Let $\mathbb{C}$ be the set of complex numbers which, as viewed in $\mathbb{R}^2$, are endowed with multiplication $(a,b)\cdot(x,y) = (ax-by, bx+ay)$, which is inspired by the idea that $(a,b)$ means $a + b\sqrt{-1}$, where we have $a = \operatorname{Re} z$, $b = \operatorname{Im} z$. This rule obeys

$$(\alpha_1 z_1 + \alpha_2 z_2)(\alpha_3 z_3 + \alpha_4 z_4) = (\alpha_1\alpha_3)z_1 z_3 + (\alpha_1\alpha_4)z_1 z_4 + (\alpha_2\alpha_3)z_2 z_3 + (\alpha_2\alpha_4)z_2 z_4$$

for all $\alpha_i \in \mathbb{R}$ and $z_i \in \mathbb{C}$, $i = 1,2,3,4$. This rule is associative: $(z_1z_2)z_3 = z_1(z_2z_3)$. The multiplication rule has a left and right identity: $(1,0)\cdot(a,b) = (a,b) = (a,b)\cdot(1,0)$. This means that $\mathbb{C}$ is an associative unital algebra over $\mathbb{R}$. Another example is the $n\times n$ real matrices. Notice that the linear transformation of (left) multiplication by $z = (a,b)$, with matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We can do this for any finite-dimensional algebra over $\mathbb{R}$. The replacement of abstract algebra by algebras of matrices (linear transformation) is called the (left) regular representation. We make the following observations:

- The linear matrix determines the complex number (because the algebra has an identity).
- Matrix addition/multiplication is consistent with algebra addition/multiplication.
- Associative and distributive properties hold.

**Theorem** (Cayley-Hamliton Theorem)**.** *We observe*

$$\det\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2 + b^2.$$

*This is zero if and only if we have $a = b = 0$. The inverse of the matrix is*

$$\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

*which is a complex number.*

This property makes $\mathbb{C}$ a division algebra. The quality $a^2+b^2$ is called the square modulus $|z|^2$ of $z \in \mathbb{C}$ and $|z| = \sqrt{a^2+b^2}$ is the modulus. From what we know of the definitions, we have that $|zw| = |z||w|$ is invertible. In fact, as stated above,

$$\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is a complex number. And

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is also a complex number. It is called the complex conjugate of $z = a+ib$, written $\bar{z} = a - ib$. So we have $z^{-1} = \frac{1}{|z|^2}\bar{z}$ and $\overline{zw} = \overline{wz}$. Thinking about this leads to the introduction of polar coordinates in $\mathbb{C} = \mathbb{R}^2$. That is, we can write $z = re^{i\theta}$, where $r = a^2+b^2$, $a = r\cos\theta$, $b = r\sin\theta$. As a matrix equation, we can identify $z = a+ib$ by

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The first matrix on the right-hand side of the above equation scales a complex number by a factor $r = |z|$ (homothetic). The second matrix on the right-hand side of the above equation rotates the complex number counterclockwise by angle $\theta$ (argument). As all rotations and scalaings commute with one another, we can use this fact to see that complex multiplication is commutative. Those matrices are conformal; that is, they preserve angles. From this, we can find all square roots of $-\,\mathrm{id}$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which are rotations by $\pm 90^\circ$. These are indistinguishable. Now, returning back to our original matrix equation representing $a+ib$, any $z \neq 0$ has $n$ precisely distinct $n^{\text{th}}$ roots $z^{\frac{1}{n}}$, which we identify by

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{\frac{1}{n}} = \begin{bmatrix} r^{\frac{1}{n}} & 0 \\ 0 & r^{\frac{1}{n}} \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix},$$

where we define $\phi := \frac{\theta+2\pi k}{n}$ for all $k = 0,1,\ldots,n-1$. The associated characteristic polynomial of the mtarix is

$$\begin{aligned}\chi\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) &= \det\left(\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) \\ &= (\lambda - a)^2 + b^2 \\ &= \lambda^2 - 2a\lambda + a^2 + b^2.\end{aligned}$$

The Cayley-Hamilton Theorem says that the matrix is a root of this polynomial, which comes “close” to the Fundamenatl Theorem of Algebra since the missing quadratic polynomials.

**Definition.** *For any square matrix $A$ over $\mathbb{R}$ (a square matrix $A$ with entries in $\mathbb{R}$), we define*

$$e^A := \sum_{l=0}^{\infty} \frac{A^l}{l!}.$$

Note that the series that defines $e^A$ converges because, if $\|\cdot\|$ denotes the operator norm of the matrix, which is defined by finding the maximum entry in the matrix, we have

$$\begin{aligned}\|e^A\| &= \left\|\sum_{l=0}^{\infty} \frac{A^l}{l!}\right\| \\ &\leq \sum_{l=0}^{\infty} \frac{\|A\|^l}{l!} \\ &= e^{\|A\|} \\ &< \infty\end{aligned}$$

because any square matrix $A$ with entries in $\mathbb{R}$ satisfies $\|A\| < \infty$. Note that, if two square matrices $A$ and $B$ commute; that is, $A$ and $B$ satisfy $AB = BA$, then the Binomial Theorem asserts $e^A e^B = e^B e^A$. If $A$ and $B$ do not commute, then $e^A$ and $e^B$ do not necessarily commute. This tells us $e^{z+w} = e^z e^w$ and $\overline{e^z} = e^{\bar{z}}$. The matrix representation of $e^{a+ib}$ is

$$\begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix} \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}.$$

Euler's formula states that this is a tautology if we can define the cosine and sine functions by their respective power series. If $z = 0$, then $z = e^w$ has no solutions; if $z \neq 0$, then $z = e^w$ has many solutions.

**Example.** *We consider the set*

$$\{w = \log|z| + i\theta : \theta \textit{ is any argument of } z\}.$$

*We introduce a "branch cut" on* $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$*. This allows us to define the "principal branch of the logarithm" as* $\operatorname{Log}(z) := \log|z| + i\theta$*, where $\theta$ is the argument of $z$, as a continuous function on* $\mathbb{C} \setminus (-\infty, 0]$.

Note that $z^w = e^{w \log z}$ is a nightmare to work with!

**Proposition.** *Any monic polynomial $p \in \mathbb{R}[x]$ over $\mathbb{R}$ can be written as a product of a linear function and an irreducible quadratic.*

*Proof.* See Exercise 1(b) of Homework 3. □

**Lemma.** *If $A \in \mathcal{A}$ is not an element of* $\operatorname{span}_{\mathbb{R}}(\mathrm{Id})$*, then:*

(1) *its minimal is $p_A(x) := (x-a)^2 + b^2$ for some $a \in \mathbb{R}$ and $b > 0$,*

*Proof.* Suppose instead by contradiction that we have $p_A(x) = x - a$. Then we have $p_A(A) = 0$, or equivalently $A - a\,\mathrm{Id} = 0$, or equivalently $A = a\,\mathrm{Id}$, which means $A \in \operatorname{span}_{\mathbb{R}}(\mathrm{Id})$. But this contradicts our assumption that $A$ is not an element of $\operatorname{span}_{\mathbb{R}}(\mathrm{Id})$. The only other possibility that contradicts our claim is to suppose that $p_A(x)$ is not invertible and we can express $p_A(x) = p_1(x)p_2(x)$, where both $p_1$ and $p_2$ are not of degree zero. Then one of $p_1$ and $p_2$ is not invertible, which would imply that one of $p_1(A)$ or $p_2(A)$ is not invertible or both $p_1(A)$ or $p_2(A)$ are not invertible. □

(2) $\operatorname{span}(\mathrm{Id}, A)$ *contains $A^{-1}$ a square root of* $-\mathrm{Id}$,

*Proof.* Since we established in (1) that the minimal polynomial of $A$ is $p_A(x) = (x-a)^2 + b^2$, we have $p_A(A) = 0$, or $A^2 - 2aA + b^2\,\mathrm{Id} = 0$. So we find that $\frac{A-a}{b}$ is a square root of $-\mathrm{Id}$, we also find the matrix inverse $A^{-1} = \frac{2a-A}{a^2+b^2}$. □

(3) *if there exists $B$ that satisfies $BA = AB$, then $B \in \operatorname{span}(\mathrm{Id}, A)$.*

*Proof.* Since $p_A$ is the minimal polynomial of $A$ for all $A \in \mathcal{A}$, in particular we have

$$\begin{aligned}p_{A+B}(A+B) - p_A(A) - p_B(B) &= 0 - 0 - 0 \\ &= 0,\end{aligned}$$

which implies (how?) $AB + BA \in \operatorname{span}(\mathrm{Id}, A, B)$, and so we have

$$AB + BA = aA + bB + c\,\mathrm{Id}$$

for some $a, b, c \in \mathbb{R}$. Because $A$ and $B$ commute with each other; that is, because we have $BA = AB$, it follows that we have

$$\begin{aligned}(2A - b)B &= 2AB - bB \\ &= AB + BA - bB \\ &= aA + c\,\mathrm{Id}.\end{aligned}$$

So we get $B = (2A-b)^{-1}(aA + c\,\mathrm{Id})$. Part (1), which gives $p_A(A) = 0$, allows us to get rid of $A^2$. So we really have $B = (2A-b)^{-1}(aA + c\,\mathrm{Id}) \in \operatorname{span}_{\mathbb{R}}(\mathrm{Id}, 2A - b)$. By applying part (2), we can conclude in fact $B \in \operatorname{span}_{\mathbb{R}}(\mathrm{Id}, 2A - b) = \operatorname{span}_{\mathbb{R}}(\mathrm{Id}, A)$. □

**Theorem** (Frobenius Theorem)**.** *Every finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, where $\mathbb{H}$ is the set of quaternions.*

