ collapses to the subcone $a(\sigma \dot{B})$

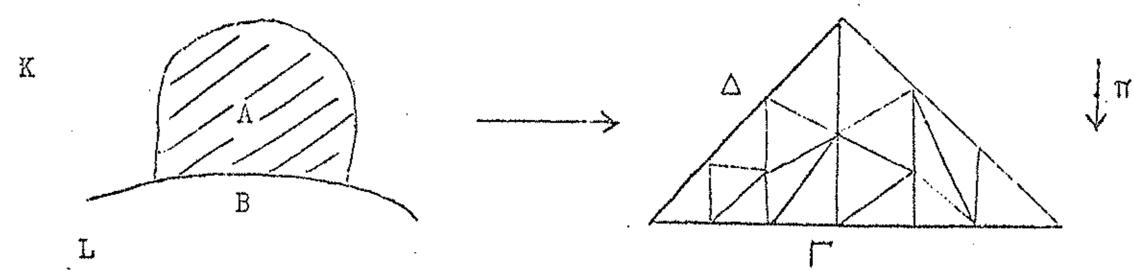
(iii) If $C < A$, but $C \not< B$, let $C = aB_1$, $B = B_1B_2$.

$$\text{Then } \sigma K \searrow \sigma L \cup \text{cone } a(c\dot{B}_1B_2)$$
$$\searrow \sigma L \cup \text{subcone } a(c\dot{B}_1B_2)^{\cdot} = \sigma L \text{ .}$$

*Lemma 13 . If $K \searrow L$ is an elementary collapse , then there exists a subdivision such that $K' \overset{s}{\searrow} L'$ and L' is stellar (but K' may not be) .

Proof . Let $A = \overline{K - L}$, and $B = A \cap L$. Then A is an n-ball and B a face . Let Δ , Γ be an n-simplex and an (n-1)-face . By Theorem 2_n choose a homeomorphism

$$h : A,B \longrightarrow \Delta , \Gamma .$$



Choose subdivisions so that h is a simplicial isomorphism $h : A',B' \xrightarrow{\cong} \Delta', \Gamma'$. Let $\pi : \Delta \longrightarrow \Gamma$ be the linear projection , mapping the vertex opposite Γ to the barycentre of Γ . Choose subdivisions Δ'' , Γ'' of Δ' , Γ' so that

$$\pi : \Delta'' \longrightarrow \Gamma''$$

is simplicial . Call such a subdivision of Δ cylindrical . Let A'' , B'' be the isomorphic subdivisions of A' , B' . Let B''' be an r^{th} derived of B , subdividing B'' , and let Γ''' the corresponding subdivision of Γ'' . By Lemma 5 , choose a subdivision Δ''' of Δ'' such that $\pi : \Delta''' \longrightarrow \Gamma'''$ simplicial , and let A''' be the corresponding subdivision of A'' . Then B''' is a stellar subdivision of B , and induces a stellar subdivision L' of L . Define $K' = A''' \cup L'$. Since Δ''' is cylindrical , $\Delta''' \overset{s}{\searrow} \Gamma'''$ cylinderwise , in decreasing order of dimension . Hence $A''' \overset{s}{\searrow} B'''$, and so $K' \overset{s}{\searrow} L'$.

Proof of Theorem 4

We are given a collapse $X \searrow Y$; that is to say a sequence of elementary collapses

$$|K| = X_r \searrow X_{r-1} \searrow \dots \searrow X_o = |L| .$$

By Theorem 1 we can find a subdivision K_r of K, such that, for each i, there is a subcomplex K_i covering X_i. Therefore we may write the elementary collapses

$$K_r \searrow K_{r-1} \searrow \dots \searrow K_o .$$

If $r = 1$ the result follows by Lemma 13. If $r > 1$ we show the result by induction. Assume we have found a subdivision K'_{r-1} of K_{r-1} such that $K'_{r-1} \overset{s}{\searrow} K'_o$. By Lemma 3 extend K'_{r-1} to a subdivision K'_r of K_r. Apply Lemma 13 to the elementary collapse $K'_r \searrow K'_{r-1}$, to obtain a simplicial collapse $K''_r \overset{s}{\searrow} K''_{r-1}$, where K''_{r-1} is a stellar subdivision of K'_{r-1}. The latter fact enables us to appeal to Lemma 12 to deduce $K''_{r-1} \overset{s}{\searrow} K''_o$, and so $K''_r \overset{s}{\searrow} K''_o$.

Full subcomplexes

If $K \subset J$ are complexes, we say K is <u>full</u> in J if no simplex of $J - K$ has all its vertices in K. We can deduce the elementary properties of fullness :

(i) If $K \subset J$, and J' a first derived complex of J then K' is full in J'.

(ii) If K full in J, and J^* any subdivision of J, then K^* full in J^*.

(iii) If K full in J, and A a simplex of J, then $A \cap K$ is empty or a face of A.

(iv) If K full in J, then there is a unique simplicial map $f : J \rightarrow I$ (the unit interval) such that $f^{-1}0 = K$.

Neighbourhoods

Let J be a complex and let $X \subset |J|$. The simplicial neighbourhood $N(X,J)$ is the smallest subcomplex of J containing a topological neighbourhood of X. It consists of all (closed) simplexes of J meeting X, together with their faces.

Now suppose X is a polyhedron in an n-manifold M. We construct derived neighbourhoods of X in M as follows. If M is compact choose a triangulation J, K of N, X. If M is not compact choose a triangulation J, K of M_o, X where M_o is a subpolyhedron containing a topological neighbourhood of X in M. Now in general M_o will not be a manifold round the edges, but it will be a manifold near X, which is all that matters. More precisely, if $A \in N(X, J)$ then $\operatorname{lk}(A,J) = \begin{cases} S^{n-1}, & A \subset \mathring{M} \\ B^{n-1}, & A \subset \dot{M} \end{cases}$

For simplicity of exposition we identify $M_o = |J|$, $X = |K|$.

Choose now an r^{th} derived complex $J^{(r)}$ of J.

Call $N = N(X,J^{(r)})$ an r^{th} derived neighbourhood of X in M.

If $r = 1$ and K full in J we call N a derived neighbourhood of X in M.

A fortiori if $r \geqslant 2$, then any r^{th} derived neighbourhood is a derived neighbourhood, because $K^{(r-1)}$ full in $J^{(r-1)}$. If J', J'' denote first and second deriveds, it is easy to show that

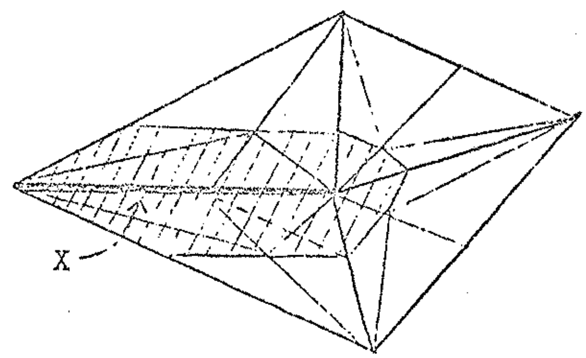
(i) $N(X, J') = \bigcup \overline{\operatorname{st}(x,J')}$, the union taken over all vertices $x \in K$,

(ii) $N(X, J'') = \bigcup \overline{\operatorname{st}(\hat{A}, J'')}$, the union taken over all simplexes $A \in K$, where $\hat{A}$ denotes the point at which A is starred in J'

Lemma 14 . Any two derived neighbourhoods of X in M are homeomorphic, keeping X fixed.

Proof . Let $N_1 = N(X, J'_1)$, $N_2 = N(X, J'_2)$ be the two given neighbourhoods. If M is compact, let J_o be a common subdivision of J_1, J_2. If M is not compact choose subdivisions of J_1, J_2 that intersect in a common subcomplex, and let J_o be this subcomplex. Choose a first derived J'_o of J_o and let $N_o = N(X, J'_o)$.

Let $f : J_1 \longrightarrow I$ be the unique simplicial map such that $f^{-1}0 = X$, which exists by the hypothesis of fullness. Choose $\varepsilon > 0$ and such that $\varepsilon < f x$, for all vertices $x \in J_o$, $x \notin X$. Let J_i^{ε} $(i = 0, 1)$ denote a first derived of J_i obtained by starring $A \in J_i$ on $f^{-1}\varepsilon$ if $fA = I$ and arbitrarily otherwise. Then $|N(X, J_i^{\varepsilon})| = f^{-1}[0, \varepsilon]$, $i = 0, 1$.



Therefore $N_1 \cong N(X, J_1^{\varepsilon})$, isomorphic

$\simeq N(X, J_o^{\varepsilon})$, homeomorphic by identity map

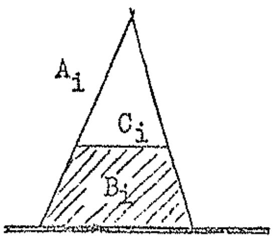
$\cong N_o$, isomorphic

$\cong N_2$, similarly.

Remark . Lemma 14 fails for first derived neighbourhoods without the fullness condition, which indicates the reason for having to pass to the second derived in general to obtain a derived neighbourhood. For example suppose X is the boundary of a 1-simplex in J. Then the first derived neighbourhood is connected, but the second derived is not.

Corollary. Any derived neighbourhood of X in M collapses to X.

Proof. By Lemma 14 it suffices to prove for one particular derived neighbourhood. Therefore choose a triangulation J, K of M, X such that K is full in J, and let $N = N(X, J^{\varepsilon})$ where J^{ε} is defined as in the proof of Lemma 14.



Order the simplexes $A_1, \dots, A_r$ of $J - K$ that meet K in order of decreasing dimension. Each A_i meets N in a convex cell B_i, with a face $C_i = A_i \cap f_{\varepsilon}^{-1}$. There is an elementary collapse of B_i from C_i, and the sequence of collapses $i = 1, \dots, r$ determines the collapses $N \searrow X$.

Lemma 15. Let $h : K \to K$ be a homeomorphism of a complex that maps each simplex onto itself, and keeps a subcomplex L fixed. Then h is ambient isotopic to the identity keeping L fixed.

Proof. The obvious isotopy moving along straight paths is not piecewise linear by the standard mistake. However it is easy to construct a piecewise linear isotopy $H : K \times I \to K \times I$ inductively on the prisms $A \times I$, $A \in K$, in order of increasing dimension. For each prism, $|H| \dot{A} \times I$ is given by induction, $H \mid A \times 0$ is the identity, and $H(a,1) = ha$. Therefore $H \mid (A \times I)^{\circ}$ is already defined, so map the centre of the prism to itself and join linearly. By construction H keeps L fixed.

Corollary 1. The isomorphism between any two first derived complexes is ambient isotopic to the identity.

Corollary 2. Any two derived neighbourhoods of X in M are ambient isotopic, keeping X fixed. If $X \subset \overset{\circ}{M}$, the isotopy can be chosen to keep $\dot{M}$ fixed.

For in the proof of Lemma 14 the homeomorphism was achieved by two isomorphisms between first deriveds, both keeping X fixed. The first deriveds can be chosen to agree outside the neighbourhoods, and so the isotopy keeps $\dot{M}$ fixed if $X \subset \overset{o}{M}$.

*<u>Theorem 5</u>. <u>A derived neighbourhood of a collapsible polyhedron is an n-manifold is an n-ball</u>.

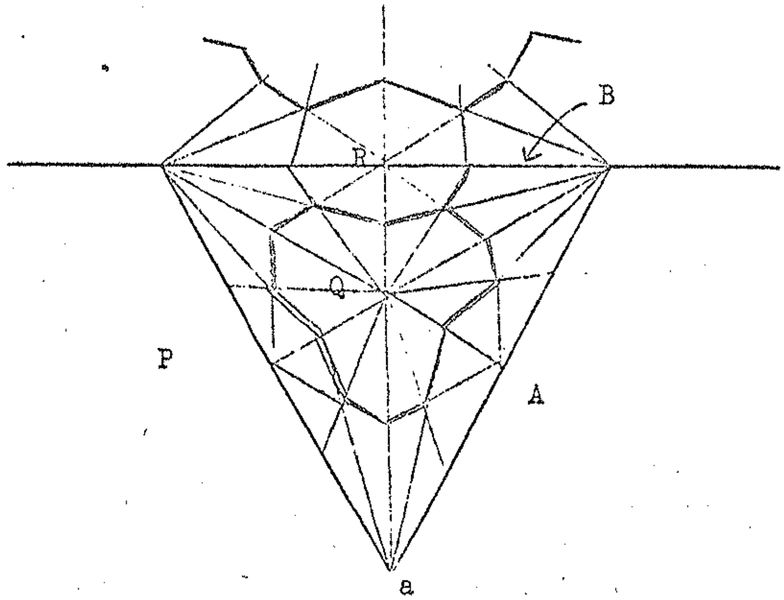
<u>Proof</u>. By induction on n, starting trivially with $n = 0$. By Lemma 14 it suffices to prove the theorem for one particular derived neighbourhood, and so we choose a second derived neighbourhood $N = N(X, J'')$, where $X = |K|$, $K \subset J$, and J'' is the second barycentric derived complex of J. Since X is collapsible, we can choose K such that $K \overset{s}{\searrow} 0$ by Theorem 4.

Let r be the number of elementary simplicial collapses involved in $K \overset{s}{\searrow} 0$. We show N is a ball by induction on r. The induction starts trivially with $r = 0$, for then K is a point, and N its closed star, which is a ball by Lemma 9. For the inductive step, let $K \searrow L$ be the first elementary simplicial collapse, collapsing a simplex A from B, say, where $A = aB$. Let $\hat{A}$, $\hat{B}$ denote the barycentres of $\hat{A}$, $\hat{B}$. Now

$$N = N(K, J'') = P \cup Q \cup R ,$$

where $P = N(L, J'')$, $Q = N(\hat{A}, J'')$, $R = N(\hat{B}, J'')$.

Now P is a ball by induction, and Q, R are balls since they are closed stars of vertices. If we can show that Q is glued onto P by a common face, then $P \cup Q$ is a ball by the Corollary to Theorem 2_n; similarly if R is glued onto $P \cup Q$ by a common face then N is a ball. Therefore the proof is reduced to showing the $P \cap Q$, and $(P \cup Q) \cap R$ are $(n-1)$-balls because if they are balls then they must be common faces, since the interiors of P, Q, R are disjoint.



Now $P \cap Q \subset \dot{Q} = \mathrm{lk}\,(\hat{A}\ , J'')$. Let

$$J_* = \mathrm{lk}\ (A',\ J') = \dot{A}'\ (\mathrm{lk}(A,J))'\ ,$$

where the prime thought this proof always denotes the barycentric first derived complex . There is an isomorphism

$$\dot{Q} \xrightarrow{\cong} J'_*$$

determined by the vertex map $\widehat{A\,C} \longrightarrow \widehat{C}$, for all $C \in J_*$. Under this isomorphism

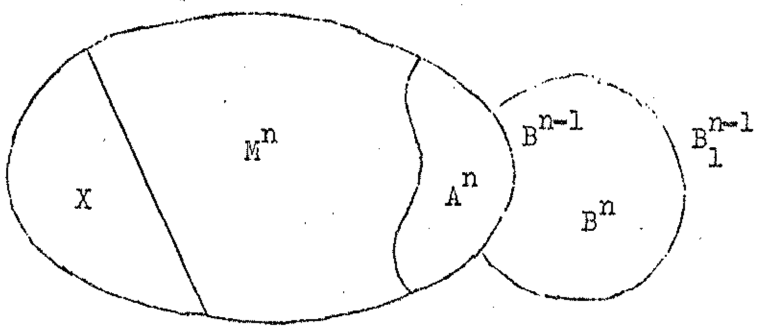
$$P \cap Q \xrightarrow{\cong} N\ (a\,\dot{B}\ ,\ J'_*\ )\ .$$

Now $a\,\dot{B}$ is collapsible , being a cone , and $(a\,\dot{B})'$ is full in J_* , which is an $(n-1)$-sphere or ball , by Lemma 9 . Therefore $N\ (a\,\dot{B}\ , J'_*)$ is a derived neighbourhood of a collapsible polyhedron , and is an $(n-1)$-ball by induction on n . Hence $P \cap Q$ is an $(n-1)$-ball .