*Proof.* If we have $\dim(\mathcal{A}) = 1$, then $\mathcal{A}$ is automatically isomorphic to $\mathbb{R}$, and so there is nothing to prove. If we have $\dim(\mathcal{A}) = 1$, then $\mathcal{A}$ is automatically isomorphic to $\mathbb{C}$, and so there is nothing to prove. Suppose instead we have $\dim(\mathcal{A}) = 3$. Then there exists $l \notin \operatorname{span}(\mathrm{Id}, i)$ with $l^2 = -1$. Consider $li + il$, which commutes with $i$ and $l$. Then, by part (3) of the lemma, we have $li + il \in \mathbb{R}$. Now, we consider $j = \alpha l + \beta \mathrm{Id}$ with $\alpha, \beta \in \mathbb{R}$. We seek $(\alpha l + \beta(\mathrm{Id}))^2 = -\mathrm{Id}$ and $i(\alpha l + \beta \mathrm{Id}) + (\alpha l + \beta \mathrm{Id})i = 0$. And so we find $\beta = \sqrt{\frac{1+(il+li)^2}{4}}$ and $\alpha = \frac{(il+li)\beta}{2}$. Thus, there exists $\alpha j$ that satisfies $j^2 = -1$, $ij + ji = 0$, and $\mathrm{Id}, i, j$ are linearly independent. Now, we will consider $k = ij$. Then we have $iki = -ji = ij = k = jkj$. So, if we have $k \in \operatorname{span}(\mathrm{Id}, i, j)$, say $k = \alpha + \beta i + \gamma j = -\alpha - \beta i - \gamma j$, which implies $\alpha = \beta = \gamma = 0$, which is a contradiction to $k^2 = -\mathrm{Id}$. So we must consider $\dim(\mathcal{A}) \geq 4$. Moreover, we have $ijk = ijij = -ijji = -1$, noting that we have $jk = jij = i$ and $ik = iij = -j$. We see that $\mathcal{A}$ "contains" a copy of $\mathcal{H}$. Suppose now, seeking a contradiction, that $\mathcal{A}$ contains some $l \in \operatorname{span}(\mathrm{Id}, i, j, k)$, with $l^2 = -\mathrm{Id}$. Moreover, we have $\gamma_1 := il + li, \gamma_2 := jl + lj, \gamma_3 := kl + lk \in \mathbb{R}$. Now consider $m := 2l + \gamma_1 i + \gamma_2 j + \gamma_3 k$, which obeys $im + mi = jm + mj + km + mk = 0$. Then we have

$$\begin{aligned} -m &= -1m1 \\ &= (ijk)m(kji) \\ &= ij(kmk)ji \\ &= ijmji \\ &= km(-k) \\ &= m, \end{aligned}$$

from which we conclude $m = 0$, and so $m \in \operatorname{span}_{\mathbb{R}}(\mathrm{Id}, i, j, k)$, which is a contradiction. □

**Definition.** *A* curve *or* path *is a continuous function $\gamma : [a, b] \to \mathbb{C}$. And a curve is a* reparameterization *of $\gamma$ if there exists a homeomoprhism $\phi : [a, b] \to [c, d]$ satisfying $\tilde{\gamma} = \gamma \circ \phi$, and $\phi(a) = c$ (preserves the orientation).*

**Definition.** *A path $\gamma : [a, b] \to \mathbb{C}$ is called* rectifiable *if*

$$\sup \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| < \infty,$$

*where the supremum is taken over partitions $a = t_0 < t_1, \cdots < t_{n-1} < t_n = b$. The value of the supremum is called the* length of $\gamma$.

**Remark.** *We are using polygonal approximation of $\gamma$. And finer partitions yield larger polygons due to the triangle inequality.*

**Lemma.** *We have the following results:*

(a) *Reparameterizations yield the same length.*

*Proof.* Homeomorphisms map partitions to partitions. □

(b) *If $a < c < b$, then we have* $\operatorname{length}(\gamma) = \operatorname{length}(\gamma|_{[a,c]}) + \operatorname{length}(\gamma|_{[c,b]})$.

*Proof.* Just add $c \in [a, b]$ to any partition. □

(c) *If $\gamma$ is piecewise $C^1$, then* $\operatorname{length}(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt$.

*Proof.* For any $a \leq t_j < t_{j+1} \leq b$, we have

$$\gamma(t_{j-1}) - \gamma(t_j) = \int_{t_j}^{t_j+1} \dot{\gamma}(t)\, dt.$$

This implies

$$\begin{aligned} |\gamma(t_{j-1}) - \gamma(t_j)| &= \left| \int_{t_j}^{t_j+1} \dot{\gamma}(t)\, dt \right| \\ &\leq \int_{t_j}^{t_j+1} |\dot{\gamma}(t)|\, dt. \end{aligned}$$

Summing over $j$ and taking supremums over partitions, we get

$$\begin{aligned}
\text{length}(\gamma) &= \sup \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| \\
&\leq \sup \sum_{j=1}^{n} \int_{t_j}^{t_j+1} |\dot{\gamma}(t)|\, dt \\
&= \sup \int_a^b |\dot{\gamma}(t)|\, dt \\
&= \int_a^b |\dot{\gamma}(t)|\, dt.
\end{aligned}$$

Now we will prove the opposite inequality. As $\dot{\gamma}$ is uniformly continuous, for all $\epsilon > 0$ there exists $\delta > 0$ such that $|t - s| < \delta$ implies $|\dot{\gamma}(t) - \dot{\gamma}(s)| < \epsilon$. We may refine any partition so that the largest spacing is smaller than $\delta$. Given $t_{j-1} < t_j$ from the partition, we have

$$\begin{aligned}
|\gamma(t_j) - \gamma(t_{j-1})| &= \left| \int_{t_{j-1}}^{t_j} \dot{\gamma}(t)\, dt \right| \\
&= \left| \int_{t_{j-1}}^{t_j} \dot{\gamma}(t_j)\, dt + \int_{t_{j-1}}^{t_j} (\dot{\gamma}(t) - \dot{\gamma}(t_j))\, dt \right| \\
&\geq \left| \int_{t_{j-1}}^{t_j} \dot{\gamma}(t_j)\, dt \right| - \left| \int_{t_{j-1}}^{t_j} (\dot{\gamma}(t) - \dot{\gamma}(t_j))\, dt \right| \\
&\geq \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t_j)|\, dt - \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t) - \dot{\gamma}(t_j)|\, dt \\
&= \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t) - (\dot{\gamma}(t) - \dot{\gamma}(t_j))|\, dt - \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t) - \dot{\gamma}(t_j)|\, dt \\
&\geq \left( \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t)|\, dt - \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t) - \dot{\gamma}(t_j)|\, dt \right) - \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t) - \dot{\gamma}(t_j)|\, dt \\
&= \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t)|\, dt - 2 \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t) - \dot{\gamma}(t_j)|\, dt \\
&= \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t)|\, dt - 2\epsilon(t_j - t_{j-1}).
\end{aligned}$$

So we can sum in $j$ to obtain, for all $\epsilon > 0$,

$$\begin{aligned}
\sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| &\geq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t)|\, dt - 2\epsilon \sum_{j=1}^{n} (t_j - t_{j-1}) \\
&= \int_a^b |\dot{\gamma}(t)|\, dt - 2\epsilon(b - a)
\end{aligned}$$

Sending $\epsilon \to 0^+$, we conclude

$$\sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| \geq \int_a^b |\dot{\gamma}(t)|\, dt,$$

proving the reverse inequality. So we conclude

$$\sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| = \int_a^b |\dot{\gamma}(t)|\, dt,$$

as desired. □

**Example.** *The function $\theta \mapsto (\cos\theta, \sin\theta)$ is injective near $\theta = 0$. It also satisfies*

$$\begin{aligned}
\text{length}(\gamma|_{[0,\delta]}) &= \int_0^{\epsilon} |\dot{\gamma}(t)|\, dt \\
&= \int_0^{\epsilon} 1\, dt \\
&= \epsilon - 0 \\
&= \epsilon.
\end{aligned}$$

*So our power series give trigonometric functions for $\theta$, where $\theta$ is measured in radians.*

**Lemma.** *If $\gamma : [a,b] \to \mathbb{C}$ is rectifiable, then $t \mapsto \operatorname{length}(\gamma|_{[a,t]})$ is continuous.*

*Proof.* See Exercise 3(?) of Homework 2(?). □

**Theorem.** *If $\gamma : [a,b] \to \mathbb{C}$ is rectifiable and $f : \mathbb{C} \to \mathbb{C}$ is continuous, then we have*

$$\int_\gamma f(z)\,dz = \lim \sum_{j=1}^n f \circ \gamma(t_j)(\gamma(t_j) - \gamma(t_{j-1}))$$

*and*

$$\int_\gamma f(z)\,|dz| = \lim \sum_{j=1}^n f \circ \gamma(t_j)|\gamma(t_j) - \gamma(t_{j-1})|,$$

*where the limit is with respect to the refinement of partitions (and this limit exists).*

*Proof.* We need to show that, for any $\epsilon > 0$, there exists $\delta > 0$ (for any fine enough partition with spacing less than $\delta$) which yields a Riemann sum that differs from the Riemann sum of any refinement by at most $\epsilon$. Note that $f \circ \gamma$ is uniformly continuous on $[a,b]$, so we may ask that $\delta$ be so small that $|t-s| < \delta$ implies $|f\circ\gamma(t) - f\circ\gamma(s)| < \epsilon$. By the uniform continuity of $f \circ \gamma$, for all $\epsilon > 0$ there exists $\delta > 0$ such that $|t-s| < \delta$ implies $|f\circ\gamma(t) - f\circ\gamma(s)| < \epsilon$. Take a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ that is finer than $\delta$. We will need to consider a refinement. We consider some interval $[t_{j-1}, t_j]$ and the nodes of the refinement therein: $t_{j-1} = s_0 < s_1 < \cdots < s_{m-1} < s_m = t_j$. So we have

$$\begin{aligned}
|\text{Riemann sum in } j - \text{Riemann sum for refinement}| &= \left| f\circ\gamma(t_j)(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{i=1}^m f\circ\gamma(s_i)(\gamma(s_i) - \gamma(s_{i-1}))\right| \\
&= \left| f\circ\gamma(t_j)(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{i=1}^m f\circ\gamma(t_j)(\gamma(s_i) - \gamma(s_{i-1})) \right. \\
&\qquad \left. + \sum_{j=1}^n f\circ\gamma(t_j)(\gamma(s_i) - \gamma(s_{i-1})) - \sum_{i=1}^m f\circ\gamma(s_i)(\gamma(s_i) - \gamma(s_{i-1}))\right| \\
&= \left| f\circ\gamma(t_j)\left((\gamma(t_j) - \gamma(t_{j-1})) - \sum_{i=1}^m f\circ\gamma(t_j)(\gamma(s_i) - \gamma(s_{i-1}))\right) \right. \\
&\qquad \left. + \sum_{i=1}^m (f\circ\gamma(t_j) - f\circ\gamma(s_i))\gamma(s_i) - \gamma(s_{i-1}))\right| \\
&= \left| f\circ\gamma(t_j)\cdot 0 + \sum_{i=1}^m (f\circ\gamma(t_j) - f\circ\gamma(s_i))\gamma(s_i) - \gamma(s_{u-1}))\right| \\
&\le \sum_{i=1}^m |f\circ\gamma(t_j) - f\circ\gamma(s_i)||\gamma(s_i) - \gamma(s_{i-1}))| \\
&< \sum_{i=1}^m \epsilon|\gamma(s_i) - \gamma(s_{i-1})| \\
&= \epsilon \operatorname{length}(\gamma|_{[t_{j-1},t_j]}).
\end{aligned}$$

So we can sum in $j$ to obtain the modulus of the difference between the Riemann sum in $j$ and the Riemann sum for the refinement:

$$\begin{aligned}
\sum_{j=1}^n \left| f\circ\gamma(t_j)(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{i=1}^m f\circ\gamma(s_i)(\gamma(s_i) - \gamma(s_{i-1}))\right| &< \epsilon \operatorname{length}(\gamma|_{[t_{j-1},t_j]}) \\
&= \epsilon \operatorname{length}(\gamma).
\end{aligned}$$

We would like to repeat this result for $\int_\gamma f(z)\,|dz|$. This mostly works, except that we used

$$\sum_{i=1}^m (\gamma(s_i) - \gamma(s_{i-1})) - (\gamma(t_j) - \gamma(t_{j-1})) = 0.$$

However, we have

$$\begin{aligned}
\text{error}_j &= \sum_{i=1}^m |\gamma(s_i) - \gamma(s_{i-1})| - |\gamma(t_j) - \gamma(t_{j-1})| \\
&\ge 0,
\end{aligned}$$

meaning that the error may not be zero. So the preceding argument would give instead

$$|\text{Riemann sum in } j - \text{Riemann sum for refinement}| \leq \epsilon \,\text{length}(\gamma) + \|f \circ \gamma\|_{\sup} \sum_{j=1}^{n} \text{error}_j$$

So the claim follows from the following lemma, which is below. ◻

**Lemma.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable curve. Given $\epsilon > 0$, there exists $\delta > 0$ such that any partition $a = t_0 < \cdots < t_n = b$ that is finer than $\delta$ satisfies*

$$\begin{aligned}\text{length}(\gamma) &\geq \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| \\ &\geq \text{length}(\gamma) - \epsilon.\end{aligned}$$

*Proof.* By definition, there is a partition $a = \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n = b$ that satisfies

$$\sum_{i=1}^{N} |\gamma(\tau_{i-1}) - \gamma(\tau_i)| \geq \text{length} - \frac{\epsilon}{2}.$$

Note that the left-hand side defines a (uniformly) continuous function of $(\tau_0, \ldots, \tau_N) \in [a, b]^{N+1} \subseteq \mathbb{R}^{N+1}$. So we can choose $\delta > 0$ so that, if $|\tau_i \tau_i'| < \delta$ for every integer $0 \leq i \leq N$, then we have

$$\sum_{i=1}^{N} |\gamma(\tau_i') - \gamma(\tau_{i-1}')| \geq \text{length} - \epsilon.$$

If necessary, we can further reduce $\delta$ so that we have $|\tau_i - \tau_{i-1}| > 2\delta$. This guarantees that the intervals $(\tau_i - \delta, \tau_i + \delta)$ are disjoint. Now, we let $a = t_0 < \cdots < t_n = b$ be a partition of $[a, b]$ that is strictly finer than $\delta$. Then every interval $(\tau_i - \delta, \tau_i + \delta)$ contains at least one $t_j$. Chose one for each interval, and so we can obtain a coarser $\tau_i$ that satisfies the second displayed equation of this proof. The partition $\{t_j\}_{j=1}^{n}$ is a refinement of this one, which means the polygonal length must be greater. ◻

**Proposition.** *If $\gamma : [a, b] \to \mathbb{C}$ is rectifiable and $f : \Omega \to \mathbb{C}$ is continuous, where $\gamma([a, b]) \subseteq \Omega \subseteq \mathbb{C}$ is open, then:*

(1) *$\int_\gamma f(z)\, dz$ and $\int_\gamma f(z)\, |dz|$ are reparameterization invariant,*

(2) *$|\int_\gamma f(z)\, dz| \leq \int_\gamma |f|\, |dz|$,*

(3) *If $\gamma$ is the concatenation of $\gamma_1$ and $\gamma_2$, then we have*

$$\int_\gamma f(z)\, dz - \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$$

*and*

$$\int_\gamma f(z)\, |dz| - \int_{\gamma_1} f(z)\, |dz| + \int_{\gamma_2} f(z)\, |dz|.$$

(4) *If $\gamma$ is (piecewise) $c^1$, then we have*

$$\int_\gamma f(z)\, dz = \int_a^b f \circ \gamma(t) \dot{\gamma}(t)\, dt$$

*and*

$$\int_\gamma f(z)\, |dz| = \int_a^b f \circ \gamma(t) |\dot{\gamma}(t)|\, dt.$$

**Definition.** *A function $f : \Omega \to \mathbb{C}$ defined on some open set $\Omega \subseteq \mathbb{C}$ is called* (complex) differentiable *at some $z_0 \in \Omega$ if there exists some $f'(z_0) \in \mathbb{C}$ that satisfies*

$$f(z) = f(z_0) + f'(z)(z - z_0) + o(|z - z_0|)$$

*as $z \to z_0$. That is, for all $\epsilon > 0$, there exists $\delta > 0$ that satisfies*

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon |z - z_0|$$

*whenever $|z - z_0| < \delta$. Equivalently, the limit $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists for all $z \in \mathbb{C} \setminus \{z_0\}$ and equals $f'(z_0)$.*

**Remark.** *We make the following remarks about $f$.*

(1) *Holomorphic means complex differentiable. Meromorphic means complex differentiable except at the poles of $f$.*

(2) *If $f'(z_0)$ exists, then it is unique.*

(3) *The standard proofs of the product and chain rules carry over to complex differentiation.*

(4) *If we split the above into real and imaginary parts via $f(x+iy) = u(x+iy)+iv(x+iy)$, then we can write the matrix equation*

$$\begin{bmatrix} u(x+iy) \\ v(x+iy) \end{bmatrix} = \begin{bmatrix} u(x_0+iy_0) \\ v(x_0+iy_0) \end{bmatrix} + \begin{bmatrix} \operatorname{Re} f(z_0) & \operatorname{Im} f(z_0) \\ \operatorname{Im} f'(z_0) & \operatorname{Re} f'(z_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}.$$

**Proposition** (Cauchy-Riemann equations)**.** *The function $f$ is holomorphic at $z_0$ if and only if we have*

$$\nabla u = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla v,$$

*where the matrix in the equation represents a rotation by $-90°$.*

**Remark.** *We say the following:*

(1) *A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ being differentiable means more than the mere existence of partial derivatives. Indeed, it means more than mere existence of every directional derivative. For instance, consider*

$$u(re^{i\theta}) := \begin{cases} 0 & \text{if } 0 \le r < \theta < 2\pi, \\ 1 & \text{otherwise}. \end{cases}$$

*In this example, all directional derivatives exist and vanish, but $u$ is not even continuous at $(0,0)$.*

(2) *The* Wirtinger derivatives *are given by*

$$\partial = \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \frac{1}{2}(\partial_x - i\partial_y),$$
$$\bar\partial = \partial_{\bar z} = \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{2}(\partial_x + i\partial_y).$$

*With this notation, $f = u + iv$ obeys the Cauchy-Riemann equations if and only if $\bar\partial f = 0$. Since $u$ and $v$ obey the Cauchy-Riemann equations, the Wirtinger derivatives imply*

$$\begin{aligned} \bar\partial f &= \frac{1}{2}(\partial_x + i\partial_y)(u+iv) \\ &= \frac{1}{2}(u_x + iv_x + iu_y - v_y) \\ &= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y) \end{aligned}$$

*and*

$$\begin{aligned} \partial f &= \frac{1}{2}(\partial_x - i\partial_y)(u+iv) \\ &= \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y) \\ &= f'(z). \end{aligned}$$

(3) *Pointwise differentiability is a bit of a dead-end concept. Form ost nice theorems (such as the Fundamental Theorem of Calculus), we need more than just $C^1$ for instance. Whenever this is too much to work with, most people turn to distributional derivatives. If we let $\Omega \subseteq \mathbb{C}$ be an open set, We say that a continuous function $f : \Omega \to \mathbb{C}$ is a* distributional *solution of the Cauchy-Riemann equations if, for all $\varphi \in C_c^\infty(\Omega)$, we have*

$$\int_\Omega f(\bar\partial\varphi)\,dx\,dy = 0.$$

*If $f$ is $C^1$, then we can integrate by parts:*

$$-\int_\Omega (\bar\partial f)\varphi\,dx\,dy = 0$$

*for all $\varphi \in C_c^\infty(\Omega)$. Thus, we have $\bar\partial f \equiv 0$, and so $f$ obeys the CAuchy-Riemann equations in the usual sense.*

(4) *The Jacobian of a holomorphic and $C^1$ function $f = u + iv$ is given by*

$$\begin{aligned} \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} &= u_x v_y - v_x u_y \\ &= u_x u_x - v_x(-v_x) \\ &= u_x^2 + v_x^2 \\ &= |\nabla u|^2 \\ &= |\nabla v|^2 \\ &= |f'(z)|^2. \end{aligned}$$

(5) *A holomorphic function is conformal at points where its derivative is nonzero.*

We consider $f : \mathbb{C} \to \mathbb{C}$ given by $\cdot z \mapsto z^2$, which is equivalent to saying $x + iy \mapsto (x^2 - y^2 + i(2xy)$. This implies $|f'(z)|^2 = 4|z|^2$.

**Theorem** (Goursat's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open. If $f : \Omega \to \mathbb{C}$ is (pointwise) holomorphic (and thus continuous), and if $T \subseteq \Omega$ is a solid closed triangle with an arc-length parametrization along $\partial T$ and oriented counterclockwise, then we have*

$$\int_{\partial T} f(z)\, dz = 0.$$

*Proof.* If we assume the case area$(T) = 0$, then we have $\int_{\partial T} f(z)\, dz = 0$ for any continuous function $f$, since we are simply traversing the same path once in this direction. So insetad we assume the case area$(T) > 0$. To achieve our goal, it suffices to show

$$\left| \int_{\partial T} f(z)\, dz \right| < \epsilon \operatorname{area} T$$

for all $\epsilon > 0$. By bisecting each side, we can express any triangle as the union of four similar triangles $T_i$ for $i = 1, 2, 3, 4$. Orient each smaller triangle $T_i$ countereclockwise to see that we get

$$\int_{\partial T} f(z)\, dz = \sum_{i=1}^{4} \int_{\partial T_i} f(z)\, dz.$$

This leads us to consider subdiviging recursively each traingle, starting with the original triangle $T$, subject to the stopping rule: If $T'$ satisfies

$$\left| \int_{\partial T'} f(z)\, dz \right| \le \epsilon \operatorname{Area}(T'),$$

then we terminate our algorithm of subdividing the triangles. We claim that this algorithm terminates in finitely many steps. If not, then there would exist an infinite nested sequence of triangles $T \supset T_1 \supset T_2 \supset T_3 \supset \cdots$. If we denote by diameter$(T)$ to be the supremum of the distance between any two points in $T$, then we observe diameter$(T_k) = 2^{-1}$ diameter$(T_{k-1}) = \cdots = 2^{-k}$ diameter$(T) \to 0$ as $k \to \infty$. The Cantor Intersection Theorem then states that there exists a unique point $z_0 \in \bigcap_{k=1}^{\infty} T_k$. (This is because, if $K$ is compact and closed sets $C_\alpha \subset K$ have the finite intersection property, then we have $\bigcap_\alpha C_\alpha \neq \emptyset$. Any finite subfamilyl has nonempty intersection.) As $f$ is differentiable at $z_0$ (because we assumed that $f$ is holomorphic on $\Omega$), there exists $\delta > 0$ such that $|z - z_0| < \delta$ implies