Similarly $(P \cup Q) \cap R \subset \dot{R}$, and if we now choose $J_* = \mathrm{lk}\ (\hat{B}\ , J')$, then there is an isomorphism $\dot{R} \xrightarrow{\cong} J'_*$, throwing $(P \cup Q) \cap R$ onto $N(\hat{A}\,\dot{B}\,,\,J'_*)$. For the same reason as before we deduce $(P \cup Q) \cap R$ is an $(n-1)$-ball . This completes the proof of Theorem 5 .

<u>*Theorem 6 .</u> <u>Suppose the manifold M^n and the ball B^n meet in a common face . Let X be a closed subset of M^n not meeting B^n . Then there is a homeomorphism $M^n \longrightarrow M^n \cup B^n$ keeping X fixed</u> .

<u>Proof</u> . Since X is closed , $M^n - X$ is a manifold . Let B^{n-1} be the common face , and let A^n be a derived neighbourhood of B^{n-1} in $M^n - X$, which is a ball by Theorem 5 . Since $\mathring{A}^n \subset \mathring{M}^n$, B^{n-1} does not meet $\mathring{A}^n$,



and so B^{n-1} is a face of A^n . Since A^n , B^n meet in the common face B^{n-1}, their union is a ball by the Corollary to Theorem $2n$. Let $B_1^{n-1} = \dot{B}^n - \mathring{B}^{n-1}$, which is a ball by Theorem 3_{n-1} .

We now construct the homeomorphism h . Define h to be the identity on $(M^n - A^n) \cup (\dot{A}^n - \mathring{B}^{n-1})$. In particular h is the identity on X . Extend $h = 1 : \dot{B}^{n-1} \longrightarrow \dot{B}_1^{n-1}$ to a homeomorphism $B^{n-1} \longrightarrow B_1^{n-1}$ by Lemma 10 . Similarly extend $h : \dot{A}^n \longrightarrow (A^n \cup B^n)^{\cdot}$ to the interiors . Then h has the desired properties .

<u>Lemma 16 .</u> <u>Any homeomorphism of a ball onto itself keeping the boundary fixed is isotopic to the identity keeping the boundary fixed</u> .

Proof . It suffices to prove for a simplex . Given $h : \Delta \longrightarrow \Delta$, we construct the isotopy $f : \Delta \times I \quad \Delta \times I$ as follows . Let

$$f(x, t) = \begin{cases} hx, & t = 0 \\ x, & t = 1 \text{ or } x \in \dot{\Delta} . \end{cases}$$

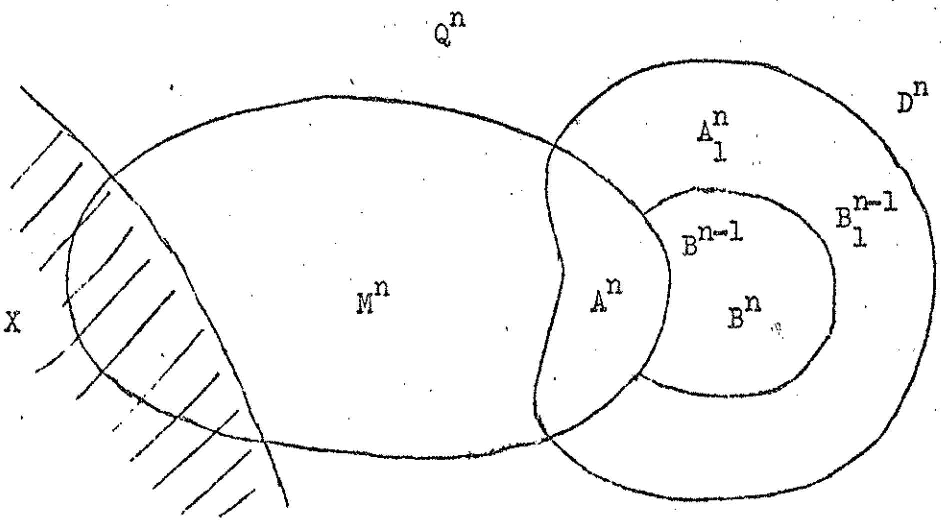
This gives f level preserving on $(\Delta \times I)^{\cdot}$. Define f level preserving on $\Delta \times I$ by mapping the centre of the prism to itself, and joining to the boundary linearly . Then f is the desired isotopy .

*Lemma 17 . Suppose $M^n \subset Q^n$ are manifolds , and that M^n is a closed subset of $\overset{\circ}{Q}{}^n$. Then $\overline{Q^n - M^n}$ is a manifold .

Proof . Let $M_1^n = \overline{Q^n - M^n}$. We have to show that every point $x \in M_1^n$ has a ball neighbourhood in M_1^n . If $x \in Q^n - M^n$, then x has a ball neighbourhood in Q^n that is contained in M_1^n , because M^n is closed in Q^n . If , on the other hand , $x \in M^n \cap M_1^n$, then $x \in \overset{\circ}{Q}{}^n$ by hypothesis , and so x lies in the interior of a ball in Q^n . Triangulate this ball so that x is a vertex , and so that its meets M^n in a subcomplex . If S^{n-1} is the link of x , then $S^{n-1} \cap M^n$ is a ball by Lemma 9 . Therefore the closure of the complement , $S^{n-1} \cap M_1^n$, is a ball by Theorem 3_{n-1} . Hence x has a ball neighbourhood in M_1^n .

*Theorem 7 . Suppose $M^n \subset Q^n$ are manifolds , and that M^n is a closed subset of $\overset{\circ}{Q}{}^n$. Let B^n be an n-ball in $\overset{\circ}{Q}{}^n$ meeting M^n in a common face . Let X be a closed subset of Q^n not meeting B^n . Then there is an ambient isotopy of Q^n moving M^n onto $M^n \cup B^n$, and keeping $X \cup \dot{Q}^n$ fixed .

Proof .



Let B^{n-1} be the common face , and let $B_1^{n-1} = \dot{B}^n - \mathring{B}^{n-1}_1$, which is a ball by Theorem 3_{n-1} . Let $M_1^n = \overline{Q^n - (M^n \cup B^n)}$, which is a manifold by Lemma 17 , since $M^n \cup B^n$ is a manifold by Theorem 6 . Let D^n be a derived neighbourhood of B^n in the manifold $Q^n - \dot{Q}^n - X$. Then D^n is a ball by Theorem 5 . Let $A^n = D^n \cap M^n$, $A_1^n = D^n \cap M_1^n$. If when constructing D^n we choose a triangulation that meets M^n , B^n in subcomplexes , this ensures that A^n , A_1^n are respectively derived neighbourhoods of B^{n-1} , B_1^{n-1} in M^n , M_1^n and therefore are balls . A^n meets B^n in the common face B^{n-1} , and A_1^n , meets B^n in the common face B_1^{n-1} . Therefore $A^n \cup B^n$, $A_1^n \cup B^n$ are balls by Corollary to Theorem 2_n .

Next we construct a homeomorphism h of D^n onto itself as follows . Define $h = 1$ on $\dot{D}^n \cup (\dot{A}^n - \mathring{B}^{n-1})$. Extend $h : \dot{B}^{n-1} \longrightarrow \dot{B}_1^{n-1}$ to the interiors by Lemma 10 . Similarly extend $\dot{A}^n \longrightarrow (A^n \cup B^n)^{\cdot}$ and $(B^n \cup A_1^n)^{\cdot} \longrightarrow \dot{A}_1^n$ to the interiors . By Lemma 16 the identity is isotopic to h , keeping $\dot{D}^n$ fixed . Extend this to an ambient isotopy of Q^n keeping fixed $Q^n - \mathring{D}^n$ (in particular $X \cup \dot{Q}^n$) . By construction this isotopy moves M^n onto $M^n \cup B^n$.

Regular neighbourhoods .

The definition of regular neighbourhood is more powerful than that of derived neighbourhood because it is intrinsic , and leads at once to an existence and uniqueness theorem .

Let X be a polyhedron in a manifold M . A regular neighbourhood N of X in M is a polyhedron such that

i) N is a neighbourhood of X in M .

ii) N is an n-manifold ($n = \dim M$)

iii) $N \searrow X$.

*Theorem 8

(1) Any derived neighbourhood of X in M is regular .

(2) Any two regular neighbourhoods of X in M are homeomorphic , keeping X fixed ,

(3) If $X \subset \mathring{M}$, then any two regular neighbourhoods of X in $\mathring{M}$ are ambient isotopic keeping $X \cup \dot{M}$ fixed .

Remark .

Clearly (3) is stronger than (2) . However it is valuable to have (2) in cases where (3) does not apply . For example suppose X is a spine of M in the interior of M ; then by (2) M is homeomorphic to any regular neighbourhood N of X in M . But obviously M and N are not ambient isotopic .

Proof of Theorem 8 .

Part (1) . Let $N = N(X, J')$ be a derived neighbourhood of X in M . We have to verify the three conditions for regularity . Condition (i) follows from the definition, and (iii) from the Corollary to Lemma 14 . To verify (ii) we check the link of each vertex $x \in N$. Let $L = \mathrm{lk}\,(x, J')$. If $x \in X$, then $\mathrm{lk}\,(x , N) = L$, which is a sphere or ball . If $x \notin X$, then $x \in \overset{\circ}{A}$, where A is a unique simplex in $J - K$, K being the subcomplex of J convering X . By the fullness of K in J , $A \cap K = B$, a face of A .

Now $L = \dot{A}'S$, where S is isomorphic to $(\mathrm{lk}\,(A,J))'$, and so is a ball or sphere . Since S lies in the interior of $\mathrm{st}\,(A,K)$ it does not meet X , and therefore $L \cap X = \dot{A}' \cap X = B'$. Therefore

$$\begin{aligned} \mathrm{lk}\,(x,N) &= N\,(B',L) \\ &= N\,(B',\dot{A}'S) \\ &= N\,(B',\dot{A}')S \end{aligned}$$

which is a ball , because $N(B',\dot{A}')$ is a ball by Theorem 5 , being a derived neighbourhood of B in $\dot{A}$. The proof of part (1) is complete .

For part (2) it suffices by Lemma 14 to show that any regular neighbourhood is homeomorphic to a derived neighbourhood , keeping X fixed . If N is the regular neighbourhood , use Theorem 4 to choose a triangulation J , K of N , X such that J collapses simplicially to K

$$J = K_r \searrow K_{r-1} \searrow \dots \searrow K_o = K$$

Let J'' be the barycentric second derived of J , and let $N_i = N\,(K_i , J'')$. Then N_o is a derived neighbourhood of X in M , and $N_r = N$. As in the proof of Theorem 5 , N_i is obtained from N_{i-1} by glueing on two balls . Neither of these balls meets X , because N_{i-1} is a neighbourhood of X , and so by Theorem 6 there is a homeomorphism $N_{i-1} \to N_i$ keeping X fixed . Composing these , we have the desired homeomorphism $N_o \to N$.

For part (3) we make the same construction as for part (2) , and instead of Lemma 14 and Theorem 6 we use Corollary 2 to Lemma 15 and Theorem 7 to show that the two neighbourhoods are ambient isotopic keeping $X \cup \dot{M}$ fixed . The proof of Theorem 8 is complete .

Proof of Theorem 3_n

At last we come to the end of our mammoth induction . We recall that in the proofs of Theorem 4 - 8 we have used Theorem 2_r , $r \leq n$ and Theorem 3_r , $r < n$, but not Theorem 3_n . We now use Theorem 8 to prove Theorem 3_n . This will make Theorem 2 - 8 and the accompanying lemmas valid for all n .

Given $B^n \subset S^n$ we have to show that $\overline{S^n - B^n}$ is a ball . Choose a homeomorphism $f : \dot{\Delta}^{n+1} \to S^n$ throwing a vertex x of Δ^{n+1} onto a point $y \in \overset{\circ}{B}{}^n$. Let $A^n = f(\mathrm{st}(x, \dot{\Delta}^{n+1}))$. Then the balls A^n , B^n are both regular neighbourhoods of y in S^n , and so by Theorem 8 Part 3 are ambient isotopic . Therefore the closures of their complements are homeomorphic . But $\overline{S^n - A^n} = f\ \Delta^n$, where Δ^n is the face of Δ^{n+1} opposite x . Hence $\overline{S^n - B^n}$ is a ball .

We conclude the chapter with some useful corollaries to Theorem 8 .

Corollary 1 . A manifold is collapsible if and only if it is a ball . For if it is collapsible , then it is a regular neighbourhood (in itself) of any point , and therefore a ball by Theorem 5 .

Corollary 2 . If $X \subset \overset{\circ}{M}$, and N , N_1 are regular neighbourhoods of X in M , such that $N_1 \subset \overset{\circ}{N}$, then $N - \overset{\circ}{N}_1 \cong \dot{N} \times I$.

Proof . Construct two derived neighbourhoods as in the proof of Lemma 14 .

$$N^* = f^{-1}[0, \varepsilon] , \quad N_1^* = f^{-1}[0 , \delta]$$

where $0 < \delta < \varepsilon < 1$. Then

$$N^* - \overset{\circ}{N}{}_1^* = f^{-1}[\delta, \varepsilon] \cong f^{-1}\varepsilon \times I = \dot{N}^* \times I .$$

Therefore the result is true for N^* , N_1^* . By Theorem 8 (2) choose a homeomorphism $h : N^* \to N$ keeping X fixed. Now hN_1^*, N_1 are both regular neighbourhoods of X in $\mathring{N}$, and so by Theorem 8 (3) we can ambient isotope hN_1^* onto N_1 keeping $\dot{N}$ fixed. Therefore

$$N - \mathring{N}_1 \cong N^* - \mathring{N}_1^* = \dot{N}^* \times I = \dot{N} \times I .$$

<u>Corollary 3</u>. <u>The combinatorial annulus theorem. If A, B are two n-balls such that $\mathring{A} \supset B$, then $A - \mathring{B} \cong S^{n-1} \times I$</u>. Proof by Corollary 2.

<u>Corollary 4</u>. <u>Suppose $X, Y \subset \mathring{M}$, and suppose X is a spine of M (i.e. $M \searrow X$). If $X \searrow Y$ or $Y \searrow X$ then Y is also a spine of M</u>.

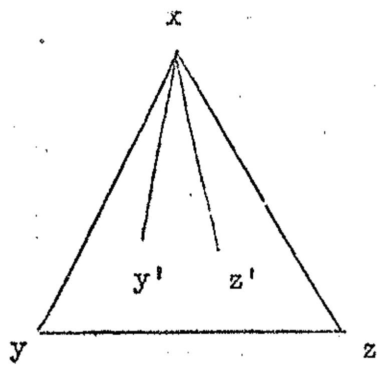
<u>Proof</u>. If $X \searrow Y$ the result is trivial, because then $M \searrow X \searrow Y$.

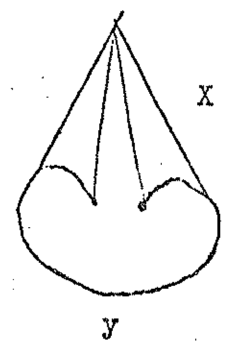
If $Y \searrow X$, let N be a regular neighbourhood of Y in $\mathring{M}$. Then $N \searrow Y \searrow X$, and so N is also a regular neighbourhood of X. By Corollary 2, $M - \mathring{N} \cong \dot{M} \times I$, and so $M \searrow N$. Therefore $M \searrow N \searrow Y$, and so Y is a spine of M.

<u>Remark 1</u>. Corollary 4 is a form of factorization of the collapsing process. However such factorization is only true for manifolds, and not true for polyhedra in general. For instance

$$\left.\begin{array}{l} X \searrow 0 \\ Y \searrow 0 \\ X \supset Y \end{array}\right\} \not\Rightarrow X \searrow Y$$

Consider the following example. Let $x\,y\,z$ be a triangle, and let y', z' be two interior points not concurrent with x. Let X be the space obtained by identifying the intervals $xy = xy'$, $xz = xz'$, and let Y be the image of $y\,z$ in X.





Then $X \searrow 0$ conewise , and $Y \searrow 0$ because Y is an arc . But $X \not\searrow Y$ because any initial elementary simplicial collapse of any triangulation of X must have its free face in Y , and so must remove part of B. Similarly can build examples to show that

$$\left.\begin{array}{r} X \searrow 0 \\ Y \searrow 0 \\ X \cap Y \searrow 0 \end{array}\right\} \not\Rightarrow X \cup Y \searrow 0$$

<u>Remark 2</u> . Corollary 4 is useful for simplifying spines . For example the spine of a bounded 3-manifold can be <u>normalised</u> in the following sense : we can find a spine , which is a 2-dimensional cell complex in which every edge bounds exactly 3 faces , and every vertex bounds exactly 4 edges and 6 faces . For choose a spine in the interior; expand each edge like a banana and collapse from one side; then expand each vertex like a pineapple and collapse from one face . By Corollary 4 any sequence of expansions and collapsesleaves us with a spine, and the process described makes it normal .