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \frac{\epsilon}{100} \frac{\operatorname{area}(T)}{\operatorname{diameter}(T)^2}.$$

Now choose $k \in \mathbb{N}$ that satisfies diameter$(T_k) < \delta$. As we have $z_0 \in T_k$ for each positive integer $k$, we have $|z - z_0| < \delta$ for all $z \in T_k$. As $z_0 \in T_k$, for all $z \in T_k$ we have $|z - z_0| < \delta$. Finally, as similar triangles perserve ratios, we have $\frac{\operatorname{area}(T_k)}{\operatorname{diameter}(T_k)^2} = \frac{\operatorname{area}(T)}{\operatorname{diameter}(T)^2}$. So we get

$$\begin{aligned}
\left| \int_{\partial T_k} f(z)\, dz - \int_{\partial T_k} f(z_0) + f'(z_0)(z - z_0)\, dz \right| &\le \frac{\epsilon}{100} \frac{\operatorname{area}(T_k)}{\operatorname{diameter}(T_k)^2} \operatorname{diameter}(T_k) \\
&= \frac{\epsilon}{100} \frac{\operatorname{area}(T)}{\operatorname{diameter}(T)^2} \operatorname{diameter}(T_k) \\
&\le \frac{\epsilon}{100} \frac{\operatorname{area}(T)}{\operatorname{diameter}(T)^2} \operatorname{perimeter}(T_k) \\
&\le \frac{\epsilon}{100} \frac{\operatorname{area}(T)}{\operatorname{diameter}(T)^2} (3 \operatorname{diameter}(T_k)) \\
&= \frac{3\epsilon}{100} \frac{\operatorname{area}(T)}{\operatorname{diameter}(T)^2} \operatorname{diameter}(T_k),
\end{aligned}$$

suggesting that this recursion process holds for all positive integers $k$. But this contradicts the earlier assumption that our recursion should have stopped at or before $T_k$. This finishes our proof of the claim. Finally, by the stopping rule for the second inequality below, we have

$$\begin{aligned}
\left| \int_{\partial T} f(z)\, dz \right| &\le \sum_{\text{finite subdivision}} \left| \int_{\partial T_i} f(z)\, dz \right| \\
&\le \sum_{\text{subdivision}} \left| \int_{\partial T_i} f(z)\, dz \right| \\
&= \epsilon \operatorname{area}(T),
\end{aligned}$$

as desired. ◻

Now, let $\varphi_\epsilon$ be the standard mollifier for all $\epsilon > 0$. we claim that $f * \varphi_\epsilon$ is $C^1$ and holomorphic (distributionally and also pointwise $C^1$). We have

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}^d} f(x-y)g(y)\, d\mathrm{vol}(y) \\ &= \int_{\mathbb{R}^d} g(x-w)f(w)\, d\mathrm{vol}(w) \\ &= g * f(x). \end{aligned}$$

A *mollifer* $\varphi \in C_c^\infty(\mathbb{R}^d)$ satisfies $\int_{\mathbb{R}^d} \varphi\, d\mathrm{vol} = 1$. We then set $\varphi_\epsilon(x) := \epsilon^{-d}\varphi(\frac{x}{\epsilon})$ for all $\epsilon > 0$, and we notice

$$\int_{\mathbb{R}^d} \varphi_\epsilon\, d\mathrm{vol} = \int_{\mathbb{R}^d} \varphi\, d\mathrm{vol}.$$

Replace $f$ by $f * \varphi_\epsilon$; this is a process called *mollification*.

**Proposition** (Distributional Goursat Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be continuous and distributionally holomorphic. Then, for any solid closed triangle $T \subseteq \Omega$, we have*

$$\int_{\partial T} f(z)\, dz = 0.$$

*Proof.* Pick a mollifer $\varphi$ with $\operatorname{supp}\varphi \subseteq B(0,1)$. Consider $\varphi_\epsilon$ with $\epsilon < \operatorname{dist}(\Omega^c, T)$. Then we have that $f * \varphi_\epsilon$ is well-defined at every point in some neighborhood of $T$. If $|\zeta| < \epsilon$ and $z \in T$, then we have $z - \zeta \in \Omega$. Now, we claim that $f * \varphi_\epsilon$ is $C^1$ and holomorphic (distributionally, and therefore $C^1$ and also pointwise). By Goursat's Theorem, we alreay have

$$\int_{\partial T} f(z)\, dz = 0.$$

So we have

$$\begin{aligned} (f*\varphi)(x+h) - (f*\varphi)(x) - h \cdot (\nabla\varphi * f)(x) &= \int_{\mathbb{R}^d} \varphi_\epsilon(x+h-y)f(y)\, d\mathrm{vol}(y) - \int_{\mathbb{R}^d} \varphi_\epsilon(x-y)f(y)\, d\mathrm{vol}(y) \\ &\quad - \int_{\mathbb{R}^d} h \cdot \nabla(x-y)f(y)\, d\mathrm{vol}(y) \\ &= \int_{\mathbb{R}^d} (\varphi_\epsilon(x+h-y) - \varphi_\epsilon(x-y) - h \cdot \nabla_\epsilon(x-y))f(y)\, d\mathrm{vol}(y) \\ &= \int_{\mathbb{R}^d} O(h^2)f(y)\, d\mathrm{vol}(y) \\ &= O(h^2) \end{aligned}$$

as $h \to 0$. So $\varphi * f$ is differentiable and satisfies $\nabla(\varphi * f) = (\nabla\varphi) * f$. It remains to show $\bar{\partial}(\varphi_\epsilon * f) \equiv 0$; this is equivalent to showing

$$\begin{aligned} \iint_{\mathbb{R}^d} (\partial_{x'} + i\partial_{y'})\varphi_\epsilon(x-x', y-y')f(x',y')\, d\,\mathrm{area}(x',y') &= -\iint_{\mathbb{R}^d} \left(\frac{\partial}{\partial x'} + i\frac{\partial}{\partial y'}\varphi_\epsilon(x-x', y-y')f(x',y')\, d\,\mathrm{area}\right) \\ &= 0, \end{aligned}$$

by the definition of distributionally holomorphic, as applied to the test function $(x', y') \mapsto \varphi_\epsilon(x-x', y-y') \in C_c^\infty(\Omega)$ whenever $\epsilon < \operatorname{dist}(\Omega^c, T)$. $\square$

**Remark.** *The* Cauchy-Goursat Integral Theorem *asserts that if $f$ is holomorphic inside of and on the closed curve $\gamma$, then we have $\int_\gamma f(z)\, dz = 0$.*

**Theorem** (Cauchy Integral Formula)**.** *Fix $0 < r < R < \infty$ and $z_0 \in \mathbb{C}$. If $\Omega = \overline{B(z,R)}$ is open, $f$ is holomorphic on $\Omega$, and $|w - z| < R$, then we have*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\, dz,$$

*where $\gamma : [0, 2\pi] \to \mathbb{C}$ is the curve given by $\theta \mapsto z_o + Re^{i\theta}$ (i.e. $\partial B(z_0, R)$). More generall, if $f$ is holomorphic in a neighborhood of $\{z \in \mathbb{C} : r \le |z - z_0| \le R\}$ and $r < |w - z_0| < R$, then we have*

$$f(w) = \frac{1}{2\pi i}\int_{\gamma+\sigma} \frac{f(z)}{z-w}\, dz,$$

*where $\sigma$ is the clockwise contour $\theta \mapsto z_0 + re^{-i\theta}$ for all $\theta \in [0, 2\pi]$.*

*Proof.* As $f$ is continuous at $w \in \Omega$, we can choose the contour $\sigma_\delta$ given by $\theta \mapsto \omega + \delta e^{i\theta}$ to get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_\delta}\frac{f(z)}{z-w}\,dz &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(w+\delta e^{i\theta}}{\delta e^{i\theta}}i\delta e^{i\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(w) + (f(w+\delta e^{it\theta} - f(w))\,d\theta \\ &= \frac{1}{2\pi}f(w) + \int_0^{2\pi} o(1)\,d\theta \\ &\to f(w)\end{aligned}$$

as $\delta \to 0$. So it remains to show

$$\frac{1}{2\pi i}\int \frac{f(z)}{z-w}\,dz = 0,$$

where this integral is evaluated over a pair of rectifiable contours (think of a point $w$ inside a smaller clockwise-oriented contour $\sigma_\delta$, which is in turn completely inside a larger counterclockwise-oriented contour). By simple computation, the value is the limit of integrals over polygonal contours. This allows us to consider a contour contained completely inside a polygonal shape, as refining partitions only makes the approximations better. Note that nodes of triangles share rays from $w$, like spokes of a bicycle wheel. These additions are canceled by the adjacent quadrilaterals. □

**Corollary.** *Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Also fix $z_0 \in \Omega$. Then we can expand $f$ in a power series about $z_0$ with a radius of convergence that is no less than* $\operatorname{dist}(z_0, \partial\Omega^c)$.

*Proof.* Choose some $0 < r < \operatorname{dist}(z_0, \Omega^c)$. We observe

$$\begin{aligned}\frac{1}{(z_0 + re^{i\theta}) - w} &= \frac{1}{1 - \frac{w-z_0}{re^{i\theta}}}\frac{1}{re^{i\theta}} \\ &= \sum_{l=0}^{\infty}\frac{(w-z_0)^l}{r^{l+1}e^{i(l+1)\theta}}\end{aligned}$$

as a uniformly (and absolutely) convergent series of functions on $[0, 2\pi] \times \{w \in \mathbb{C} : |z_0 - w| < \delta\}$ for any $\delta < r$. For any such $w$, we have

$$\begin{aligned}f(w) &= \frac{1}{2\pi i}\int_{\theta\mapsto z_0 + re^{i\theta}}\frac{f(z)}{z-w}\,dz \\ &= \sum_{l=0}^{\infty}\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{r^l e^{il}}\,d\theta,\end{aligned}$$

which implies

$$\begin{aligned}|f(w)| &\le \sum_{l=0}^{\infty}\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{r^l e^{il}}\,d\theta\right| \\ &\le \sum_{l=0}^{\infty}\frac{1}{2\pi}\frac{2\pi\sup\{|f(z)| : |z-z_0| < r\}}{r^l} \\ &= \sum_{l=0}^{\infty}\frac{\sup\{|f(z)| : |z-z_0| < r\}}{r^l}.\end{aligned}$$

So the series of integrals converges for all $|w - z_0| \le \delta$ if $\delta < r$. So we have just represented $f$ as a power series with a radius of convergence of at least $\delta$. By the freedom in choosing $\delta$ and $r$, this gives the radius of convergence no smaller than $\operatorname{dist}(z_0, \Omega^c)$. Now, differentiating the series representation repeatedly, we find that $f$ is smooth on $\Omega$ and satisfies

$$\frac{f^{(k)}(w)}{k!} = \frac{1}{2\pi i}\int_{\theta\mapsto z_0+re^{i\theta}}\frac{f(z)}{(z-w)^{k+1}}\,dz.$$

From our previous estimates which we used to prove convergence of the series, we conclude

$$\left|\frac{f^{(k)}(w)}{k!}\right| \le \frac{\sup\{|f(re^{i\theta})| : 0 \le \theta < 2\pi}{|w-z_0|^{k+1}},$$

as desired. □

The preceding proof implies immediately the following corollary.