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

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Seminar on Combinatorial Topology

by E.C. ZEEMAN

Chapter 4 : UNKNOTTING BALLS AND SPHERES

Suppose $M^m \subset M^q$ are manifolds; we say the embedding is <u>proper</u> if $\dot{M}^m \subset \dot{M}^q$ and $\overset{o}{M}{}^m \subset \overset{o}{M}{}^q$. A <u>(q,m)-manifold pair</u> $M^{q,m} = (M^q, M^m)$ is a pair such that $M^m \subset M^q$ properly. The <u>codimension</u> of the pair is $c = q - m$. The <u>boundary</u> $\dot{M}^{q,m} = (\dot{M}^q, \dot{M}^m)$ is a pair of the same codimension. We write $M^{p,n} \subset M^{q,m}$ if $M^p \subset M^q$ and $M^n = M^p \cap M^m$.

In this chapter we are interested in <u>sphere pairs</u> $S^{q,m}$ and <u>ball pairs</u> $B^{q,m}$. The boundary of a ball pair is a sphere pair. If $B^{q,m} \subset \dot{M}^{q+1,m+1}$ we call $B^{q,m}$ a <u>face</u> of $M^{q+1,m+1}$.
The <u>standard</u> (q,m)-ball pair is $\Delta^{q,m} = (\sum^{q-m} \Delta^m, \Delta^m)$ where Δ^m is the standard m-simplex, and $\sum^{q-m}$ denotes (q-m)-fold suspension. The standard (q,m)-sphere pair is $\dot{\Delta}^{q+1,m+1}$. We say a sphere or ball pair is <u>unknotted</u> if it is homeomorphic to a standard pair. The cone on an unknotted (q,m) ball or sphere pair gives an unknotted (q+1,m+1) ball pair.

<u>Theorem 9</u>. <u>Any sphere or ball pair of codimension ≥ 3 is unknotted.</u>

<u>Remark 1</u>. In codimension 2 the theorem fails for both spheres and balls. The (3,1) sphere pairs give classical knot theory, and in higher dimensions knots can be tied for example by suspending and spinning (3,1) knots.

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<u>Conjecture 1</u> . <u>The sphere pair (S^q, S^{q-2}) is unknotted if $S^q - S^{q-2}$ is a homotopy S^1</u> . If $q = 3$ the result is true by a theorem of Papakyriakopoulos . If $q \geqslant 5$ an analogous topological theorem of Stallings says that if the sphere is topologically locally unknotted then it is topologically unknotted.

<u>Conjecture 2</u> . <u>Sphere and ball pairs unknot in codimension 1</u> . This is the Schönfiles conjecture which is true for $q \leqslant 3$, and unsolved for $q \geqslant 4$.

<u>Conjecture 3</u> . <u>If B is a ball pair contained in an unknotted sphere pair of the same dimension, then B is unknotted</u> . This is true for codimension $\geqslant 3$ by Theorem 9 . It is true for codimension 2 when $q = 3$ by the unique factorization of classical knot theory (an unknotted curve is not the sum of two knots) . It is true for codimension 1 when $q \leqslant 3$ by the Schönflies Theorem . But otherwise in codimensions 1 and 2 is unsolved .

A modified result is that $B \subset S$ are both unknotted then the complementary ball pair $\overline{S - B}$ is also unknotted . This proved by Theorem 8 part 3 generalised to <u>relative</u> regular neighbourhoods .

<u>Remark 2</u> . In differential theory Theorem 9 is no longer true because Haefliger has knotted S^{4k-1} differentially in S^{6k} . Above this critical dimension, in the stable range, he has unknotted all sphere pairs .

<u>Plan of the proof of Theorem 9</u> .

Most of this chapter is devoted to proving Theorem 9 . The proof is by induction on m keeping the codimension $c = q - m$ fixed . We eventually show that

$$\text{Theorem } 9_{q-1,m-1} \Longrightarrow \text{Theorem } 9_{q,m}$$

The induction starts trivially with $m = 0$; for, given c , then a$(c,0)$ ball pair is a ball B^c with an interior point B^o , which is homeomorphic to a standard pair .

Next we observe that :

<u>Unknotting of (q,m)-ball pairs implies unknotting of (q,m)-sphere pairs</u> .

<u>Proof</u> . Given $S^{q,m}=(S^q,S^m)$, triangulate the pair and choose a vertex $x \in S^m$. Let

$$B^{q,m} = (S^q - st(x,S^q) \ , \ S^m - st(x,S^m)) \quad .$$

If $\Delta^{m+1} = y \ \Delta^m$ is the standard simplex, then

$$\Delta^{q+1,m+1} = \Delta^{q,m} \cup y \ \Delta^{q,m} \quad .$$

By hypothesis choose an unknotting homeomorphism $B^{q,m} \longrightarrow \Delta^{q,m}$; then map x to y and extend linearly to an unknotting $S^{q,m} \longrightarrow \Delta^{q+1,m+1}$.

<u>Lemma 18</u> . <u>Let (B^q, B^m) and (C^q, C^m) be two unknotted ball pairs</u> . <u>Then any homeomorphisms $f : \dot{B}^q \longrightarrow \dot{C}^q$ and $g : B^m \longrightarrow C^m$ that agree on $\dot{B}^m$ can be extended to a homeomorphism $h : B^q \rightarrow C^q$</u> .

<u>Proof</u> . Extend f conewise to $\bar{f}: B^q \rightarrow C^q$ as in the proof of Lemma 10 . Let $e: C^m \rightarrow C^m$ be the composition

$$\begin{tikzcd} C^m & B^m \arrow[l, "\bar{f}"'] \arrow[r, "g"] & C^m \\ {} \arrow[rr, "e"'] & & {} \end{tikzcd}$$

Then e keeps $\dot{C}^m$ fixed , since f , g agree on $\dot{B}^m$. By the unknottedness we can suspend e to a homeomorphism $\bar{e}: C^q \longrightarrow C^q$ fixed on $\dot{C}^q$. Then $h = \bar{e}\bar{f} : B^q \rightarrow C^q$ agrees with both f and g , and proves the Lemma .

<u>Corollary</u> . <u>Any homeomorphism between the boundaries of two unknotted ball pairs can be extended to the interiors</u> .

<u>Lemma 19</u> . <u>Assume Theorem $9_{q-1,m-1}$</u> . <u>Then if two unknotted (q,m) ball pairs meet in a common face their union is a unknotted ball pair</u> .

<u>Proof</u> . Let B_1, B_2 be the ball pairs meeting in the face F . Let $\Sigma\ \Delta$ be the suspension of the standard $(q-1,m-1)$ ball pair Δ , with suspension points x_1 , x_2 say . Choose an unknotting $F \longrightarrow \Delta$ by hypothesis. Extend $\dot{F} \longrightarrow \dot{\Delta}$ to unknottings $\dot{B}_i - \overset{o}{F} \longrightarrow x_i\ \dot{\Delta}$ by the above corollary . Similarly extend $\dot{B}_i \longrightarrow (x_i\ \Delta\ )^{\cdot}$ to the interiors . Then $B_1 \cup B_2$ is unknotted by the homeomorphism onto $\Sigma\,\Delta$.

<u>Lemma 20</u> . <u>If (B^q, B^m) is a ball pair of codimension $\geqslant 3$ then $B^q \searrow B^m$</u> .

<u>Remark</u> . Lemma 20 fails in codimension 2 ; for example a knotted arc properly embedded in B^3 is not a spine of B^3 . The proof of Lemma 20 involves some geometrical construction, and we postpone it until after Lemma 23, which is the crux of the matter . First let us show how Lemma 20 implies Theorem 9 .

<u>P r o o f o f T h e o r e m 9 a s s u m i n g L e m m a 20</u>

We assume Theorem $9_{q-1,m-1}$, where $q-m \geqslant 3$. By the observation that unknotting balls implies unknotting spheres, it suffices to show that a given ball pair $B = (B^q, B^m)$ is unknotted .

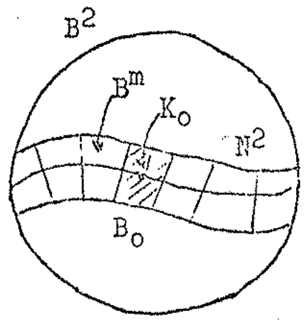
Choose a triangulation J , K of B^q, B^m such that K is simplicially collapsible

$$K = K_r \searrow K_{r-1} \searrow \dots \searrow K_o = \text{point} .$$

Let J'' be the second barycentric derived of J, and let B_i be the ball pair

$$B_i = \left(N(K_i , J'') , \; N(K_i , K'') \right) .$$

We show inductively that B_i is unknotted.



The induction starts with $i = 0$, because B_o is a cone on the ball or sphere pair $(lk(K_o,J''),lk(K_o,K''))$ which is unknotted by Theorem $9_{q-1,m-1}$. For the inductive step assume B_{i-1} unknotted. As in Theorem 5, we notice that B_i is obtained by glueing on two more small ball pairs, each by a common face, and each of which being unknotted like B_o. Hence B_i is unknotted by Lemma 19. At the end of the induction B_r is unknotted.

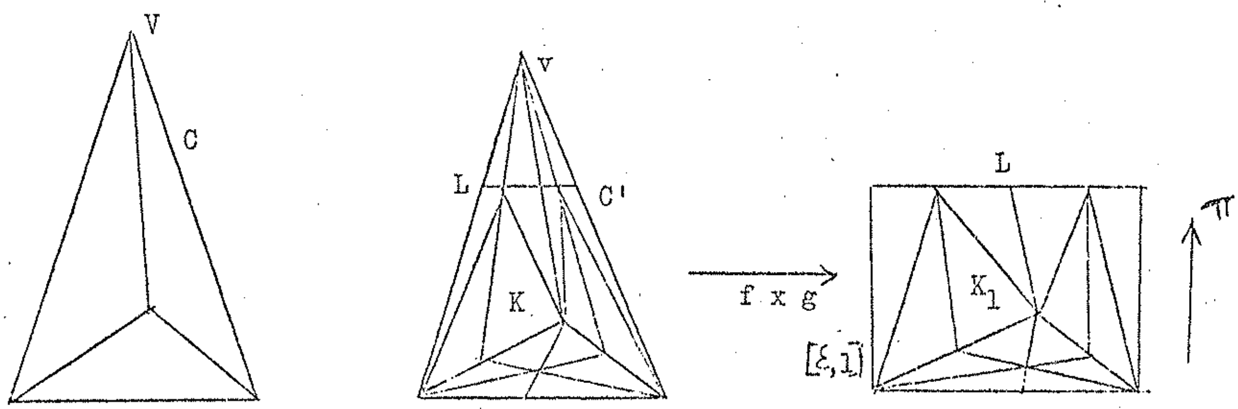
Now $B_r = (N^q,B^m)$, where N^q is a regular neighbourhood of B^m in B^q. But by Lemma 20, B^q itself is another regular neighbourhood. Therefore by Theorem 8 Part 2 there is a homeomorphism $B^q \to N^q$ keeping B^m fixed, or, in other words, a homeomorphism $B \to B_r$, showing B unknotted.

Conical subdivisions

We shall need a lemma about subdividing cones. Let $C = vX$ be a cone on a polyhedron X, with vertex v. If $Y \subset C$, the <u>subcone through Y</u> is the smallest subset of C containing Y of the form vZ, $Z \subset X$. For example a subcone through a point is a <u>generator</u> of the cone. A triangulation of C is called <u>conical</u> if the subcone through each simplex is a subcomplex.

<u>Lemma 21</u>. <u>Any triangulation of C has a conical subdivision</u>.

<u>Proof</u> . Let C also denote the given triangulation . Let $f: C \to I$ denote the piecewise linear map such that $f^{-1}(0) = v$, $f^{-1}(1) = x$, and such that f maps each generator linearly . Choose $\mathcal{E} > 0$, and such that $\mathcal{E} < fx$ for every vertex $x \in C$, $x \neq v$. Choose a first derived C' of C such that each simplex of C meeting $f^{-1}(\mathcal{E})$ is starred on $f^{-1}(\mathcal{E})$. Then $f^{-1}[\mathcal{E},1]$, $f^{-1}(\mathcal{E})$ are subcomplexes , K , L say , of C' , and $C' = K \cup vL$. Let $g : K \to L$ be radial projection , which is a projective map and not piecewise linear . Then $f \times g : K \to [\mathcal{E},1] \times L$ is a projective homeomorphism ,



that maps K projectively onto an isomorphic complex , K_1 say , triangulating $[\mathcal{E}, 1] \times L$. The projection $\pi : K_1 \to L$ onto the second factor is piecewise linear, and so there are subdivisions such that $\pi : K_1' \to L'$ is simplicial . Let $K' = (f \times g)^{-1} K_1'$. Then K' is a subdivision of K , containing L' as a subcomplex , because $\pi(f \times g) : L \to L$ is the identity . Let $C'' = K' \cup vL'$. Then C'' is a subdivision of C , and is conical because K_1' is cylindrical .

<u>S h a d o w s</u>

Let I^q be the q-cube . We single out the last coordinate for special reference and write $I^q = I^{q-1} \times I$. Intuitively we regard I as vertical , and I^{q-1} as horizontal , and identify I^{q-1} with the base of the cube . Let X be a polyhedron in I^p . Imagine the sun vertically overhead , causing X to cast a shadow; a point of I^p lies in the <u>shadow of X</u> if it is vertically below some point of X .

<u>Definition</u> . Let X^* be the closure of the set of points of X that lie in the same vertical line as some other point of X (i.e. the set of points of X that either overshadow , or else are overshadowed by , some other point of X) . Then X^* is a subpolyhedron of X .

<u>Lemma 22</u> . <u>Given a ball pair (B^q, B^m) of codimension ≥ 3 , then there is a homeomorphism $B^q, B^m \longrightarrow I^q, X$ such that</u>

i) <u>X does not meet the base of the cube</u>

ii) <u>X meets each vertical line finitely</u>

iii) <u>$\dim X^* \leq m - 2$</u> .

<u>Proof</u> . First choose the homeomorphism to satisfy i) , which is easy . Now triangulate I^q, X . Then shift all the vertices of this triangulation by arbitrary small moves into general position , in such a way that any vertex in the interior of I^p remains in the interior, and any vertex in a face of I^p remains inside that face . If the moves are sufficiently small, the new positions of vertices determine an isomorphic triangulation , and a homeomorphism of I^p onto itself . The general position ensures that conditions (ii) and (iii) are satisfied , because $m \leq q - 3$,

$$\text{and so} \quad \dim X^* \leq (m + 1) + m - q \leq m - 2 \quad .$$

<u>Remark</u> . The "general position" of the above proof may be analysed more rigorously as follows . Each vertex is in the interior of some face , and has coordinates in that face . The set of all such coordinates of all vertices determine a point $\mathcal{X}$ some high dimensional euclidean space , and

the sufficient smallness means that x is permitted to vary in an open set, U say. To satisfy the conditions (ii) and (iii) we merely have to choose $x \in U$, so as to avoid a certain finite set of proper linear subspaces.

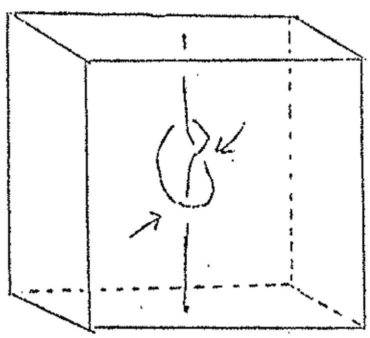
Sunny collapsing

Suppose we are given polyhedra $I^q \supset X \supset Y$, such that $X \searrow Y$ is an elementary collapse. We call this collapse sunny if no point of $X - Y$ lies in the shadow of X. We call a sequence of elementary sunny collapses a sunny collapse, and if Y is a point we call X sunny collapsible.

Corollary to Theorem 4. If X is sunny collapsible then some triangulation is simplicially sunny collapsible. For each elementary sunny collapse is factored in a sequence of elementary simplicial sunny collapses.

Lemma 23. If (I^q, X) is (q,m)-ball pair of codimension $\geqslant 3$ satisfying the conditions of Lemma 22 then X is sunny collapsible.

Remark. Lemma 23 fails with codimension 2. The classical example of a knotted arc in I^3 gives a good intuitive feeling for the obstruction to a sunny collapse: looking down from above it is possible to start collapsing away until we hit underpasses, which are in shadow and so prevent any further progress.



Definition. A principal k-complex is a complex in which every principal simplex is k-dimensional.

Proof of Lemma 23 .

We shall construct inductively a decreasing sequence of subpolyhedra

$$X = X_o \supset X_1 \supset \ldots \supset X_m = \text{a point} ,$$

and , for each i , a homeomorphism

$$f_i : X_i \longrightarrow v\, K^{m-i-1}$$

onto a cone on a principal $(m-i-1)$-complex, satisfying the three conditions :

1) $f_i X_i^*$ does not contain the vertex of the cone , and meets each generator of the cone finitely .

2) $\dim X_i^* \leqslant m-i-2$.

3) There is a sunny collapse $X_{i-1} \searrow X_i$.

The induction starts with $X = X_o$.

Condition (2) is by hypothesis and (3) is vacuous . Choose a homeomorphism $f_o : X_o \longrightarrow \Delta$, where Δ is the standard m-simplex . Since $f_o X^*$ is a sub-polyhedron of dimension $\leqslant m-2$, we can choose a point $v \in \mathring{\Delta} - f_o X^*$, and in general position relative to $f_o X^*$. Starring Δ at v makes Δ into the cone $v\ \dot{\Delta}$ on $\dot{\Delta}$, which is principal . Condition (2) is satisfied by our choice of v .

The induction finishes with X_m = a point , and so we shall have a sunny collapse

$$X \searrow X_1 \searrow X_2 \searrow \ldots \searrow X_m$$

which will prove the lemma .

The hard part is the inductive step .

Suppose we are given $f_{i-1} : X_{i-1} \longrightarrow v\, K^{m-i}$, satisfying the three conditions, we have to construct f_i , X_i , K^{m-i-1} and prove the three conditions .

Let C , L triangulate $v\,K^{m-i}$, $f_{i-1}\,X^*_{i-1}$. By Lemma 21 we can choose C to be conical . In particular C contains a subdivision $(K^{m-i})'$ of K^{m-i} . Define

K^{m-i-1} = the (m-i-1)-skeleton of $(K^{m-i})'$, which is a principal complex since K^{m-i} was principal . Let C_o be the subcomplex of C triangulating the subcone $v\,K^{m-i-1}$. Let $e_o : C_o \to C$ be the inclusion map . We shall construct another embedding

$$e : C_o \longrightarrow C$$

that differs slightly , but significantly , from e_o . Having chosen e , then there is a unique subpolyhedron X_i , and homeomorphism f_i , such that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\subset"] \arrow[d, "f_i"] & X_{i-1} \arrow[d, "f_{i-1}"] \\ C_o \arrow[r, "e"] & C \end{tikzcd}$$

is commutative .

It is no good choosing $e = e_o$, because then $X^*_i = X^*_{i-1}$ which would be of too high a dimension . In fact this is the crux of the matter : we must arrange some device for collapsing away the top-dimensional shadows of X^*_{i-1} . The first thing to observe is that the triangulation L of $f_{i-1}\,X^*_{i-1}$ is in no way related to the embedding of X^*_{i-1} in the cube I^q . The inverse images of simplexes of L may wrap around and overshadow each other hopelessly, so our next task is to take a subdivision that remedies this confusion . We have piecewise linear maps

$$L \xrightarrow{f_{i-1}^{-1}} X_{i-1}^{*} \xrightarrow{\pi} I^{q-1}$$

where the first is a homeomorphism , and π is vertical projection onto the base of the cube I^q . By Theorem 1 we can choose subdivisions L' of L , $(I^{q-1})'$ of I^{q-1} and a triangulation M of X_{i-1}^{*} such that the maps

$$L' \xleftarrow{f_{i-1}} M \xrightarrow{\pi} (I^{q-1})'$$

are simplicial .

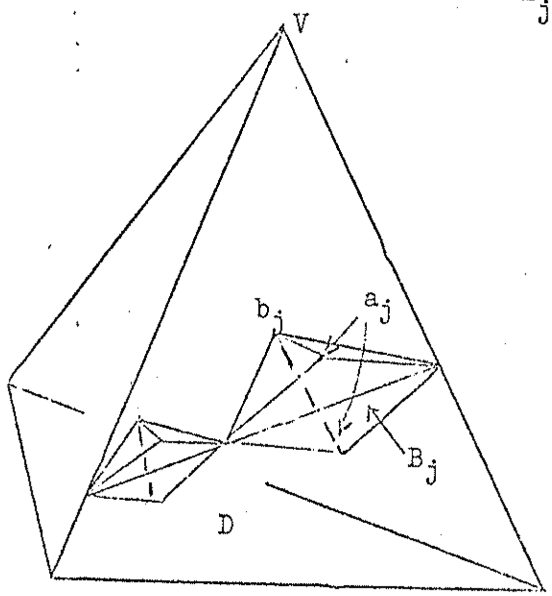
Recall that $\dim M \leqslant m-i-1$, by induction on i . Let A_1 $A_1, A_2, \ldots, A_r$ be the $(m-i-1)$-simplexes of M . Each A_j is projected non-degenerately by π , because of Lemma 22 (ii) . If $j \neq k$ there are two possibilities : either $\pi A_j \neq \pi A_k$ or $\pi A_j = \pi A_k$. In the first case π maps $\mathring{A}_j$, $\mathring{A}_k$ disjointly and so no point of $\mathring{A}_j \cup \mathring{A}_k$ overshadows any other . In the second case either $\mathring{A}_j$ overshadows $\mathring{A}_k$ or vice versa . Consequently overshadowing induces a partial ordering amongst the A'_s , and we choose the ordering $A_1, A_2, \ldots, A_r$ to be compatible with this partial ordering . We summarise the conclusion :

<u>Sublemma</u> . <u>All points of X that overshadow $\mathring{A}_k$ are contained in</u> $\bigcup_{j<k} \mathring{A}_j$

We now pass to L' . Let $B_j = f_{i-1} A_j \in L'$. The next step in the proof is to construct a little $(m-i+1)$-dimensional blister Z_j about each B_j in the cone C . The blisters are the device that enable us to make the sunny collapse, and the fact that there is just sufficient room to construct them is an indication of why codimension $\geqslant 3$ is a necessary and sufficient condition for unknotting .

Fix j. Let $\hat{B}_j$ be the barycentre of B_j. Since the base of the cone is principal, there is simplex $D^{m-i} \in (K^{m-i})'$ such that B_j is contained in the subcone $v\dot{D}$. There are two cases depending on whether or not B_j lies in $\dot{D}$, in the base of the cone. If $B_j \subset \dot{D}$, let b_j be a point in $\overset{o}{D}$ near $\hat{B}_j$, and let a_j be a point in the generator vB_j near B_j. If $B_j \not\subset \dot{D}$, let b_j be a point in $v\overset{o}{D}$ near $\hat{B}_j$, and let a_j be a pair of points on the generator through $\hat{B}_j$, near $\hat{B}_j$ and either side of $\hat{B}_j$. In either case define the blister

$$Z_j = a_j b_j B_j .$$



We choose the points sufficiently near to the barycentres so that no two blisters meet more than is necessary (i.e. $Z_j \cap Z_k = \dot{B}_j \cap \dot{B}_k$). The bottom of the blister is $a_j B_j$ and the top is $a_j b_j \dot{B}_j$. Let e_j ne the map

$$e_j : a_j B_j \longrightarrow a_j b_j \dot{B}_j$$

that raises the blister, and is given by mapping $\hat{B}_j \longrightarrow b_j$.

Now C_o meets each blister in its bottom. Therefore we can define the embedding $e : C_o \rightarrow C$ by choosing $e = e_j$ on the intersection with each blister, and $e = 1$ otherwise. In other words e is a map that raises all the blisters. Having defined e, we have completed the definition of X_i and $f_i : X_i \longrightarrow v\,K^{m-i-1}$.

There remains to verify the three conditions. Condition (2) holds because by our construction $X_i^* = X_{i-1} \cap X_i$

$$= \text{the } (m-i-2)\text{-skeleton of } M .$$

Condition (1) holds , because

$$f_i X_i^* = f_{i-1} X_i^*$$

$$\subset f_{i-1} \quad X_{i-1}^* \quad ,$$

for which the condition holds by induction .

Finally we come to condition (3) . Let $Z = \bigcup Z_j$. For each $(m-i)$-simplex $D \in (K^{m-i})'$, $\overline{v\,D - Z}$ is a ball because it is a simplex with a few blisters pushed in round the edge, and $\overline{D - Z}$ is a face . Therefore collapse each $v\,\overline{D - Z}$ from $\overline{D - Z}$. We have collapsed

$$C \searrow e\,C_o \cup Z \quad .$$

and the inverse image under f_{i-1} determines a sunny collapse

$$X_{i-1} \searrow X_i \cup f_{i-1}^{-1} Z \quad ,$$

sunny because we have not yet removed any point of X_{i-1}^* .

We now collapse the blisters as follows . Each blister meets $e\,C_o$ in its top , and so by collapsing each blister onto its top in turn, $j=1,\ldots,r$, we effect a collapse

$$e\,C_o \cup Z \searrow e\,C_o \quad .$$

If $Y_j = f_{i-1}^{-1} Z_j$, then the inverse image of this collapse determines a sequence of elementary collapses

$$X_i \cup \bigcup_1^r Y_j \searrow X_i \cup \bigcup_2^r Y_j \searrow \ldots \searrow X_i \quad .$$

Each of these elementary collapses is sunny by the Sublemma, because by the time we come to collapse Y_k , say , the only points that might have been in shadow are those in $\overset{o}{A}_k$, but these are sunny for we have already removed everything that overshadows them . We have demonstrated the sunny collapse $X_{i-1} \searrow X_i$, which completes the proof of Lemma 23 .

Proof of Lemma 20 .

We can now return to the proof of Lemma 20 , which will conclude the proof of Theorem 9 . Given a ball pair (B^q, B^m) of codimension $\geqslant 3$, we have to show $B^q \searrow B^m$. By Lemma 22 it suffices to show for a ball pair (I^q, X) satisfying the conditions of Lemma 22 .

By Lemma 23 and the Corollary to Theorem 4 we can choose a triangulation K of X that is simplicially sunny collapsible .

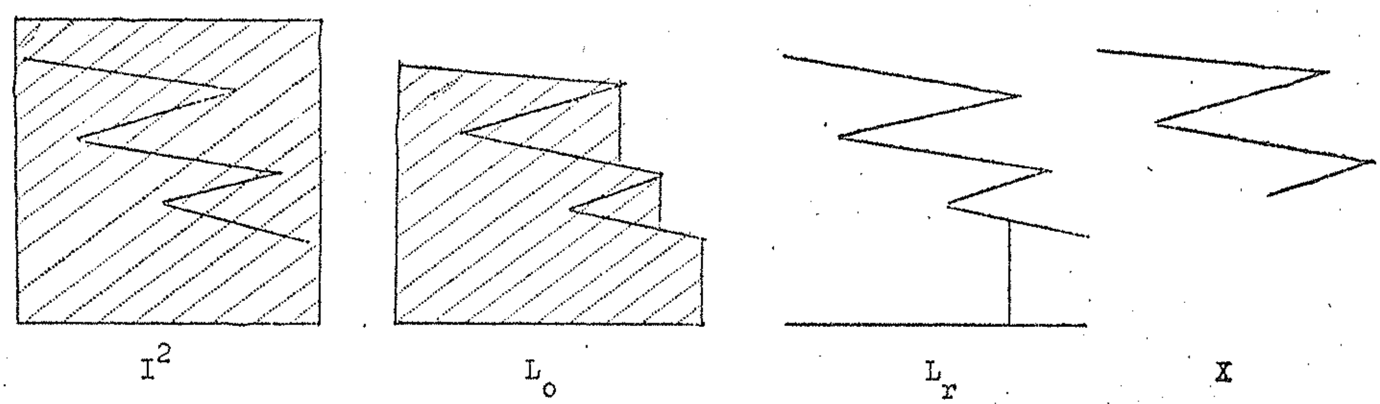
$$K = K_0 \searrow K_1 \searrow \ldots \searrow K_r = \text{a point} .$$

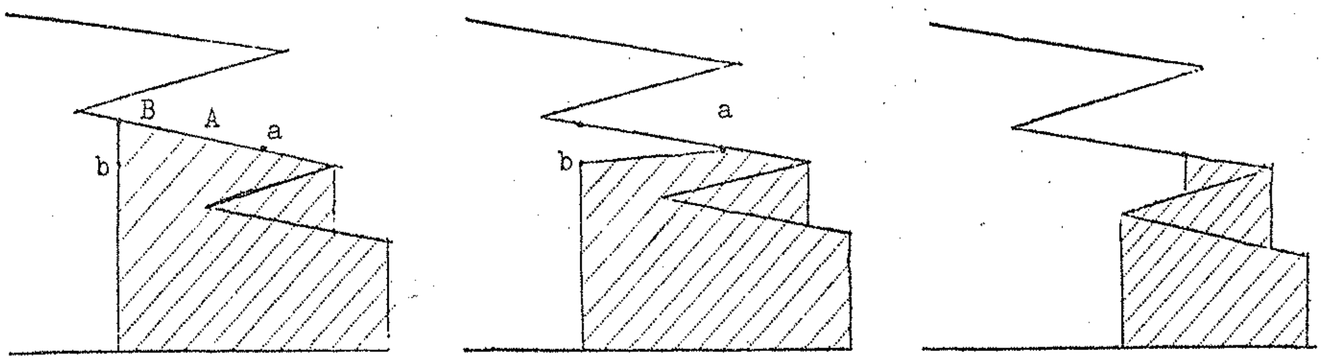
Let L_i be the polyhedron consisting of $I^{q-1} \cup X$ together with all points in the shadow of K_i . We shall show that

$$I^q \searrow L_0 \searrow L_1 \searrow \ldots \searrow L_r \searrow X .$$

The first step is as follows . Choose a cylindrical subdivision $(I^q)'$ of I^q containing a subdivision L_0' of L . Then collapse $(I^q)' \searrow L_0'$ prismwise from the top, in order of decreasing dimension of the prisms .

The last step is easy , because L_r consists of $I^{q-1} \cup X$ joined by a single arc . Collapse I^{q-1} onto the bottom of this arc, and then collapse the arc . There remain the intermediate steps $L_{i-1} \searrow L_i$, $i = 1 , \ldots , r$.





L_{i-1} $\quad$ T $\quad$ L_i

Fix i, and suppose the elementary simplicial sunny collapse $K_{i-1} \searrow K_i$ collapses A from B, when $A = aB$. Choose a point b below the barycentre $\hat{B}$ of B, and sufficiently close to $\hat{B}$ for $bA \cap X = A$ (this is possibly by Lemma 22 (ii)). Let $T = L_i$ together with all points in the shadow of $ab\dot{B}$. Since the collapse is sunny, no points of K_i overshadow $\mathring{A} \cup \mathring{B}$, and so $T \cap bA = a(bB)^{\cdot}$, $T \cup bA = L_{i-1}$. In other words collapsing bA from bB gives an elementary collapse

$$L_{i-1} \searrow T .$$

Finally collapse $T \searrow L_i$ prismwise downwards from $ab\dot{B}$, as in the first case. This completes the proof of Lemma 20 and Theorem 9.

Isotopies of balls and spheres

Recall that Lemma 16 proved that any homeomorphism of a ball onto itself keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.

Corollary 1 to Theorem 9 . *If $q-m \geqslant 3$, then any two proper embeddings $B^m \subset B^q$ that agree on $\dot{B}^m$ are ambient isotopic keeping $\dot{B}^q$ fixed.*

Proof . Let f, g be the embeddings . By Lemma 18 we can extend $1 : \dot{B}^q \to \dot{B}^q$ and $gf^{-1} : f B^m \to g B^m$, which agree on $\dot{B}^m$, to a homeomorphism between the ball pairs

$$h : (B^q, f B^m) \longrightarrow (B^q, g B^m) .$$

By construction $hf = g$, and, by Lemma 16, h is ambient isotopic to the identity keeping $\dot{B}^q$ fixed .