**Corollary** (Cauchy estimates)**.** *If $f : \Omega \to \mathbb{C}$ is holomorphic and $z_0 \in \Omega$, and we have $r < \operatorname{dist}(z_0, \Omega^c)$, then*

$$\left|\frac{f^{(k)}(z)}{k!}\right| \le \frac{\sup\{|f(re^{i\theta})| : 0 \le \theta < 2\pi}{(r - |w-z_0|)^{k+1}},$$

**Theorem.** *Suppose $f : \Omega \to \mathbb{C}$ is continuou and $\Omega \subseteq \mathbb{C}$ is open. If $f$ satisfies $\int_{\partial T} f(z)\, dz = 0$ for every closed solid triangle $T \subseteq \Omega$, then $f$ is holomorphic.*

*Proof.* This is really a corollary of the previous two proofs: the proof of the Cauchy Integral Formula only assumed a hypothesis, which says that $f$ is holomorphic, in order to apply Goursat's Theorem. Now, the output of Goursat's Theorem is a hypothesis. Secondly, we already proved that, if $f$ can be represented by a Cauchy Integral Formula, then it is analytic, and so $f$ is infinitely differenitable, which implies $f$ is holomorphic. □

**Corollary.** *If $f$ is holomorphic on $\Omega$, then so is $f'$.*

*Proof.* Since $f$ is holomorphic on $\Omega$, it is analytic on $\Omega$. So we can consider a sequence $\{f_n\}_{n=1}^{\infty}$ written as

$$f_n(z) = \sum_{k=0}^{n} \frac{f^{(n)}(z_0)}{k!}(z - z_0)^k.$$

Then we have $f_n \to f$ in some neighborhood of $z_0$ as $n \to \infty$. This signifies that $f'$ exists and satisfies $f' = \lim_{n\to\infty} f_n'$. In particular, we have

$$f'(z) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(z_0)}{k!}(z - z_0)^k,$$

which means $f'$ is analytic, and therefore holomorphic. □

**Theorem** (Uniqueness Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a connected set, and suppose $f : \Omega \to \mathbb{C}$ is holomorphic. If there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$ that converges to some point $z_0 \in \Omega$ such that we have $f(z_n) = 0$ for each positive integer $n$, then we have $f \equiv 0$.*

*Proof.* We know that $f$ is analytic near $z_0$. So, for all small enough $\delta > 0$ such that the set described by $|z - z_0| < \delta$ is a subset of $\Omega$, we have the power series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

But we have $f(z_0) = 0$ by continuity. And we have

$$\begin{aligned} f'(z_0) &= \lim_{z_n \to z_0} \frac{f(z_n) - f(z_0)}{z_n - z_0} \\ &= 0. \end{aligned}$$

We already assumed $f(z_n) = 0$ in the hypothesis. So we get $z_n \to z_0$ by hypothesis. But then we have

$$\begin{aligned} \frac{f''(z_0)}{z!} &= \lim_{z \to z_0} \frac{f(z_n) - f(z_0)}{(z_n - z_0)^2} \\ &= 0. \end{aligned}$$

Now let us consider the set $O \subseteq \Omega$, as these points $z \in \Omega$ on which $f$ vanishes in a neighborhood of $z$. For insteance, we have $z_0 \in O$, so that we can ensure $O \neq \emptyset$. It remains to show that $O$ is closed and open ("clopen"). By our definition of $O$, we have that $O$ is open. And our recent argument also shows that $O$ is closed. Since we established $O \neq \emptyset$, we must conclude $O = \Omega$, which signifies that $f$ vanishes on $\Omega$. □

**Remark.** *We will make some remarks about the Uniqueness Theorem (also called the Identity Principle).*

(1) *We must require $z_0 \in \Omega$, rather than $z_0 \in \partial\Omega$, for the Uniqueness Theorem to work. For example, the function $z \mapsto \sin(\frac{1}{z})$ is defined on $\mathbb{C} \setminus \{0\}$. There is a sequence $\{z_n\}_{n=1}^{\infty}$ with $z_n = \frac{1}{2\pi n}$ for each positive integer $n$ and $z_n \to 0 \in \{0\} = \partial(\mathbb{Z} \setminus \{0\})$ as $n \to \infty$. But we have $\sin(\frac{1}{z_n}) = \sin(2\pi n) = 0$ for each positive integer $n$ but the limit $\lim_{z_n \to z_0} \sin(\frac{1}{z_n}) = \sin(\infty)$ does not exist.*

(2) *An equivalent assertion of the Uniqueness Theorem says that, if two function $f$ and $g$ are holomorphic on $\Omega$ and there exists $\{z_n\}_{n=1}^{\infty} \subset \Omega$ such that we have $f(z_n) = g(z_n)$ for each positive integer $n$, then we have $f \equiv g$ on $\Omega$.*

(3) *As $\Omega \subset \mathbb{C}$ is an open set, saying that $\Omega$ is connected is equivalent to saying $\Omega$ is path connected. This does mean that the only closed and open ("clopen") subsets of $\Omega$ are $\emptyset$ and $\Omega$.*

**Definition.** *A function $f$ is said to be* meromoprhic *on $\Omega \subseteq \mathbb{C}$ if there exists a discrete (and therefore countable) subset $\{z_k\}_{k=1}^{n} \subset \Omega$ for any positive integer $n$ such that $f$ is holomorphic on $\Omega \setminus \{z_n\}$ and $\lim_{z \to z_k} |f(z)| = \infty$ for each positive integer $k$. Such points $z_k$ are called* poles.

**Remark.** *Note that we can extend $f$ to a continuous function on $\Omega$ by taking in values of the Riemann sphere. From this perspective, the poles are the preimages of the north pole on the Riemann sphere.*

**Proposition.** *If $z \in \Omega$ is a pole of a meromorphic function $f$, then there is a positive integer $k$ and numbers $a_1, \ldots, a_k \in \mathbb{C}$ that satisfy $z \mapsto f(z) - \sum_{j=1}^{k} a_j (z - z_0)^{-j}$, where the summation term is the* principal part *of the function.*

*Proof.* Consider the map $z \mapsto \frac{1}{f(z)}$. Since $p$ is a pole of $f$, it follows that we have $\frac{1}{f(z)} \to 0$ as $z \to p$. So Riemann's Removable Singularity Theorem allows us to extend $\frac{1}{f}$ holomorphically to a neighborhood of $p$. Such an extension admits the power series expansion

$$\frac{1}{f(z)} = \sum_{k=1}^{\infty} c_k (z-p)^k$$

about $p$, in a neighborhood of $p$. Note that $\frac{1}{f}$ is not the zero series because we have $f(z) \in \mathbb{C} \setminus \{0\}$ in some deleted neighborhood of $p$. Let $n$ be the index of the first nonzero term ($c_n \neq 0$). We have $(z-p)^n f(z) \to \frac{1}{c_n} \in \mathbb{C}$ as $z \to p$. By Riemann's Removable Singulairty Theorem, $(z-p)^k f(z)$ extends holomorphically to a neighborhood of $p$. So we have

$$(z-p)^n f(z) = \sum_{k=0}^{\infty} b_k (z-p)^k$$

with $b_0 \neq 0$. Equivalently, we can write

$$f(z) = \sum_{k=0}^{\infty} b_k (z-p)^{k-n},$$

as desired. □

**Definition.** *If $f$ is holomorphic in a deleted neighborhood of $z_0$ and $z_0$ is not removable and not a pole, then we say that $z_0$ is an* essential singularity.

**Example.** *If we deifne $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := e^{\frac{1}{z}}$, then $z = 0$ is an essential singularity of $f$ because we can achieve different values of $\lim_{z \to 0} f(z)$, depending on how we send $z \to 0$. For instance, along the path of the positive $x$-axis, we have $e^{\frac{1}{z}} \to \infty$, but along the path of the negative $x$-axis, we have $e^{\frac{1}{z}} \to 0$.*

**Theorem** (Casorati-Weierstrass Theorem)**.** *In every deleted neighborhood of an essential singularity, a function achieves a dense set of values in $\mathbb{C}$.*

*Proof.* See Exercise 2 of Homework 3. □

**Theorem.** *If $f$ is holomorphic in some disk $B \subseteq \mathbb{C}$, then there exists a primitive $F : B \to \mathbb{C}$; i.e. there exists a holomorphic function $F$ that satisfies $F' = f$. Moreover, for any rectifiable curve $\gamma : [a,b] \to B$, we have*

$$\int_\gamma f(z)\, dz = F \circ \gamma(b) - F \circ \gamma(a).$$

*In particular, we have $\int_\gamma f(z)\, dz = 0$ if $\gamma : [a,b] \to B$ is a loop.*

*Proof.* Without loss of generality, we can center the ball $B$ at $z = 0$. Choose a primitive $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) := \int_{\gamma_z} f(w)\, dw,$$

where $\gamma_z : [0,1] \to B$ is defined by $\gamma_z(t) := tz$. (We are choosing this primitive $F$ to satisfy $F(0) = 0$.) For all $z, z+h \in B$ (for any $h > 0$), we have

$$\begin{aligned} F(z+h) - F(z) &= \int_{\gamma_1} f(w)\, dw + \int_{\gamma_2} f(w)\, dw \\ &= 0 + (f(z)h + o(h)) \\ &= f(z)h + o(h) \\ &\to 0, \end{aligned}$$

as $h \to 0^+$, where $\gamma_1$ is the closed curve going from 0 to $z+h$, then $z+h$ to $z$, then $z$ to 0, and $\gamma_2$ is the line going from $z$ to $z+h$. Therefore, $\lim_{\substack{h\to 0 \\ h \neq 0}} \frac{F(z+h)-F(z)}{h}$ exists and satisfies

$$f(z) = \lim_{\substack{h\to 0 \\ h \neq 0}} \frac{F(z+h) - F(z)}{h}.$$

Therefore, $F$ is differentiable and satisfies $F' = f$. Now, it remains to show

$$\int_\gamma f(z)\, dz = F \circ \gamma(b) - F \circ \gamma(a).$$

If $\gamma$ is of class $C^1$, then this statement is just the Fundamental Theorem of Calculus. But then polygons follow by direct comparison, and rectifiable curves follow by polygonal approximation. □

**Remark.** *What we just proved relies on the property that the disk $B$ is convex. This result does not hold for arbitrary open sets. For instance, $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ has a primitive on the non-convex set $\mathbb{C} \setminus \{0\}$, but no such primitive of $f$ exists at $z = 0$.*

**Remark.** *Note that part of the proof which uses the assumption that $f$ is holomorphic to conclude $\in_{\partial T} f(w)\, dw = 0$ is the assertion of Morera's Theorem. So $f$ has a holomorphic primitive, which is $F$. But then, by itsp ower series representation, we know that $F' = f$ is holomorphic.*