Theorem 10 . *Any orientation preserving homeomorphism of S^n is isotopic to the identity .*

Proof : by induction on n , starting trivially with $n = 0$. Let f be the given homeomorphism . Choose a point $x \in S^n$, and ambient isotopic fx to x . This moves f to f_1 , say , where $f_1 x = x$. Choose a ball B containing x in its interior . Then $B, f_1 B$ are regular neighbourhoods of x , and so by Theorem 8 ambient isotope $f_1 B$ onto B . This moves f_1 to f_2 , say , where $f_2 B = B$. The restriction $f_2 | \dot{B}$ preserves orientation , and is therefore isotopic to the identity by induction . Extend the isotopy conewise to B and $\overline{S^n - B}$, making it into an ambient isotopy , that moves f_2 to f_3 , say where $f_3 | \dot{B} = 1$. Apply Lemma 16 to each of B , $\overline{S^n - B}$ to ambient isotope f_3 into the identity .

Corollary 2 to Theorem 9 . *If $q-m \geqslant 3$, then any two embeddings $S^m \subset S^q$ are ambient isotopic .*

Proof . If (S^q, S^m), $q > m$, is an unknotted sphere pair, then S^q is the $(q-m)$-fold suspension of S^m , and so there is

(1) an orientation reversing homeomorphism of S^q, throwing S^m

onto itself, and

(2) an orientation preserving homeomorphism of S^q , throwing S^m onto itself with reversed orientation .

Let f , $g : S^m \to S^q$ be the two given embeddings . Since $q - m \geqslant 3$, the unknotting gives a homeomorphism $(S^q, fS^m) \to (S^q, gS^m)$, which we can choose to be orientation preserving on S^2 by (1) , and which is therefore isotopic to the identity by Theorem 10 . Therefore f is ambient isotopic to f_1 , say , such that $f_1 S^m = gS^m$. Let $h = gf_1^{-1} : f_1 S^m \to f_1 S^m$. By (2) above and Theorem 10 , we can choose f_1 so that h is orientation preserving .

Now apply Theorem 10 to the smaller sphere S^m , to obtain an isotopy from the identity to h ; suspend this isotopy of S^m into an ambient isotopy of S^q moving f_1 into g .

Remark . The above two corollaries are also true for unknotted ball and sphere pairs of codimension 1 and 2 . The aim of the next four chapters is to obtain similar results for arbitrary manifolds .

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Seminar on Combinatorial Topology

by E.C. ZEEMAN

Chapter 5 : ISOTOPY

The natural way to classify embeddings of one manifold in another is by means of isotopy. But there are several definitions of isotopy, and the purpose of this chapter is to prove three of the definitions equivalent. The three that we consider (of which the first two were mentioned in Chapter 2) are :

(1) Isotopy, sliding the smaller manifold in the larger through a family of embeddings ;

(2) Ambient isotopy, rotating the larger manifold on itself, carrying the smaller with it ;

(3) Isotopy by moves, making a finite number of local moves, each inside a ball in the larger manifold, analogous to moving a complex in Euclidean space by shifting the vertices, like the moves of classical knot theory .

Since any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity it follows at once that

$$\text{isotopy by moves} \Longrightarrow \text{ambient isotopy} \Longrightarrow \text{isotopy} .$$

In Theorems 11 and 12 we shall show that these arrows can be reversed . To reverse the second arrow, that is to cover an isotopy by an ambient isotopy, it is necessary to impose a local unknottedness condition on the isotopy .

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For otherwise the knots of classical knot theory give counterexamples of embeddings that are mutually isotopic but not ambient isotopic. However the results of Chapter 4 show that this phenomenon occurs only in codimension 2, and possibly codimension 1 .

Throughout this chapter we shall be considering embeddings of a compact m-manifold M in a q-manifold Q , which may or may not be compact . We restrict attention to proper embeddings $f : M \to Q$; recall that f is proper provided $f^{-1}\dot{Q} = \dot{M}$; in particular if M is closed , then any embedding of M in $\mathring{Q}$ is proper . By a homeomorphism of Q , we mean a homeomorphism of Q onto itself ; in particular a homeomorphism is a proper embedding .

Definitions of Isotopy

Recall definitions that have been given in previous chapters .

(1) A homeomorphism h of M is a homeomorphism of M onto itself . If $Y \subset M$ and $h \mid Y$ = the identity we say h keeps Y fixed .

(2) An isotopy of M in Q is a proper level preserving embedding

$$F : M \times I \to Q \times I \ .$$

Denote by F_t the proper embedding $M \to Q$ defined by $F(x,t) = (F_t x, t)$, all $x \in M$. The subspace $\bigcup_{t \in I} F_t M$ of Q is called the track left by the isotopy .

If $X \subset M$, we say F keeps X fixed if $F(x,t) = F(x,0)$, all $x \in X$ and $t \in I$.

(3) The embeddings f , $g : M \to Q$ are isotopic if there exists an isotopy F of M in Q with $F_0 = f$, $F_1 = g$.

(4) An <u>ambient isotopy</u> of Q is a level preserving homeomorphism $H : Q \times I \to Q \times I$ such that $H_0 =$ the identity, where as above H_t is defined by $H(x,t) = (H_t x, t)$, all $x \in Q$. We say that H <u>covers</u> the isotopy F if the diagram

$$\begin{tikzcd} Q \times I \arrow[rd, "H"] & \\ & Q \times I \\ M \times I \arrow[uu, "F_0 \times I"] \arrow[ru, "F"'] & \end{tikzcd}$$

is commutative; in other words $F_t = H_t F_0$, all $t \in I$.

(5) The embeddings $f, g : M \to Q$ are <u>ambient isotopic</u> if there is an ambient isotopy H of Q such that $H_1 f = g$.

<u>Remark</u>. If $M = Q$, then a proper embedding $M \times I \to Q \times I$ is the same as a homeomorphism $Q \times I \to Q \times I$. Therefore since we have restricted attention to proper embeddings, the only difference between an isotopy of Q in Q, and an ambient isotopy of Q, is that the latter has to start with the identity; consequently two homeomorphisms of Q are isotopic if and only if they are ambient isotopic.

(6) A homeomorphism or ambient isotopy of Q is said to be <u>supported</u> by X if it keeps $Q - X$ fixed. By continuity the frontier $\overline{X} \cap \overline{Q - X}$ of X in Q must also be kept fixed.

(7) An <u>interior move</u> of Q is a homeomorphism of Q supported by a ball keeping the boundary of the ball fixed. A <u>boundary move</u> of Q is a

homeomorphism of Q supported by a ball that meets $\dot{Q}$ in a face ; the complementary face is the frontier of the ball that is kept fixed by continuity.

(8) The embeddings f, g are isotopic by moves if there is a finite sequence $h_1, h_2, \ldots, h_n$ of moves of Q such that $h_1 h_2 \ldots h_n f = g$.

Locally unknotted embeddings

Let $f : M \to Q$ be a proper embedding . Let Q_o be a regular neighbourhood of fM in Q . Let K, L be triangulations of M, Q_o such that $f : K \to L$ is simplicial . We say that f is a <u>locally unknotted embedding</u> if, for each vertex $v \in K$, the pair

$$\left(lk\,(fv, L), f(lk(v,K) \right)$$

is unknotted . Notice that since the embedding is proper, the pair is either a sphere or ball pair according to whether $v \in \overset{o}{M}$ or $v \in \dot{M}$.

<u>Corollary 3 to Lemma 9</u> . <u>Any proper embedding of codimension ≥ 3 is locally unknotted</u> . Therefore then we say "locally unknotted" in future we refer only to the cases of codimension 1 or 2 .

<u>Remark 1</u> . The definition is independent of Q_o , and the triangulations, because if all the links are unknotted, then the same is true for any subdivisions of K , L , and hence also true for any other triangulations .

<u>Remark 2</u> . An equivalent condition is to say that the closed stars of vertices are unknotted ball pairs, but in codimensions 1 and 2 the equivalence, for a boundary vertex , depends upon a result that we have

quoted, but not proved, that if an unknotted ball pair has an unknotted face then the complementary face is also unknotted .

Remark 3 . If $f : M \to Q$ is locally unknotted embedding, then so is the restriction to the boundaries $f : \dot{M} \to \dot{Q}$.

Remark 4 . We say a ball pair (B^q, B^m) is locally unknotted if the inclusion is so ; for example this always happens in codimension $\geqslant 3$ or in the classical case $(q,m) = (3,1)$. Suppose (B^q, B^m) is locally unknotted , and let N^q be a regular neighbourhood of B^m in B^q . Then although (B^q, B^m) may be (globally) knotted, it can be shown that (N^q, B^m) is unknotted , by adapting the proofs of Lemma 19 and Theorem 9 .

Locally unknotted isotopies

We say an isotopy $F : M \times I \to Q \times I$ is locally unknotted if

(i) each level $F_t : M \to Q$ is a locally unknotted embedding , and

(ii) for each subinterval $J \subset I$, the restriction $F : M \times J \to Q \times J$ is a locally unknotted embedding .

Remark 1 . If F is a locally unknotted isotopy, then so is the restriction to the boundaries $F : \dot{M} \times I \to \dot{Q} \times I$. The proof is non-trivial (as in Remark 2 above) and is omitted . As we need to use the fact in Corollary 1 to Theorem 12 below , we should either accept it without proof , or else add it as an additional condition in the definition of locally unknotted isotopy .

Remark 2 . Any isotopy of codimension $\geqslant 3$ is locally unknotted .

Remark 3 . The above definition is tailored to our needs . There is an alternative definition as follows ; we say an isotopy is locally *trivial* if, for each $(x,t) \in M \times I$, there exists an m-ball neighbourhood A of x in M, and an interval neighbourhood J of t in I, and a commutative diagram

$$\begin{tikzcd}
A \times J \arrow[r, "\subset"] \arrow[d, "\subset"] & \Sigma A \times J \arrow[d, "G"] \\
M \times I \arrow[r, "F"] & Q \times I
\end{tikzcd}$$

where Σ denotes (q-m)-fold suspension, and G is a level preserving embedding onto a neighbourhood of $F(x,t)$. It is easy to verify that

$$\begin{array}{c}
F \text{ is a locally trivial isotopy} \\
\Downarrow \\
F \text{ is a locally unknotted isotopy} \\
\Downarrow \\
F \text{ is an isotopy and a locally unknotted embedding.}
\end{array}$$

We shall prove in Corollary to Theorem 12 that the top arrow can be reversed. Therefore a locally trivial isotopy is the same as a locally unknotted isotopy. We conjecture the bottom arrow can also be reversed - it is a problem depending upon the Schönflies problem, and the unique factorisation of sphere knots.

We now state the theorems, and then prove them in the order stated.

Theorem 11. Let H be an ambient isotopy of Q with compact support keeping Y fixed. Then H_1 can be expressed as the product of a finite number of moves keeping Y fixed.

Addendum. Given any triangulation of a neighbourhood of X, then the moves can be chosen to be supported by the vertex stars. Therefore the moves can be made arbitrarily small.

Corollary. Let M be compact, let $f : M \to Q$ be a proper locally unknotted embedding, and let g be a homeomorphism of M that is isotopic to the identity keeping $\dot{M}$ fixed. Then g can be covered by a homeomorphism h of Q keeping $\dot{Q}$ fixed; in other words the diagram is commutative:

$$\begin{tikzcd} Q \arrow[r, "h"] & Q \\ M \arrow[u, "f"] \arrow[r, "g"] & M \arrow[u, "f"'] \end{tikzcd}$$

Remark.

In fact the corollary is improved by Theorem 12 below, to the extent of covering not only the homeomorphism but the whole isotopy. However we need to use the corollary in the proof of Theorem 14, in the course of proving Theorem 12.

Theorem 12. (Covering isotopy theorem).

Let $F : M \times I \to Q \times I$ be a locally unknotted isotopy keeping $\dot{M}$ fixed, and let N be a neighbourhood of the track left by the isotopy. Then F can be covered by an ambient isotopy supported by N keeping $\dot{Q}$ fixed.

Addendum. Let X be a compact subset of $\dot{Q}$ and N a neighbourhood of X in Q. Then an ambient isotopy of $\dot{Q}$ supported by X can be extended to an ambient isotopy of Q supported by N.

Corollary 1 . Theorem 2 remains true if we omit "keeping $\dot{M}$ fixed" from the hypothesis and "keeping $\dot{Q}$ fixed" from the thesis .

Corollary 2 . Let $f, g : M \to Q$ be two proper locally unknotted embeddings . Then the following four conditions are equivalent :

(1) f, g are isotopic by a locally unknotted isotopy

(2) f, g are ambient isotopic

(3) f, g are ambient isotopic by an ambient isotopy with compact support

(4) f, g are isotopic by moves .

Corollary 3 . An isotopy is locally trivial if and only if it is locally unknotted .

Proof of Theorem 11

We are given an ambient isotopy $H : Q \times I \to Q \times I$ with compact support, and have to show that H_1 is a composition of moves . We first prove the theorem for the case when Q is a combinatorial manifold , namely a simplicial complex in E^n, say . Then $Q \times I$ is a cell complex in $E^n \times I$. We regard E^n as horizontal and I as vertical.

Let K , L be subdivisions of $Q \times I$ such that $H : K \to L$ is simplicial (in fact a simplicial isomorphism) . Let A be a principal simplex of L , and B a vertical line element in A . Define $\theta(A)$ to be the angle between $H^{-1}(B)$ and the vertical . Since $H : K \to L$ is simplicial, this does not depend upon the choice of B . Since H is level preserving , $\theta(A) < \frac{\pi}{2}$. Define $\theta = \max \theta(A)$, the maximum taken over all principal simplexes of L . Then $\theta < \frac{\pi}{2}$.

Now let W denote the set of all linear maps $Q \longrightarrow I$ (i.e. maps that map each simplex of Q linearly into I) . Let

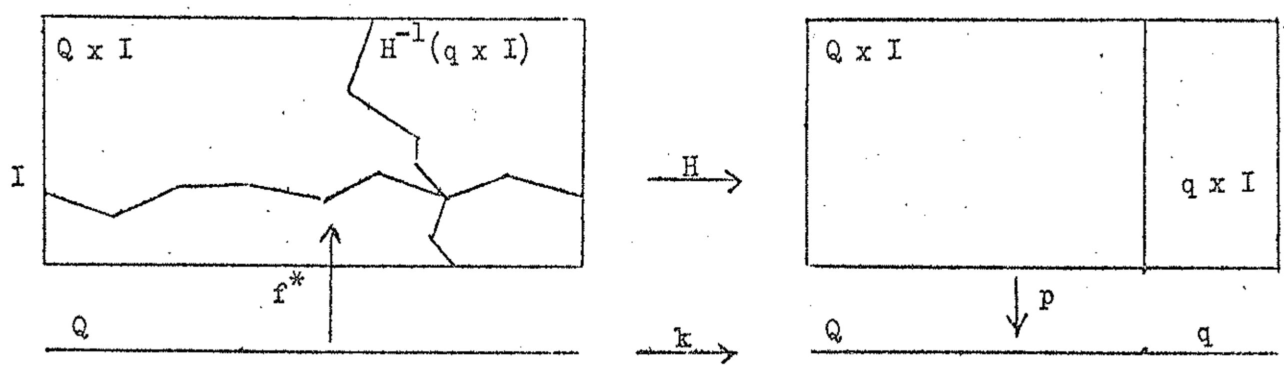
$$W_\delta = \left\{ f \in W \; ; \; \max f = \min f < \delta \right\}.$$

If $f \in W$, denote by f^* the graph of f , given by

$$f^* = 1 \times f : Q \longrightarrow Q \times I \quad .$$

Then f^* maps each simplex of Q linearly into $E^n \times I$. Let $\varphi(f)$ be the maximum angle that any simplex of $f^* Q$ makes with the horizontal . Given $\varepsilon > 0$, there exists $\delta > 0$, such that if $f \in W_\delta$, then $\varphi(f) < \varepsilon$, for choose δ sufficiently small compared with the 1-simplexes of Q . Choose $\varepsilon < \frac{\pi}{2} - \theta$, and choose δ accordingly .

Now let f be a map in W_δ, and q a point of Q . Consider the intersections of the arc $H^{-1}(q \times I)$ with $f^* Q$; we claim there is exactly one intersection .



For since f^* is a graph , $f^* Q$ separates the complement $(Q \times I) - f^* Q$ into points above and below the graph . If there were no intersection , then the arc would connect the below-point $H^{-1}(q,0)$ to the above-point $H^{-1}(q,1)$, contradicting their separation . At each intersection, since $\varphi(f) + \theta < \frac{\pi}{2}$

the arc , oriented by I , passes from below to above . Hence there can be at most one intersection .

Let $p : Q \times I \longrightarrow Q$ denote the projection onto the first factor . Then

$$k = p\,H\,f^{*} : Q \longrightarrow Q$$

is a 1 - 1 map by the above claim , and so is a (piecewise linear) homeomorphism of Q .

By the compactness of Q and I , choose a sequence of maps $f_0, f_1, \ldots, f_n$ in W_δ , such that $f_0(Q) = 0$, $f_n(Q) = 1$, and for each , f_{i-1} and f_i agree on all but one , v_i say , of the vertices of Q . Define $k_i = p\,H\,f_i^{*}$. Then $k_0 = H_0 =$ the identity, and $k_n = H_1$. Define $h_i = k_i k_{i-1}^{-1}$. Then h_i is a homeomorphism of Q supported by the ball $k_i\,(\overline{st}(v_i, Q))$, keeping $k_i\,(lk(v_i, Q))$ fixed , and so is a move . Therefore $H_1 = h_n h_{n-1} \ldots h_1$, a composition of moves .

If H keeps Y fixed , then $k_i \mid Y = k_0 \mid Y$, for each i , and so each move h_i keeps Y fixed . In particular the moves keep $Q - X$ fixed , and are supported by X .

Suppose now that Q is a compact manifold ; let $T \rightarrow Q$ be a triangulation in the structure . We have proved the theorem for T and so it also follows for Q .

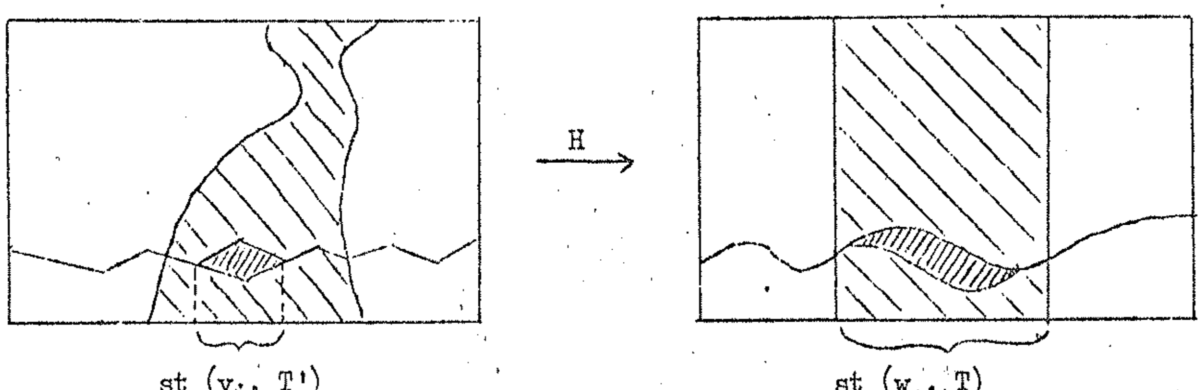
Suppose now that Q is non-compact . Let N be a regular neighbourhood of X in Q . Then N is a compact submanifold , and $N \cap (\overline{Q - N}) \subset Y$. Therefore $H \mid N \times I$ is an ambient isotopy of N keeping $N \cap Y$ fixed , and by the compact case $H_1 \mid N$ is a composition of moves supported by X keeping $N \cap Y$ fixed . The moves can be extended by the identity to moves of Q keeping Y fixed , and so H_1 is composition of moves of Q . The proof of Theorem 11 is complete .

?roof of the Addendum to Theorem 11

We are given a triangulation $T \to N$ of a neighbourhood of X, and have to show that the moves chosen to be supported by the vertex stars of T. Without loss of generality we can assume N is a regular neighbourhood, because any neighbourhood contains a regular neighbourhood. Therefore T is a combinatorial manifold. Let β denote the open covering of $N \times I$:

$$\beta = \{ st(w, T) \times I ; w \in T \},$$

where w runs over the vertices of T. Let λ be the Lebesgue number of the covering $H^{-1}\beta$ of $N \times I$. Choose a subdivision T' of T such that the mesh of the star covering of T' is less than $\lambda/2$. In the above proof of Theorem 11 use T' instead of Q, and choose δ with the additional restriction that $\delta < \lambda/2$.



Continuing with the same notation as in the proof of Theorem 11, for each i the ball $f_i^*(st(v_i, T'))$, is of diameter less than λ, and so is contained in $H^{-1}(st(w_i, T) \times I)$ for some vertex $w_i \in T$. Therefore

$$\text{support } h_i \subset k_i\,(\overline{st}(v_i, T')) \subset st(w_i, T)$$

as desired.

Proof of the Corollary to Theorem 11 .

Given $M \xrightarrow{g} M \xrightarrow{f} Q$, where g is isotopic to the identity keeping $\dot{M}$ fixed , we leave to cover g by a homeomorphism h of Q .

Let N be a regular neighbourhood of fM in Q , and choose triangulations of M , N - call them by the same names - such that $f : M \to N$ is simplicial . By the Addendum we can write

$$g = g_1 g_2 \dots g_n \ ,$$

where g_i is supported by the ball $B_i^m = \overline{st}(v_i, M)$, for some vertex $v_i \in M$. Let $B_i^q = \overline{st}(fv_i, Q)$. Then the ball pair $(B_i^q , f B_i^m)$ is unknotted , because f is locally unknotted , and therefore the homeomorphism $fg_i f^{-1}$ of the smaller ball can be suspended to a homeomorphism , h_i say, of the larger ball . Since g keeps $\dot{M}$ fixed , g_i keeps $\dot{B}_i^m$ fixed , and so h_i keeps $\dot{B}_i^q$ fixed. Therefore h_i extends to a move of Q keeping $\dot{Q}$ fixed . The composition $h = h_1 h_2 \dots h_n$ covers g .

Collars

Before proving Theorem 12, it is necessary to prove a couple of theorems about collars of compact manifolds . The theorems can be generalised to non-compact manifolds, but we shall only need the compact case . Define a collar of M to be an embedding

$$c : \dot{M} \times I \longrightarrow M$$

such that $c(x,0) = x$, all $x \in \dot{M}$.

Let $f : M \to Q$ be a proper locally-unknotted embedding between two compact manifolds, and let c , d be collars of M , Q . We say c , d are

<u>compatible with f</u> if the diagram

$$\begin{tikzcd} \dot{Q} \times I \arrow[r, "d"] & Q \\ \dot{M} \times I \arrow[u, "f \times I"] \arrow[r, "c"] & M \arrow[u, "f"'] \end{tikzcd}$$

is commutative and $\operatorname{im} d \cap \operatorname{im} f = \operatorname{im} fc$.

<u>Lemma 24</u> . <u>Given a proper locally unknotted embedding between compact manifolds, there exist compatible collars</u> .

<u>Corollary</u> . <u>Any compact manifold has a collar</u> . (For in the lemma choose the smaller manifold to be a point) .

<u>Proof of Lemma 24</u> .

Let M^+ denote the mapping cylinder of $\dot{M} \subset M$. Then $M^+ = \dot{M} \times I \cup M$, with the identification $(x, 1) = x$, and the induced structure . Then M^+ has a natural collar . The given proper embedding $f : M \longrightarrow Q$ induces a proper embedding $f^+ : M^+ \longrightarrow Q^+$ with which the natural collars are compatible .

Let ρ denote the retraction maps of the mapping cylinders , shrinking the collars ; then the diagram

$$\begin{tikzcd} Q^+ \arrow[r, "\rho"] & Q \\ M^+ \arrow[u, "f^+"] \arrow[r, "\rho"] & M \arrow[u, "f"'] \end{tikzcd}$$

is commutative . We shall produce homeomorphisms c, d (not merely maps) such that

$$\begin{tikzcd} Q^+ \arrow[r, "d"] & Q \\ M^+ \arrow[u, "f^+"] \arrow[r, "c"] & M \arrow[u, "f"'] \end{tikzcd}$$

is commutative, and such that c, d agree with ρ on the boundaries . The restrictions of c, d to the natural collars will prove the lemma .

Choose triangulations of M, Q - call them by the same names - such that f is simplicial, and let M' , Q' denote the barycentric derived complexes . For each p-simplex $A \in \dot{M}$, let A^* denote its dual in $\dot{M}$; more precisely A^* is the $(m-1-p)$-ball in $\dot{M}'$ given by

$$A^* = \bigcap_{v \in A} \operatorname{st}(v, \dot{M}') \quad .$$

Using the linear structure of the prisms $A \times I$, $A \in \dot{M}$, define the m-ball $A^+ \subset \dot{M} \times I$ to be the join

$$A^+ = (A \times 0)\,(A^* \times 1) \quad .$$

The set of all such balls cover $\dot{M} \times I$ and determine a triangulation of $\dot{M} \times I$; the latter agrees with M' on the overlap , and so together with M' determines a triangulation of M^+ .

Order the simplexes $A_1, A_2, \ldots, A_r$ of K in an order of locally increasing dimension (i.e. of $A_i \prec A_j$ then $i \leq j$) . Similarly order the simplexes B_1 , $B_2, \ldots, B_s$ of L such that $fA_i = B_i$, $1 \leq i \leq r$. Define inductively ,

$$M_0 = M' , \quad M_i = M_{i-1} \cup A_i^+$$

$$Q_0 = Q' , \quad Q_i = Q_{i-1} \cup B_i^+ \quad .$$

We have ascending sequences of subcomplexes

$$M' = M_0 \subset M_1 \subset \ldots \subset M_r = M_{r+1} = \ldots = M_s = M^+$$

$$Q' = Q_0 \subset Q_1 \subset \ldots\ldots\ldots\ldots\ldots\ldots \subset Q_s = Q^+ ,$$

such that $f^+ M_i \subset Q_i$, for each i . We shall show inductively there exist homeomorphisms c_i, d_i such that

$$\begin{tikzcd} Q_i \arrow[r, "d_i"] & Q \\ M_i \arrow[u, "f^+"] \arrow[r, dash, "c_i"] & M \arrow[u, "f"'] \end{tikzcd}$$

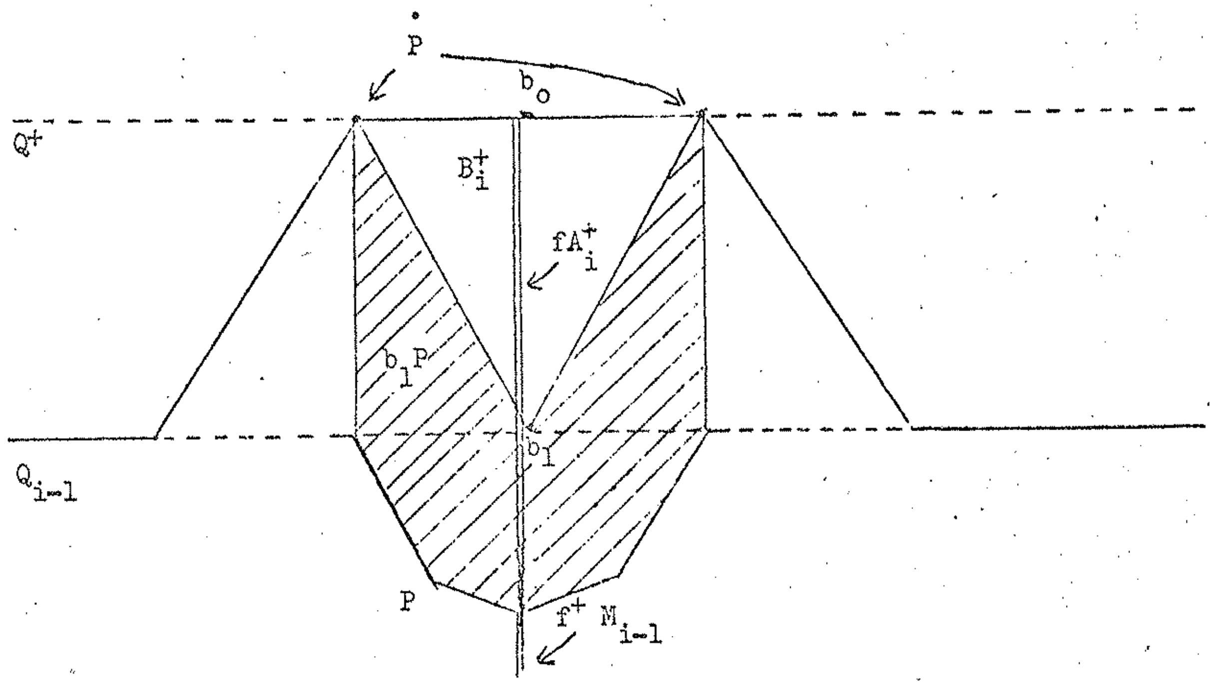
is commutative, and such that c_i, d_i agree with ρ on the boundaries . The induction begins trivially with $c_o = d_o =$ identities, and ends with what we want .

For the inductive step, fix i , and assume c_{i-1} , d_{i-1} to be defined . There are two cases .

<u>Case (i), $i \leqslant r$</u> . For $j = 0, 1$ let a_j denote the barycentre of $A_i \times j$, and let $b_j = f^+ a_j$. Let P denote the $(q-1, m-1)$ ball pair

$$P = \Big(lk\,(b_1 , Q_{i-1}) , lk(b_1, f^+ M_{i-1}) \Big) ,$$

which is unknotted because by hypothesis $f : M \to Q$ is locally unknotted , and so by induction $f^+ : M_{i-1} \longrightarrow Q_{i-1}$ is also . Then $b_1 P$ is the cone pair on P , and $b_0\, b_1\, \dot{P}$ the cone pair on $b_1\, \dot{P}$.



<u>Sublemma</u> . <u>There exists a homeomorphism $h : b_0 b_1 \dot{P} \cup b_1 P \to b_1 P$ that maps $b_0 \to b_1$, is the identity on P , and maps $b_0 \dot{P} \to b_1 \dot{P}$ linearly</u> .

<u>Proof</u> . If P were a standard ball pair , and $b_0 P$ a cone on P , and b_1 the barycentre of (the smaller of the pair) $b_0 P$, then the proof would be trivial by linear projection . An unknotting homeomorphism from P onto a standard pair maps the given set-up onto the standard set-up , and the sublemma follows by composition .

Returning to the proof of the lemma, notice that $b_0 b_1 \dot{P}$ is none other than the ball pair $(B_i^+ , f^+ A_i^+)$, and so h extends by the identity to a homeomorphism of manifold pairs

$$h : (Q_i , f^+ M_i) \longrightarrow (Q_{i-1} , f^+ M_{i-1}) .$$

Define $c_i = c_{i-1}(f^+)^{-1}hf^+$ and $d_i = d_{i-1}h$. Then c_i , d_i are homeomorphisms satisfying the commutativity condition .

Finally we have to show that c_i, d_i agree with ρ on the boundaries. For points outside A_i^+ , B_i^+ this follows by induction . For points in A_i^+ , B_i^+ it follows from the diagram

$$\begin{tikzcd} b_0 \dot{P} \arrow[rr, "h"] \arrow[dr, "d_i = \rho"'] & & b_1 \dot{P} \arrow[dl, "\rho = d_{i-1}"] \\ & b_1(\rho \dot{P}) & \end{tikzcd}$$

which is commutative by the linearity of the sublemma .

Case (ii) $i > r$. This case is simplex, because only the larger manifold Q is concerned . In case (i) ignore the smaller ball ; the proof gives a homeomorphism $h : Q_i \longrightarrow Q_{i-1}$ keeping $M_i = M_{i-1}$ fixed . Define $c_i = c_{i-1}$ and $d_i = d_{i-1} h$. The proof of Lemma 24 is complete .

Our next task is to improve Lemma 24 in Theorem 14 to the extent of moving the smaller collar from thesis to hypothesis . First it is necessary to show , in Theorem 13 , that any two collars of the same manifold are ambient isotopic , and for this we need three lemmas . Lemma 24 is about shortening a collar ; Lemma 25 is about isotoping a homeomorphism which is not level preserving into one which is level preserving over a small subinterval ; and Lemma 26 is about isotoping an isotopy . In each lemma an isotopy is constructed, and we must be careful to avoid the standard mistake and make sure that it is a polymap (i.e. piecewise linear) .

Notation . Suppose $0 < \varepsilon \leq 1$. Let I_ε denote the interval $[0, \varepsilon]$. Given a collar c of M , define the shortened collar .

$$c_\varepsilon : \dot{M} \times I \longrightarrow M$$

by $c_\varepsilon(x,t) = c(x, \varepsilon t)$, all $x \in \dot{M}$, $t \in I$.