**Example.** *Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^k$ for any integer $k$. If $\gamma$ is the unit circle, then we have*

$$\int_\gamma z^k\, dz = \begin{cases} 0 & \text{if } k \in \mathbb{Z} \setminus \{-1|, \\ 2\pi & \text{if } k = -1. \end{cases}$$

*Also, we have $F(z) := \frac{z^{k+1}}{k+1}$ for all positive integers $k$, which means $f$ has a primitive globally on $\mathbb{C} \setminus \{0\}$. In fact, $f$ has a primitive on $\mathbb{C}$ if $k$ is nonnegative, but only on $\mathbb{C} \setminus \{0\}$ is $k$ is negative. In particular, $f$ has a primitive on any ball $B \subseteq \mathbb{C} \setminus \{0\}$; that is, on any ball that does not contain or pass through the origin $z = 0$.*

**Definition.** *Let $\Omega \subseteq \mathbb{C}$ be an open set. We say that the two continuous paths $\gamma_0, \gamma_1 : [0, 1] \to \Omega$ are* homotopic*, with endpoints fixed, in $\Omega$ if there exists a* homotopy*, which is a continuous map $H : [0, 1] \times [0, 1] \to \Omega$ that satisfies $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$, and also $H(0, t)$ and $H(1, t)$ are constant (beginning and end, respectively, are fixed).*

**Example.** *The two paths $\gamma_0, \gamma_1$ that share their endpoints are homotopic in $\mathbb{C}$. But the same two paths $\gamma_0, \gamma_1$ are not homotopic on $\mathbb{C} \setminus \{0\}$.*

**Definition.** *We say that two loops $\gamma_0, \gamma_1 : [0, 1] \to \mathbb{Z}$ are* freely homotopic in $\Omega$ *if there exists a continuous map $H : [0, 1] \times [0, 1] \to \Omega$ that satisfies $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$ and also requires that we have $H(0, t) = H(1, t)$ for all $t \in [0, 1]$.*

**Definition.** *A loop that is freely homotopic to a constant path is called* contractible.

**Theorem.** *Let $\Omega \subseteq \mathbb{C}$ be open, suppose $f : \Omega \to \mathbb{C}$ is homolorphic, and consider the two paths/loops $\gamma_0, \gamma_1$ in $\Omega$. Then:*

(1) *If $\gamma_0, \gamma_1$ are homotopic (with endpoints fixed), then we have*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

*Proof.* Notice that $H([0, 1] \times [0, 1])$ is compact. Hence, there exists $r > 0$ that saitsfies $\operatorname{dist}(H(s, t), \Omega^c) > 10r$ for every $s, t \in [0, 1]$. We have seen that there are polygons $\tilde{\gamma}_0, \tilde{\gamma}_1$ with notes on $\gamma_0, \gamma_1$ that lie inside $\Omega$ and yield line integrals approximating those of $\gamma_0, \gamma_1$. In particular, given $\epsilon > 0$, there exists $\delta > 0$ so that any polygons built with mength smaller than $\delta$ obey

$$\left| \int_{\gamma_0} f(z)\, dz - \int_{\tilde{\gamma}_0} f(z)\, dz \right| + \left| \int_{\gamma_1} f(z)\, dz - \int_{\tilde{\gamma}_1} f(z)\, dz \right| < \epsilon.$$

If necessary, we can further reduce $\delta$ so that, for all $t \in [0, 1]$ satisfying $|t - t'| + |s - s'| < 2\delta$, we have $|H(s, t) - H(s', t')| < r$; that is, the distance between samples is small relative to their distances to $\Omega^c$. Now, we build a bunch of polygonal paths based on the nodes $0 = s_0 < s_1 < \cdots < s_n = 1$ on $\gamma_0$ and $0 = t_0 < t_1 <)m = 1$ on $\tilde{\gamma}_0$. ◻

(2) *If $\gamma_0, \gamma_1$ are freely homotopic (loops) in $\Omega$, then we have*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

(3) *If the loop $\gamma$ is contractible, then we have*

$$\int_\gamma f(z)\, dz = 0.$$

**Corollary.** *Let $\Omega \subseteq \mathbb{C}$ be open and simply connected. If $f : \Omega \to \mathbb{C}$ is holomorphic, then there exists a (holomorphic) primitive $F : \Omega \to \mathbb{C}$ that satisfies $F' = f$.*

*Proof.* By working on connected components, we may assume that $\Omega$ is connected, which implies that $\Omega$ is polygonally path connected. Choose $z_0 \in \Omega$ and define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) := \int_{\gamma_z} f(z)\, dz,$$

where $\gamma_z$ is a rectifiable path in $\Omega$ from $z_0$ to $z$. To check that $F$ is well-defined, concatenate $\gamma_z$ with the inverse of another rectifiable path $\tilde{\gamma}_z$ from $z_0$ to $z$, wiritng $\tilde{\gamma}_z^{-1} \circ \gamma_z = \Gamma$. Then $\Gamma$ is a loop in $\Omega$. Since $f$ is holomorphic on $\Omega$, there are no singularities on $\Gamma$. This implies that $\Gamma$ is contractible, which means we have

$$\int_\Gamma f(z)\, dz = 0.$$

In fact, we have

$$\begin{aligned}0 &= \int_\Gamma f(z)\,dz\\ &= \int_{\gamma_z} f(z)\,dz + \int_{\tilde{\gamma}_z^{-1}} f(z)\,dz\\ &= \int_{\gamma_z} f(z)\,dz - \int_{\tilde{\gamma}_z} f(z)\,dz,\end{aligned}$$

from which we conclude

$$\int_{\gamma_z} f(z)\,dz = \int_{\tilde{\gamma}_z} f(z)\,dz$$

thereby establishing that $F$ is well-defined. Now it remains to verify $F' = f$. Consider a segment from $z$ to $z+h$ for all $|h| > 0$, and let $\gamma_{z+h}$ be $\gamma_z$ joined by a line segment from $z$ to $z+h$. Then we have

$$\begin{aligned}F(z+h) - F(z) &= \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw\\ &= \int_{[z,z+h]} f(w)\,dw\\ &= hf(z) + O(h^2),\end{aligned}$$

and so we have

$$\frac{F(z+h)-F(z)}{h} = f(z) + \frac{O(h^2)}{h},$$

from which we can send $h \to 0$ both sides to obtain $F'(z) = f(z)$ for all $z \in \Omega$. □

**Corollary.** *If $\Omega \subseteq \mathbb{C} \setminus \{0\}$ is simply connected, then for any given $z_0 \in \Omega$, and $w_0 \in \mathbb{C}$ with $z_0 = e^{w_0}$, there exists a holomorphic function $f : \Omega \to \mathbb{C}$ that satisfies $f(z_0) = w_0$ and $z = e^{f(z)}$. In other words, we can build a branch of* Log *on $\Omega$ that respects $z_0 = e^{w_0}$.*

*Proof.* Define $f : \Omega \to \mathbb{C}$ by

$$f(z) := w_0 + \int_{[z_0,z]} \frac{dw}{w},$$

where $[z_0, z]$ denotes a rectifiable path from $z_0$ to $z$. (Recall that the value of the line integral is the same over *any* rectifiable path from $z_0$ to $z$.) From a previous argument, we get $f'(z) = \frac{1}{z}$. Now, we have

$$\begin{aligned}\frac{d}{dz}\left(\frac{1}{z}e^{f(z)}\right) &= -\frac{1}{z^2}e^{f(z)} + \frac{1}{z}e^{f(z)}f'(z)\\ &= -\frac{1}{z^2}e^{f(z)} + \frac{1}{z}e^{f(z)}\frac{1}{z}\\ &= 0,\end{aligned}$$

which means that $\frac{1}{z}e^{f(z)}$ is constant on $\Omega$. In fact, because we have

$$\begin{aligned}\frac{1}{z_0}e^{f(z_0)} &= \frac{1}{e^{w_0}}e^{w_0 + \int_{z_0}^{z_0}\frac{dw}{w}}\\ &= \frac{1}{e^{w_0}}e^{w_0}\\ &= 1,\end{aligned}$$

which means we have in fact $\frac{1}{z}e^{f(z)} = 1$, or equivalently $z = e^{f(z)}$, for all $z \in \Omega$. Now, we will need to show $F' = f$. Let $\gamma_{z+h}$ be $\gamma_z$ joined by the line segment from $z$ to $z+h$. Then we have

$$\begin{aligned}F(z+h) - F(z) &= \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw\\ &= \int_{[z,z+h]} f(w)\,dw\\ &= hf(z) + O(h^2),\end{aligned}$$

and so we have

$$\frac{F(z+h)-F(z)}{h} = f(z) + \frac{O(h^2)}{h},$$

from which we can send $h \to 0$ both sides to obtain $F'(z) = f(z)$ for all $z \in \Omega$. Now, to show uniqueness, we consider another holomorphic function $g : \Omega \to \mathbb{C}$ that satisfies $g(z_0) = w_0$ and $z = e^{g(z)}$. Then we can take the derivative of both sides of $1 = \frac{1}{z}e^{g(z)}$ (from the equivalent equation $z = e^{g(z)}$) to get

$$\begin{aligned} 0 &= \frac{d}{dz}\left(\frac{1}{z}e^{g(z)}\right) \\ &= -\frac{1}{z^2}e^{g(z)} + \frac{g'(z)}{z}e^{g(z)} \\ &= \left(-\frac{1}{z} + g'(z)\right)\frac{e^{g(z)}}{z}, \end{aligned}$$

from which we conclude $g'(z) = \frac{1}{z}$. So we have

$$\begin{aligned} \frac{d}{dz}(e^{f(z)-g(z)}) &= (f'(z) - g'(z))e^{f(z)-g(z)} \\ &= \left(\frac{1}{z} - \frac{1}{z}\right)e^{f(z)-g(z)} \\ &= 0e^{f(z)-g(z)} \\ &= 0, \end{aligned}$$

and so we see that $e^{f(z)-g(z)}$ is constant on $\Omega$, since $\Omega$ is connected. In fact, as $z_0 \in \Omega$ satisfies $f(z_0) = g(z_0)$, we have in particular $e^{f(z_0)-g(z_0)} = e^0 = 1$, meaning that we have $e^{f(z)-g(z)} = 1$ for all $z \in \Omega$. This implies that we must have $f(z) - g(z) = 2\pi k$ for any $k \in \mathbb{Z}$. In particular, we have $0 = f(z_0) - g(z_0) = 2\pi k$, forcing $k = 0$. Therefore, we conclude $f(z) - g(z) = 0$, or $f(z) = g(z)$, proving uniqueness. ◻