Lemma 25 . *The collars c , c_ε are ambient isotopic keeping $\dot{M}$ fixed* .

Proof . First lengthen the collar c as follows . The image of c is a submanifold of M , and so the closure of the complement is also a submanifold (by Lemma 17), with boundary $c(\dot{M} \times 1)$. Therefore the latter has a collar , which we can add to c to give a collar , d say , of M such that $c = d_{1/2}$. Then $c_\varepsilon = d_{\varepsilon/2}$.

By Lemma 16 there is an ambient isotopy G of I , keeping $\dot{I}$ fixed , and finishing with the homeomorphism that maps $[0, 1/2]$, $[1/2, 1]$ linearly onto $[0, \varepsilon/2]$, $[\varepsilon/2, 1]$. Let $1 \times G$ be the ambient isotopy of $\dot{M} \times I$, and let H be the image of $I \times G$ under d . Since $1 \times G$ keeps $\dot{M} \times \dot{I}$ fixed , we can extend H by the identity to an ambient isotopy H of M keeping $\dot{M}$ fixed . Then $H_1 c = c_\varepsilon$, proving the lemma .

Lemma 26 . *Let X be a polyhedron, and $f : X \times I_\varepsilon \rightarrow X \times I$ an embedding such that $f | X \times 0$ is the identity . Then there exists δ, $0 < \delta < \varepsilon$, and an embedding $g : X \times I \longrightarrow X \times I$ such that* :

(i) *g is level preserving in I_δ* ;

(ii) *g is ambient isotopic to f keeping $X \times \dot{I}$ fixed* ;

(iii) *If $Y \subset X$, and $f | Y \times I_\varepsilon$ is already level preserving, then we can choose g to agree with f on $Y \times I_\varepsilon$* , *and the ambient isotopy to keep $f(Y \times I_\varepsilon)$ fixed* .

Proof . Let K , L be triangulations of $X \times I_\varepsilon$, $X \times I$ such that $f : K \rightarrow L$ is simplicial (in fact a simplicial embedding) . Choose δ ,

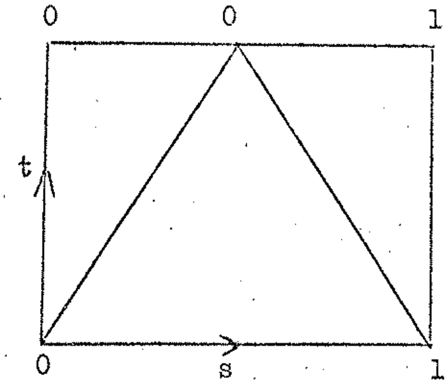
$0 < \delta < \varepsilon$, so small that no vertices of K or L lie in the interval $0 < t \leq \delta$. Choose first deriveds K' , L' of K , L according to the rule : if the interior of a simplex meets the level $X \times \delta$, then star it at a point on $X \times \delta$; otherwise star it barycentrically . Let $g : K' \longrightarrow L'$ be the first derived map . We verify the three properties :

Property (i) holds because by construction g is level preserving at the levels 0 and δ , and any point in between these two levels lies on a unique interval that is mapped linearly onto another interval , both intervals beginning (at the same point) in $X \times 0$ and ending in $X \times \delta$.

To prove property (ii) define another first derived L'' of L by the rule : if a simplex lies in fK then star it so that $f : K' \to L''$ is simplicial: otherwise star it barycentrically . The isomorphism $L'' \longrightarrow L'$ is isotopic to the identity by Lemma 15 Corollary 1 , and so f , g are ambient isotopic . The isotopy keeps fixed any subcomplex of L on which L' and L'' agree , and in particular keeps $X \times \dot{I}$ fixed .

To prove property (iii) we put extra conditions on the choices of K and L' . Choose K so as to contain $Y \times I_\varepsilon$ as a subcomplex . Having chosen K , K' , and therefore L'' , then choose L' so as to agree with L'' on $f(Y \times I_\varepsilon)$, this being compatible with the condition of starring on the δ-level because $f | Y \times I_\varepsilon$ is already level preserving . Therefore H keeps $f(Y \times I_\varepsilon)$ fixed .

<u>Lemma 27</u> . <u>Let $g : X \times I \longrightarrow X \times I$ be an ambient isotopy of X</u> . <u>Let h be the ambient isotopy of X that consists of the identity for half the time followed by g at twice the speed . Then g , h are ambient isotopic keeping $X \times \dot{I}$ fixed</u> .



Proof. Triangulate the square I^2 as shown, and let $u : I^2 \longrightarrow I$ be the simplicial map determined by mapping the vertices to 0 or 1 as shown. Define $G : (X \times I) \times I \longrightarrow (X \times I) \times I$ by

$$G\Big((x,s),t\Big) = \Big((g_{u(s,t)}x,s),t\Big) .$$

Then (i) G is a level preserving homeomorphism, and

(ii) G is piecewise linear, because the graph ΓG of G is the intersection of two subpolyhedra of $(X \times I^2)^2$:

$$\Gamma G = \Big((1 \times u)^2\Big)^{-1} \Gamma g \cap \Big(X^2 \times \Gamma 1\Big) ,$$

where $(1 \times u)^2$ denotes the map $(X \times I^2)^2 \longrightarrow (X \times I)^2$, where Γg is the graph of g , and $\Gamma 1$ is the graph of the identity on I^2 .

Therefore G is an isotopy of $X \times I$ in itself. By the construction of u, G moves g to h keeping $X \times \dot{I}$ fixed. Therefore g, h are ambient isotopic keeping $X \times \dot{I}$ fixed.

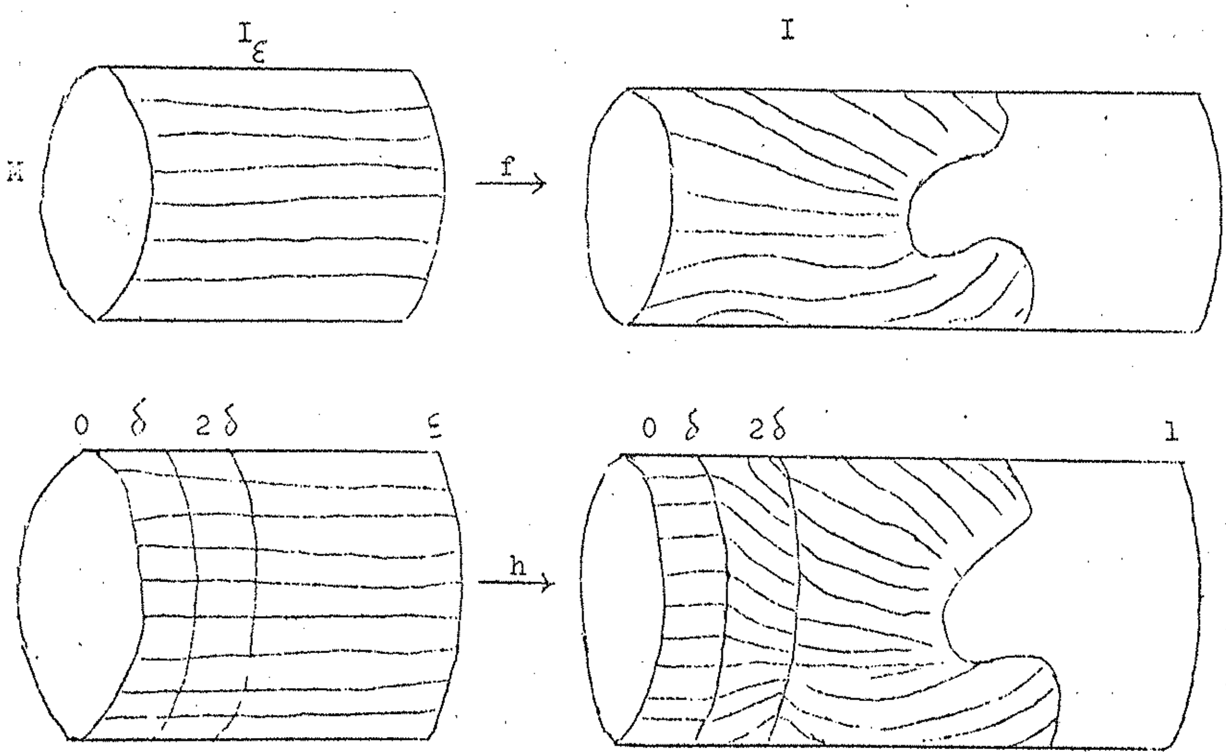
Theorem 13. If M is compact, then any two collars of M are ambient isotopic keeping $\dot{M}$ fixed.

Proof. Given two collars, the idea is to (i) ambient isotope one of them until it is level preserving relative to the other on a small interval, (ii) isotope it further until it agrees with the other on a smaller interval, and then (iii) isotope both onto this common shortened collar.

Let $c,d : \dot{M} \times I \rightarrow M$ be the two given collars . Since each maps onto a neighbourhood of $\dot{M}$ in M , we can choose $\varepsilon > 0$, such that $c(\dot{M} \times I_\varepsilon) \subset d(\dot{M} \times I)$. Since c,d are embeddings, we can factor $c = d\,f$, where f is an embedding such that the diagram

$$\begin{tikzcd}
\dot{M} \times I_\varepsilon \arrow[d, "f"'] \arrow[dr, "c"] & \\
\dot{M} \times I \arrow[r, "d"'] & M
\end{tikzcd}$$

is commutative, and $f \mid \dot{M} \times 0$ is the identity .



By Lemma 26 there exists δ , $0 < 2\delta < \varepsilon$, and an ambient isotopy F of $\dot{M} \times I$ moving f to g keeping $\dot{M} \times \dot{I}$ fixed , and such that g is level preserving for $0 \leqslant t \leqslant 2\delta$. The reason for making g level preserving is that we can now apply Lemma 27 and obtain an ambient isotopy G of $\dot{M} \times I_{2\delta}$ moving $g \mid \dot{M} \times I_{2\delta}$ to h keeping $\dot{M} \times \dot{I}_{2\delta}$ fixed, and such that h is the identity for $0 \leqslant t \leqslant \delta$. Extend h to an embedding $h : \dot{M} \times I_{\varepsilon} \to \dot{M} \times I$ by by making it agree with g outside $M \times I_{2\delta}$, and extend G by the identity to an ambient isotopy of $\dot{M} \times I$.

Then $G\,F$ is an ambient isotopy moving f to h keeping $\dot{M} \times \dot{I}$ fixed. Let H be the image of $G\,F$ under d. Since $G\,F$ keeps $\dot{M} \times 1$ fixed, we can extend H by the identity to an ambient isotopy H of M keeping $\dot{M}$ fixed. Let $e = H_1 c$. Then e is a collar ambient isotopic to c, and agreeing with the beginning of d, because of $x \in \dot{M}$ and $t \in I$ then

$$\begin{aligned} e_{\delta}(x,t) &= e(x, \delta t) \\ &= H_1 c(x, \delta t) \\ &= d\ G_1 F_1 d^{-1} c(x, \delta t) \\ &= d\ G_1 F_1 f(x, \delta t) \\ &= d\ h\ (x, \delta t) \\ &= d(x, \delta t) \\ &= d_{\delta}(x, t)\ . \end{aligned}$$

Therefore $e_{\delta} = d_{\delta}$, and so by Lemma 25 there is a sequence of ambient isotopic collars c, e, e_{δ}, d. The proof of Theorem 13 is complete.

<u>Theorem 14</u>. <u>Given a proper locally unknotted embedding $f : M \to Q$ between compact manifolds, and a collar c of $\dot{M}$, then there exists a compatible collar d of Q</u>.

<u>Proof</u>. Lemma 24 furnishes compatible collars, c^*, d^* say, of M, Q. By Theorem 13 there is an ambient isotopy G of M keeping $\dot{M}$ fixed, such that $G_1 c^* = c$. By Corollary to Theorem 11 we can cover G_1 by a homeomorphism h of Q keeping $\dot{Q}$ fixed. Let $d = hd^*$.

Then the commutativity of the diagram

$$\begin{tikzcd}
\dot{Q} \times I \arrow[rr, "d"] \arrow[dr, "d^*"'] & & Q \\
& Q \arrow[ur, "h"'] & \\
& M \arrow[u, "f"] \arrow[dr, "G_1"] & \\
\dot{M} \times I \arrow[uuu, "f \times 1"] \arrow[ur, "c^*"] \arrow[rr, "c"'] & & M \arrow[uuu, "f"']
\end{tikzcd}$$

and the fact that

$$\begin{aligned}
\operatorname{im} d \cap \operatorname{im} f &= \operatorname{im} h d^* \cap \operatorname{im} h f \\
&= h\,(\operatorname{im} d^* \cap \operatorname{im} f) \\
&= h\,(\operatorname{im} f c^*) \\
&= \operatorname{im} fc\ ,
\end{aligned}$$

ensure that the collars c , d are compatible with f . The proof of Theorem 14 is complete .

We now prove the critical lemma for the covering isotopy theorem , Theorem 12 .

Lemma 28 . Let M , Q be compact, and F a locally unknotted isotopy of M in Q keeping $\dot{M}$ fixed . Then there exists $\varepsilon > 0$, and a short ambient isotopy $H : Q \times I_\varepsilon \to Q \times I_\varepsilon$ of Q that keeps $\dot{Q}$ fixed and covers the beginning of F . In other words the diagram

$$\begin{tikzcd}
Q \times I_\varepsilon \arrow[dr, "H"] & \\
& Q \times I_\varepsilon \\
M \times I_\varepsilon \arrow[uu, "F_0 \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

<u>is commutative</u> .

Proof. For the convenience of the proof of this lemma we assume that $F_0 = F_1$. For , if not , replace F by F^* where F

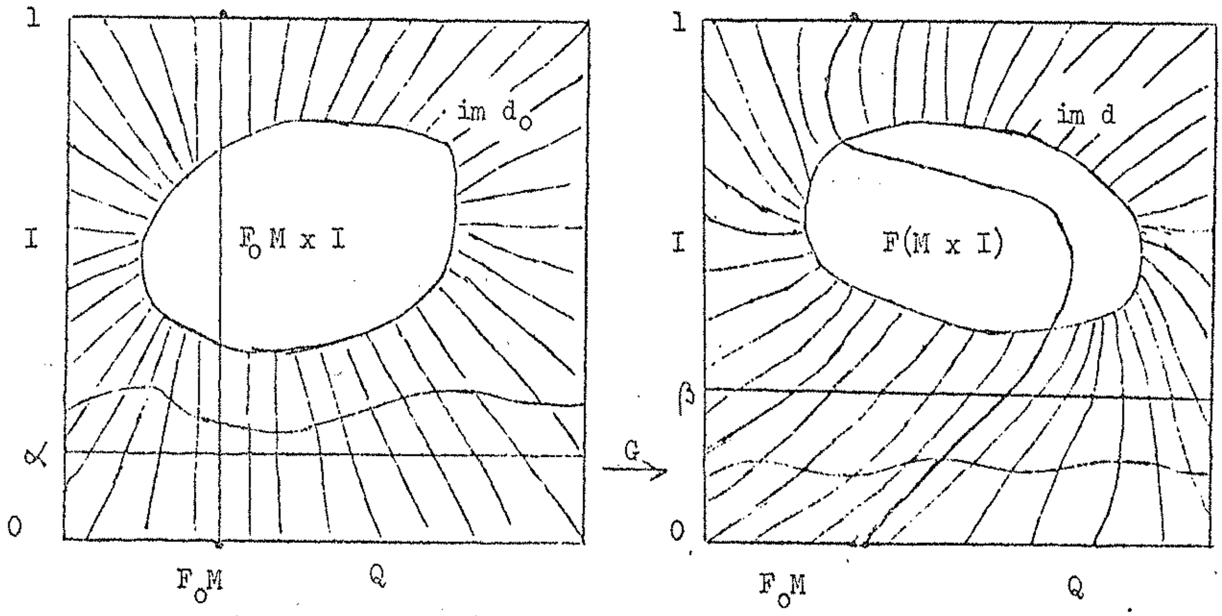
$$F^*_t = \begin{cases} F_t & t \leqslant 1/2 \\ F_{1-t} & t \geqslant 1/2 \end{cases} .$$

Then, since $F^*_0 = F^*_1$, the proof below gives H covering the beginning of F^* , which is the same as the beginning of F if $\varepsilon \leqslant 1/2$.