**Example.** *Define the function $F : \mathbb{C} \to \mathbb{C}$ by*

$$F(s) := \int_\gamma \frac{z^{-s}}{e^{-2\pi z} - 1}\,dz.$$

*where $\gamma$ is the path described by the map $t \mapsto \frac{1}{2} - |t| + it$ for all $t \in \mathbb{R}$ (this path is the line $y = x - \frac{1}{2}$ below the x-axis and $y = -x + \frac{1}{2}$ above the x-axis). Then, in the strip $\{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$, the integrand $\frac{z^{-s}}{e^{-2\pi z}-1}$ has poles at $z = 2ki$ for all $k \in \mathbb{Z}$, but this integrand is periodic (with period $i$). If we maintain a distance of $\frac{1}{4}$ from the poles, then there exists a uniform bound of the integrand. The integral defining $F$ is absolutely convergent for any $s \in \mathbb{C}$; indeed, along $\gamma$ we have*

$$\begin{aligned} \int_{-\infty}^{\infty} \left|\frac{z^{-s}}{e^{-2\pi z-1}}\right| dt &= \int_{-\frac{3}{2}}^{\frac{3}{2}} \left|\frac{z^{-s}}{e^{-2\pi z-1}}\right| dt + \int_{|t|>\frac{3}{2}} \left|\frac{z^{-s}}{e^{-2\pi z-1}}\right| dt \\ &\le \int_{-\frac{3}{2}}^{\frac{3}{2}} C\,dt + \int_{|t|>\frac{3}{2}} (30|t|)^{|s|}e^{-r+\frac{1}{2}+\pi\operatorname{Im} s}\,dt \\ &< \infty, \end{aligned}$$

*where $C > 0$ is some independent constant. Moreover, $F$ is holomorphic on $\mathbb{C}$ (in other words, $F$ is an entire function) because it is a uniform limit of Riemann sums, which are self-evidently holomorphic. For absolute convergence, we observe that $\int_\gamma f(z)\,dz$ is the limit of integrals over $-T \le t \le T$. We will choose $T \in \{\frac{1}{2}, 1 + \frac{1}{2}, 2 + \frac{1}{2}, 3 + \frac{1}{2}, \dots\}$. Throughout $C_T$, we have $|z| \ge T$; moreover, we have $|e^{-2\pi z} - 1| \ge C > 0$, by considering $\operatorname{Re} z \in (-\infty, -1) \cup [-1, 1) \cup (1, \infty)$ separately. Moreover, we have*

$$\begin{aligned} |z^{-s}| &= |z^{-i\operatorname{Im} s}||z^{-\operatorname{Re} s}| \\ &= |e^{-i\operatorname{Im} s\log z}||z|^{-\operatorname{Re} s} \\ &\le e^{\pi|\operatorname{Im} s|}T^{-\operatorname{Re} s}, \end{aligned}$$

*where $C(s)$ is a constant depending on $s$, and so we have*

$$\begin{aligned} \left|\int_{C_T} \frac{z^{-s}}{e^{-2\pi z} - 1}\,dz\right| &\le \int_{C_T} \frac{|z^{-s}|}{|e^{-2\pi z} - 1|}\,|dz| \\ &\le \int_{C_T} \frac{e^{\pi|\operatorname{Im} s|}T^{-\operatorname{Re} s}}{C}\,|dz| \\ &\lesssim_s T^{-\operatorname{Re} s}T \\ &\to 0 \end{aligned}$$

*as $T \to \infty$, because $\operatorname{Re} z > 1$. (The notation "$\lesssim_s$" means "$\le C_s$".) This leaves us with*

$$\begin{aligned} \sum_{T\ge|n|\ge1} - \int_{right\ way} \frac{z^{-s}}{e^{-2\pi z} - 1}\,dz &= \sum_{T\ge|n|\ge0} -2\pi i(\text{residueat } z = in) \\ &= -2\pi(z - in) + O(|z - in|^2). \end{aligned}$$

*So the residue is*

$$\begin{aligned}\text{residue} &= -\frac{1}{2\pi}e^{-s\log(in)}\\ &= -\frac{1}{2\pi}e^{-s(\log|\pi|+\frac{\pi}{2}\operatorname{signum}(n))}.\end{aligned}$$

*This leads us to saying*

$$\begin{aligned}F(s) &= i\sum_{n=1}^{\infty} n^{-s}(e^{-\frac{i\delta\pi}{2}} + e^{\frac{i\delta\pi}{2}})\\ &= 2i\cos\left(\frac{\pi s}{2}\right)\zeta(s),\end{aligned}$$

*which shows that $F$ is absolutely convergent.*

**Definition.** *Given a rectifiable curve $\gamma : [a,b] \to \mathbb{C}$ and a point $z \in \mathbb{C} \setminus \gamma([a,b])$, we define the* Winding number

$$\operatorname{Wind}_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}.$$

**Proposition.** *The Winding number satisfies the following:*

*(i)* $\operatorname{Wind}_\gamma(z) \in \mathbb{Z}$,

*Proof.* There exists a homotopy $\gamma$ in $\mathbb{C} \setminus \{z\}$ which is $C^1$. Specifically, define

$$G(t) := \int_a^t \frac{\dot\gamma(s)}{\gamma(s)-z}\,ds,$$

and then define

$$H(t) := (\gamma(t)-z)e^{-G(t)}.$$

Then we have

$$\begin{aligned}\frac{dH}{dt} &= \frac{d}{dt}((\gamma(t)-z)e^{-G(t)})\\ &= \dot\gamma(t)e^{-G(t)} - (\gamma(t)-z)e^{-G(t)}\frac{dG}{dt}\\ &= \dot\gamma(t)e^{-G(t)} - (\gamma(t)-z)e^{-G(t)}\frac{\dot\gamma(t)}{\gamma(t)-z}\\ &= 0.\end{aligned}$$

Therefore, as $G(a) = 0$ by definition of $G(t)$, we obtain

$$\begin{aligned}\gamma(a)-z &= (\gamma(a)-z)e^0\\ &= (\gamma(a)-z)e^{-G(a)}\\ &= H(a)\\ &= H(b)\\ &= (\gamma(b)-z)e^{-G(b)}.\end{aligned}$$

As $\gamma$ is a loop, which means $\gamma(a) = \gamma(b)$, we can remove $\gamma(a)-z$ and $\gamma(b)-z$ from both sides of our latest equality to obtain

$$e^{-G(b)} = 1.$$

In other words, we require $G(b) = 2\pi i k$ for some $k \in \mathbb{Z}$, and so we must conclude

$$\begin{aligned}\operatorname{Wind}_\gamma(z) &= \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}\\ &= \frac{1}{2\pi i}\int_a^b \frac{\dot\gamma(s)}{\gamma(s)-z}\,ds\\ &= \frac{1}{2\pi i}G(b)\\ &= \frac{1}{2\pi i}(2\pi i k)\\ &= k,\end{aligned}$$

establishing (i). □

*(ii)* $\mathrm{Wind}_\gamma(z)$ *is constant on connected components of* $\mathbb{C} \setminus \gamma([a,b])$*,*

*Proof.* We note that the map $z \mapsto \frac{1}{\gamma(t)-z}$ is continuous uniformly in $t$, which means the map

$$\begin{aligned} z \mapsto \int_\gamma \frac{w}{w-z}\,dw \\ &= \int_a^b \frac{\dot\gamma(t)}{\gamma(t)-z}\,dt \\ &= G(b) \end{aligned}$$

depends continuously on $z \in \mathbb{C} \setminus \gamma([a,b])$. By part (i), $G(z)$ is an integer. The only continuous integer-valued functions such as $G(z)$ are constant functions, implying in particular that

$$\mathrm{Wind}_\gamma(z) = \frac{1}{2\pi i} G(b)$$

is a constant function. □

*(iii)* $\mathrm{Wind}_\gamma(z) \to 0$ *as* $z \to \infty$*.*

*Proof.* Notice that we have $\frac{\dot\gamma(t)}{\gamma(t)-z} \to 0$ uniformly as $|z| \to \infty$. We recall that, if terms of a sequence converges uniformly to its limit, then the integral of terms of the same sequence converges to the integral of its limit. In particular, we have

$$\begin{aligned} \mathrm{Wind}_\gamma(z) &= \frac{1}{2\pi i} G(b) \\ &= \frac{1}{2\pi i}\int_a^b \frac{\dot\gamma(t)}{\gamma(t)-z}\,dt \\ &\to \int_a^b 0\,dt \\ &= 0 \end{aligned}$$

as $|z| \to \infty$, as desired. □

**Theorem** ("Super Cauchy Theorem"). *Let* $\Omega \subset \mathbb{C}$ *be open and let* $\gamma : [a,b] \to \Omega$ *be a rectifiable loop. Then the following are equivalent:*

*(a) If* $f : \Omega \to \mathbb{C}$ *is holomorphic, then*

$$\oint f(z)\,dz = 0.$$

*(b) If* $f : \Omega \to \mathbb{C}$ *is holomorphic, then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \mathrm{Wind}_\gamma(w) f(w)$$

*for all* $w \in \Omega \setminus \gamma$*.*

*(c)* $\mathrm{Wind}_\gamma(w) = 0$ *for all* $w \in \mathbb{C} \setminus \Omega$

*(d) (Residue Theorem) If* $g : \Omega \to \mathbb{C}$ *is meromorphic, then*

$$\frac{1}{2\pi i}\oint_\gamma g(z)\,dz = \sum_{z \in \Omega} \mathrm{Res}(g,z)\,\mathrm{Wind}_\gamma(z),$$

*where the sum on the right-hand side is finite.*

**Remark.** *We make three remarks here:*

- *It is a good idea to intepret this theorem concretely. Think of the point* $w$ *as the location of a lamp post. Suppose you are walking with your dog; that is, if your dog is on a leash, then you are walking along* $\gamma$ *and the dog is walking along* $\sigma$*. If the leash is small enough, then Rouchè's Theorem will say that the winding numbers of you and your dog are the same:* $\mathrm{Wind}_\gamma(z) = \mathrm{Wind}_\sigma(z)$*.*
- *If* $\gamma$ *is contractible in* $\Omega$*, then (a) holds, as homotopy is invariant. This implies that (b), (c), (d) also hold.*
- *Some curves obey items (a), (b), (c), (d) without being contractible. Curves that do satisfy all these properties are called* null homologous. *An example that shows this distinction is the "Pochammer Contour" in* $\mathbb{C} \setminus \{0,1\}$*. The Pochammer Contour has a winding number of zero, which is (c), but it is* <u>not</u> *contractible.*

*Proof of "Super Cauchy Theorem".* First, we will prove (a) $\Rightarrow$ (b). We note that the map $z \mapsto \frac{f(z)-f(w)}{z-w}$ is holomorphic for all $z \in \Omega$. So, by part (a), we have

$$\begin{aligned}\frac{1}{2\pi i}\oint \frac{f(z)}{z-w}\,dz &= \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{z-w}\,dz + \frac{1}{2\pi i}\oint_\gamma \frac{f(z)-f(w)}{z-w}\,dz \\ &= \left(\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-w}\,dz\right) f(w) + \frac{1}{2\pi i}0 \\ &= \mathrm{Wind}_\gamma(w) f(w) + 0 \\ &= \mathrm{Wind}_\gamma(w) f(w),\end{aligned}$$

which is (b).