Therefore assume $F_0 = F_1$. This means that the two proper embeddings F , $F_0 \times 1$ of $M \times I$ in $Q \times I$ agree on the boundary $(M \times I)^{\cdot}$, because F keeps $\dot{M}$ fixed . Choose a collar c of $M \times I$, and then by Theorem 14 choose collars d , d_0 of $Q \times I$ such that c , d are compatible with F , and c , d_0 are compatible with $F_0 \times 1$. We have a commutative diagram of embeddings

$$\begin{tikzcd}
 & (Q \times I)^{\cdot} \times I \arrow[dl, "d_0"'] \arrow[dr, "d"] & \\
Q \times I & (M \times I)^{\cdot} \times I \arrow[u, "F \times 1"'] \arrow[d, "c"] & Q \times I \\
 & M \times I \arrow[ul, "F_0 \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

Notice that both the collars d , d_0 maps $(Q \times 0) \times 0$ to $Q \times 0$. Therefore im d contains a neighbourhood of $Q \times 0$ in $Q \times I$, and so contains $Q \times I_\beta$, for some $\beta > 0$. Similarly $d_0 d^{-1}(Q \times I_\beta)$ contains a neighbourhood of $Q \times 0$, and so contains $Q \times I_\alpha$, for some α , $0 < \alpha \leqslant \beta$.



Let $G = d\,d_0^{-1} : Q \times I_\alpha \longrightarrow Q \times I_\beta$. Then G has the properties :

(i) $G \mid \dot{Q} \times I =$ identity, because d , d_0 agree on $(Q \times I)^{\cdot} \times 0$;

(ii) $G \mid Q \times 0 =$ identity .

(iii) G covers the beginning of F in the sense that the diagram

$$\begin{tikzcd}
Q \times I_\alpha \arrow[dr, "G"] & \\
& Q \times I_\beta \\
M \times I_\alpha \arrow[uu, "F_0 \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

is commutative . For if $x \in M$ and $t \in I_\alpha$, then by compatibility

$$(F_0 x,\ t) \in \text{im}\,(F_0 \times 1) \cap \text{im}\, d_0 = \text{im}\,(F_0 \times 1)\, c \quad .$$

Therefore for some $y \in (M \times I)^{\cdot} \times I$,

$$(F_0 x,\ t) = (F_0 \times 1)\, cy = d_0\, (F \times 1)^{\cdot} y \quad .$$

Therefore

$$\begin{aligned} G\,(F_o x 1)(x,t) &= (d d_o^{-1})\, d_o\, (F \times 1)\, y \\ &= d(F \times 1)\, y \\ &= F\, cy \\ &= F(F_o x1)^{-1}\, (F_o x1)\, cy \\ &= F(F_o x1)^{-1}\, (F_o x, t) \\ &= F(x,t)\ . \end{aligned}$$

In other words $G(F_o x1) = F$, which proves (iii) .

By Lemma 26 there is an ε , $0 < \varepsilon < \alpha$ and an embedding $H : Q \times I_\alpha \to Q \times I_\beta$ ambient isotopic to G , such that $H \mid Q \times 0 =$ identity , and H is level preserving in I_ε . Further, since G is already level preserving on $(\dot{Q} \cup F_o M) \times I_\alpha$, we can by Lemma 26 (iii) choose H to agree with G on this subpolyhedron . In other words , the restriction $H : Q \times I_\varepsilon \to Q \times I_\varepsilon$ is a short ambient isotopic covering the beginning of F and keeping $\dot{Q}$ fixed .

<u>Proof of Theorem 12 , the covering isotopy theorem</u> .

We are given a locally unknotted isotopy $F : M \times I \to Q \times I$ keeping $\dot{M}$ fixed , and a subdivision N of the track of F , and we have to cover F by an ambient isotopy H of Q supported by N keeping $\dot{Q}$ fixed . We are given that M is compact , and we first consider the case when Q is also compact and $N = Q$.

If $0 < t < 1$, the definition of locally unknotted isotopy ensures that the restrictions of F to $[0 , t]$ and $[t , 1]$ are locally unknotted embeddings , and therefore we can apply Lemma 7 to both sides of the level t , and cover F in the neighbourhood of t . More precisely, for each $t \in I$, there exists a neighbourhood $J^{(t)}$ of t in I , and a level preserving homeomorphism $H^{(t)}$ of $Q \times J^{(t)}$ such that $H^{(t)}$ keeps $\dot{Q}$ fixed , $H_t^{(t)} = 1$, and such that the diagram

$$\begin{tikzcd}
Q \times J^{(t)} \arrow[dr, "H^{(t)}"] & \\
 & Q \times J^{(t)} \\
M \times J^{(t)} \arrow[uu, "F_t \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

is commutative . By compactness of I we can cover I by a finite number of such intervals $J^{(t)}$. Therefore we can find values $t_1, t_2, \ldots, t_n$ and $0 = s_1 < s_2 < \ldots < s_{n+1} = 1$, such that for each i , $[s_i , s_{i+1}] \subset J^{(t_i)}$. Write $H^i = H^{(t_i)}$.

We now define H by induction on i , as follows . Define $H_o = 1$. Suppose $H_t : Q \to Q$ has been defined for $0 \leqslant t \leqslant s_i$ such that $H_t F_o = F_t$. Then define

$$H_t = H_t^i (H_{s_i}^i)^{-1} H_{s_i} , \quad \text{for} \quad s_i \leqslant t \leqslant s_{i+1} .$$

Therefore

$$\begin{aligned} H_t F_o &= H_t^i (H_{s_i}^i)^{-i} H_{s_i} F_o \\ &= H_t^i (H_{s_i}^i)^{-1} F_{s_i} \\ &= H_t^i F_{t_i} \\ &= F_t . \end{aligned}$$

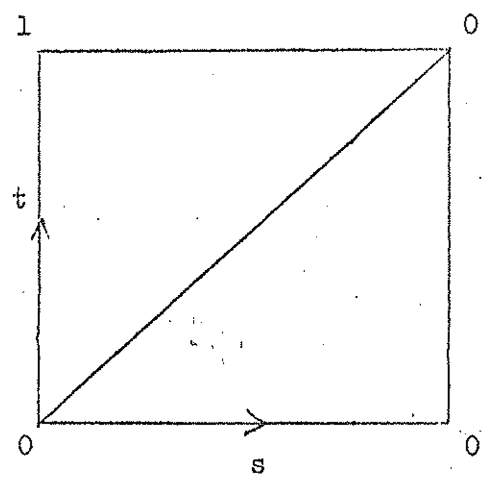
At the end of the induction we have H_t defined and $H_t F_o = F_t$, all $t \in I$.

Moreover H is piecewise linear, because it is composed of a finite number of piecewise linear pieces, and H keeps $\dot{Q}$ fixed because H^i does . Therefore the proof is complete for the case when Q is compact and $N = Q$.

We now extend the proof to the general case , when Q is not necessarily compact , and $N \subset Q$. We may assume that N is a regular neighbourhood of the tracl , because any neighbourhood contains a regular neighbourhood . Therefore N is a compact submanifold of Q . By the compact case , cover F by an ambient isotopy of Q covering F supported by N and keeping $\dot{Q}$ fixed . The proof of Theorem 12 is complete .

<u>Proof of Addendum to Theorem 12</u> .

We have to extend a given ambient isotopy H of $\dot{Q}$ with compact support X to an ambient isotopy H^* of Q supported by a given neighbourhood N of X in Q . (H is not a corollary because the embedding $\dot{Q} \times I \to Q \times I$ induced by H is not proper) . Without loss of generally we may assume the neighbourhood N of X to be regular , and therefore a compact manifold . Restrict H to X and extend by the identity to an ambient isotopy , G say , of $\dot{N}$ keeping $\dot{N} - X$ fixed .



Triangulate the square I^2 as shown, and let $n : I^2 \to I$ be the simplicial map determined by mapping the vertices to 0 or 1 as shown.

Define $G^* : (\dot{N} \times I) \times I \to (\dot{N} \times I) \times I$ by

$$G^*\Big((x,s),t\Big) = \Big((G_{u(s,t)}x,s),t\Big) .$$

As in the proof of Lemma 27, it follows that G^* is an ambient isotopy of $\dot{N} \times I$ keeping $(\dot{N} \times I) \cup (\dot{N} - X) \times I$ fixed .

Choose a collar $c : \dot{N} \times I \longrightarrow N$ and let H^* be the image of G^* under c. Since G^* keeps $\dot{N} \times 1$ fixed, H^* can be extended by the identity to an ambient isotopy of N; and since G^* keeps $(\dot{N} - X) \times 0$, H^* keeps the frontier of N fixed, and so can be further extended to an ambient isotopy H^* of Q supported by N. By construction H^* extends H, as desired.

Proof of Corollary 1 to Theorem 12.

Corollary 1 is concerned with the case when the isotopy F of M in Q does not keep $\dot{M}$ fixed. Let T be the track of F in Q, which is compact since M is compact. Let $\dot{F} : \dot{M} \times I \longrightarrow \dot{Q} \times I$ denote the restriction of F to the boundary, which is locally unknotted because F is. Let X be a regular neighbourhood of the track $T \cap \dot{Q}$ of $\dot{F}$ in $\dot{Q}$, and let N_o be a regular neighbourhood of X in Q. By choosing X, N_o sufficiently small, we can ensure that the given neighbourhood N of T is also a neighbourhood of N_o.

Use Theorem 12 to cover $\dot{F}$ by an ambient isotopy of $\dot{Q}$ supported by X, and by the Addendum extend the latter to an ambient isotopy, G say, of Q supported by N_o. Then $G^{-1}F$ is an isotopy of M in Q keeping $\dot{M}$ fixed, whose track is contained in $T \cup N_o$. But N is a neighbourhood of $T \cup N_o$, and so we can again use Theorem 12 to cover $G^{-1} F$ by an ambient isotopy, H say, of Q supported by N. Therefore GH covers F and is supported by N.

Recall Lemma 16. Any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.

Corollary. Any homeomorphism of a ball keeping a face fixed is isotopic to the identity keeping the face fixed. For by Theorem 2 the ball is homeomorphic to a cone on the complementary face . First isotope the complementary face back into position, and extend the isotopy conewise to the ball ; then isotope the ball .

Proof of Corollary 2 to Theorem 12 .

We have to show the equivalence of

(1) isotopic,

(2) ambient isotopic,

(3) ambient isotopic by an ambient isotopy with compact support,

and

(4) isotopic by moves .

(1) implies (3) by Theorem 12 and Corollary 1, because we can choose the neighbourhood N to be compact. (3) implies (4) by Theorem 11 . (4) implies (2) by Lemma 16 and Corollary, because then each move is ambient isotopic to the identity . Finally (2) implies (1) trivially .

Proof of Corollary 3 to Theorem 12 .

If F is a locally trivial isotopy , then by definition each point in $M \times I$ has a neighbourhood which is locally unknotted ; therefore F is locally unknotted . Conversely if F is a locally unknotted isotopy , then the level F_o is a locally unknotted embedding , and so the constant isotopy $F_o \times 1$ is locally trivial . By Theorem 12 cover F by H ; then $F = H(F_o \times 1)$ is locally trivial , because the homeomorphism H preserves local triviality . This completes the proofs of the theorems and corollaries stated at the beginning of the chapter .

Remarks on combinatorial isotopy.

We have framed the definitions of isotopy and proved the theorems in the polyhedral category, because that is the spirit of these seminars. In other words, there is no reference to any specific triangulations of either of the manifolds concerned. However there is a definition of isotopy in the combinatorial category when the receiving manifold Q happens to be Euclidean space, by virtue of the linear structure of Euclidean space. The manifold M is given a fixed triangulation, K say, and the isotopy is defined by moving the vertices of K. At each moment the embedding of M is uniquely determined by the positions of the vertices, and by the linear structure of Euclidean space. But a general polyhedral manifold Q has only a piecewise linear structure, not a linear structure, and so the positions of the vertices of K do not determine a unique embedding of M. It is no good picking a fixed triangulation L of Q, and considering linear embeddings $K \to L$, because this has the effect of trapping M locally, and preventing the movement of any simplex of K across the boundary of any simplex of Q. Therefore to obtain any useful form of isotopy it is essential to retain the polyhedral structure of Q, even though we may descend to the combinatorial structure of M. We now give a definition in these terms, which looks at first sight much more special than the definitions of isotopy above, but in fact turns out to be equivalent; we state the theorem without proof. The moral of the story is: stick to the polyhedral category and don't tinker about with the combinatorial category; keep the latter out of definitions and theorems, and use it only as it ought to be used, as an inductive tool for proofs.

Linear moves with respect to a triangulation

Let Δ^q be the standard q-simplex, and Δ^m an m-dimensional face, $q > m$. Let x be the barycentre of Δ^q, and y a point between x and the barycentre of Δ^m. Let $\sigma : \Delta^q \to \Delta^q$ be the homeomorphism throwing x to y, mapping the boundary by the identity, and joining linearly.

Let M be closed, K a triangulation of M, and let $f, g : M \to Q$ be proper embeddings. We say there is a <u>move from f to g linear with respect to K</u> if the following occurs:

There is a closed vertex star of K, $A = \overline{\mathrm{st}}\,(v,K)$ say, and a q-ball $B \subset Q$, and a homeomorphism $h : B \to \Delta^q$ such that

(i) f, g agree on $K - \overset{o}{A}$,

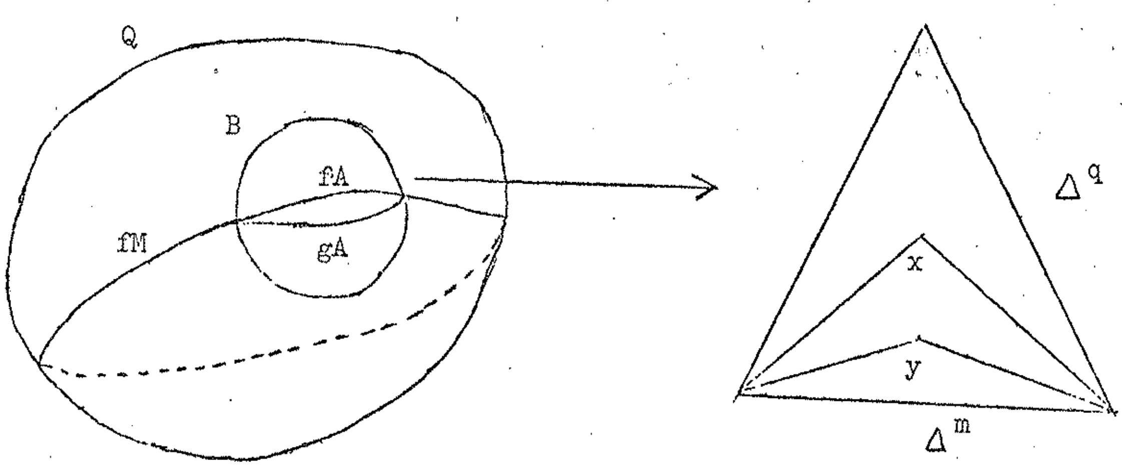
(ii) $A = f^{-1}B = g^{-1}B$,

(iii) hf maps $\mathrm{lk}(v,K) \longrightarrow \dot{\Delta}^m$, homeomorphically,

$$v \longrightarrow x \quad ,$$

$$A \longrightarrow x \qquad {}^m \text{, by joining linearly .}$$

(iv) $g \mid A = h^{-1} \sigma h(f \mid A)$.



Roughly speaking, h is a local coordinate system, chosen so that the move from f to g looks as simple as possible, just moving one vertex of K linearly in the most harmless fashion, like a move of classical knot theory.

Addendum (stated without proof). Let M be closed, and K an arbitrary fixed triangulation of M. Let $f, g : M \to Q$ be proper embeddings that are locally unknotted and ambient isotopic. If codimension > 0, then f, g are isotopic by moves linear with respect to K.

The addendum becomes surprising if we imagine embeddings of a 2-sphere in a manifold, and choose K to be the boundary of a 3-simplex, with exactly 4 vertices. Then we can move from any embedding to any other isotopic embedding by assiduously shifting just those 4 vertices linearly back and forth. All the work is secretly done by judicious choice of the balls, or local coordinate systems in the receiving manifold, in which the moves are made.

Remark. Notice the restriction codimension > 0 that occurs in the addendum (but not in Theorem 11 for example). It is an open question as to whether or not the restriction is necessary. In particular we have the problem: is a homeomorphism of a ball that keeps the boundary fixed isotopic to the identity by linear moves?

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