Next, we will prove (b) $\Rightarrow$ (a). Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $w \in \Omega$. We define

$$h(z) := (z-w)f(z),$$

which satisfies $h(w) = 0$. So, by part (b) for $h$, we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma f(z)\,dz &= \frac{1}{2\pi i}\oint_\gamma \frac{h(z)}{z-w}\,dz \\ &= \mathrm{Wind}_\gamma(w) h(w) \\ &= \mathrm{Wind}_\gamma(w)\cdot 0 \\ &= 0,\end{aligned}$$

which is (a).

Next, we will prove (a) $\Rightarrow$ (c). Suppose we have $w \in \mathbb{C} \setminus \Omega$, so that $z \mapsto \frac{1}{z-w}$ is holomorphic in $\Omega$. So, by (a), we have

$$\begin{aligned}\mathrm{Wind}_\gamma(w) &= \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-2} \\ &= \frac{1}{2\pi i}0 \\ &= 0,\end{aligned}$$

which is (c).

Next, we will prove (c) $\Rightarrow$ (b). Define the set

$$\tilde{\Omega} := \{z \in \mathbb{C} \setminus \gamma : \mathrm{Wind}_\gamma(z) = 0\}.$$

By continuity and "integer-valued-ness" of $z \mapsto \mathrm{Wind}_\gamma(z)$, we find that the set $\tilde{\Omega}$ is open. Part (c), which says $\mathrm{Wind}_\gamma(w) = 0$ for all $w \in \mathbb{C} \setminus \Omega$, implies $\tilde{\Omega} \cup \Omega = \mathbb{C}$. Now, given a holomorphic function $f : \Omega \to C$, we define

$$F(w) := \begin{cases} \frac{1}{2\pi i}\oint_\gamma \frac{f(z)-f(w)}{z-w}\,dz & \text{if } w \in \Omega, \\ \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz & \text{if } w \in \tilde{\Omega}. \end{cases}$$

Note that, if $w \in \Omega \cap \tilde{\Omega}$, then the definitions agree since $w \in \tilde{\Omega}$ implies

$$\begin{aligned}\frac{1}{2\pi i}\oint \frac{dz}{z-w} &= \mathrm{Wind}_\gamma(w) \\ &= 0.\end{aligned}$$

As $w \to \infty$, we have $\mathrm{Wind}_\gamma(w) \to 0$, which is equivalent to saying that $w$ is eventuallly in $\tilde{\Omega}$. But this just shows that $F(w) \to 0$. Therefore, by Liouville's Theorem, we get $F \equiv 0$. So we have

$$\begin{aligned}0 &= \frac{1}{2\pi i}\oint \frac{f(z)}{z-w}\,dz - \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-w} f(w) \\ &= \frac{1}{2\pi i}\oint \frac{f(z)}{z-w}\,dz - \mathrm{Wind}_\gamma(w) f(w),\end{aligned}$$

from which (b) follows.

Next, we will prove (d) $\Rightarrow$ (b). We only need to consider the function

$$g(z) := \frac{f(z)}{z-w}.$$

Then, by (d) (the Residue Theorem), we have

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz &= \sum_{z\in\Omega}\operatorname{Res}\left(\frac{f(z)}{z-w},z\right)\operatorname{Wind}_\gamma(z)\\ &= (z-w)\frac{f(z)}{z-w}\operatorname{Wind}_\gamma(z)\bigg|_{z=w}\\ &= f(w)\operatorname{Wind}_\gamma(w),\end{aligned}$$

which is (b).

Finally, we will prove (c) $\Rightarrow$ (d). Let

$$K := \gamma([a,b])\cup\{w\in\mathbb{C}\setminus\gamma : \operatorname{Wind}_\gamma(w)\neq 0\}.$$

If $w\in K$, then $w$ satisfies $\operatorname{Wind}_\gamma(w)\neq 0$, which means we have $w\in\Omega$, and so $K\subset\Omega$. We also notice that $K$ is bounded. Therefore, by (c), we have $\operatorname{Wind}_\gamma(w)\to 0$ as $w\to\infty$. And $K$ is also closed, which means $\gamma$ is as well, and so $K$ is the inverse image of a bounded integer value $F_n$. Or, instead we can consider the limit of points. So $K$ is compact. BNow, we define

$$\tilde{\Omega} := \{z\in\Omega : \frac{1}{2}\operatorname{dist}(z,\Omega^c) > \operatorname{dist}(z,K)\}.$$

We observe that $\overline{\tilde{\Omega}}$ is closed, and is also bounded bounded whenever $\Omega^c\neq\varnothing$ or $\Omega^c=\varnothing$ (if the latter, then we also have $\overline{\tilde{\Omega}}\subset\Omega$. On $\tilde{\Omega}$, we can write

$$g(z) := \sum_{j,k}\frac{a_{j,k}}{(z-z_j)^k} + f(z),$$

where the sum on the right-hand side is finite, and $f$ is holomorphic on $\tilde{\Omega}$. Integrate both sides and divide by $2\pi i$ to get

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma g(z)\,dz &= \frac{1}{2\pi i}\sum_{j,k}\oint_\gamma\frac{a_{j,k}}{(z-z_j)^k}\,dz + \frac{1}{2\pi i}\oint_\gamma f(z)\,dz\\ &= \frac{1}{2\pi i}\sum_{j,k}\oint_\gamma\frac{a_{j,k}}{(z-z_j)^k}\,dz + \frac{1}{2\pi i}0\\ &= \frac{1}{2\pi i}\sum_{j,k}\oint_\gamma\frac{a_{j,k}}{(z-z_j)^k}\,dz.\end{aligned}$$

Note that, if $k\neq 1$, then $\frac{1}{(z-z_j)^k}$ has a primitive, because, according to the Residue Theorem, the center is not allowed to go through a pole. So we have, in fact,

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma g(z)\,dz &= \frac{1}{2\pi i}\sum_j\oint_\gamma\frac{a_{j,1}}{z-z_j}\,dz + \frac{1}{2\pi i}\sum_{j,k>1}\oint_\gamma\frac{a_{j,k}}{(z-z_j)^k}\,dz\\ &= \frac{1}{2\pi i}\sum_j\oint_\gamma\frac{a_{j,1}}{z-z_j}\,dz + \frac{1}{2\pi i}\sum_{j,k>1}0\\ &= \sum_j\frac{1}{2\pi i}\oint_\gamma\frac{a_{j,1}}{z-z_j}\,dz\\ &= \sum_j a_{j,1}\operatorname{Wind}_\gamma(z_j),\end{aligned}$$

where we have used the implication (c) $\Rightarrow$ (b) in the last step. We claim that our last quantity is equal to $\sum_{z\in\Omega}\operatorname{Res}(g,z)\operatorname{Wind}_\gamma(z)$. When we defined all the $z_j$, we included every pole that has nonzero winding number because very such pole is in $\tilde{\Omega}$. Perhaps we may consider a better set

$$\tilde{\Omega} := \left\{z\in\mathbb{C} : \operatorname{dist}(z,K) < \min\left\{\frac{1}{2}\operatorname{dist}(K,\Omega^c),1\right\}\right\}.$$

If $f$ is holomorphic in some neighborhood of $z_0$, $f(z_0)=0$, and $f$ does not vanish identically (in any neighborhood of $z_0$), then (since $f$ is not identially zero about $z_0$) $f$ has a power series representation with nonzero coefficients $a_i$:

$$f(z) = a_k(z-z_0)^k + \sum_{j=k+1}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j.$$

this number $k$ is called the *order* of the zero for all integers $k\geq 1$. We could call a point in which a neighborhood of $f$ vanishes at zero of infinite order. Analogously, if a meormophic function $g$ has ap ole at some $z_0$, the order of the pole (which is finite) is the order of the pole of $\frac{1}{g}$. □

**Theorem** (Argument principle)**.** *If $f : \Omega \to \mathbb{C}$ is meromorphic and $g : \Omega \to \mathbb{C}$ is holomorphic, and if $\gamma$ is null holomogous on $\Omega$ and misses the zeros and poles of $f$, then*

$$\frac{1}{2\pi i}\oint_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_w g(w)\operatorname{Order}(f,w)\operatorname{Wind}_\gamma(w) - \sum_p g(p)\operatorname{Order}(f,p)\operatorname{Wind}_\gamma(p)$$

*where the $w$ denote the zeros of $f$ and the $p$ denote the poles of $f$.*

The following example is the version of the Argument Principle most commonly presented in complex analysis literature.

**Example.** *If we set $g \equiv 1$ and $\gamma$ is a simple closed curve oriented correctly that misses the zeros and poles of $f$, then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = N_{0,f,\gamma} - N_{\infty,f,\gamma}$$

*where $N_{0,f,\gamma}$ denotes the number of zeros of $f$ "inside $\gamma$" and $N_{\infty,f,\gamma}$ denotes the number of poles of $f$ "inside $\gamma$".*

*Furthermore, if $\gamma$ is smooth, we can compute the left-hand side as*

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_a^b \frac{f'\circ\gamma(t)}{f\circ\gamma(t)}\dot\gamma(t)\,dt\\ &= \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\log(f\circ\gamma(t))\,dt.\end{aligned}$$

*Note that* $\operatorname{Re}\log(w) = \log|w|$ *is globally defined and smooth (holomorphic) for all $w \in \mathbb{C}\setminus\{0\}$, which means we have*

$$\operatorname{Re}\left(\int_a^b \operatorname{Re}\log(\gamma(t))\,dt\right) = 0.$$

*So our left-hand side is, in fact,*

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\log(f\circ\gamma(t))\,dt\\ &= \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\left(\log|f\circ\gamma(t)| + i\operatorname{Arg}(f\circ\gamma(t))\right)\,dt\\ &= \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\log|f\circ\gamma(t)|\,dt + \frac{1}{2\pi i}\int_a^b i\frac{d}{dt}\operatorname{Arg}(f\circ\gamma(t))\,dt\\ &= 0 + \frac{1}{2\pi}\int_a^b \frac{d}{dt}\operatorname{Arg}(f\circ\gamma(t))\,dt\\ &= \frac{1}{2\pi}\operatorname{Arg}(f\circ\gamma(t))\big|_a^b\\ &= \frac{\textit{Total increment of the argument}}{2\pi}.\end{aligned}$$

*More precisely, we have*

$$\begin{aligned}\operatorname{Wind}_{f\circ\gamma}(0) &= \frac{1}{2\pi i}\oint_{f\circ\gamma}\frac{dw}{w-0}\\ &= \frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt}(f\circ\gamma(t))}{f\circ\gamma(t)}\,dt\\ &= \frac{1}{2\pi i}\int_a^b \frac{f'\circ\gamma(t)}{f\circ\gamma(t)}\dot\gamma(t)\,dt\\ &= N_{0,f,\gamma} - N_{\infty,f,\gamma},\end{aligned}$$

*as desired.*

**Example.** *If we let $\gamma$ be a simple closed curve and $f$ is a holomorphic function such that $f\circ\gamma$ is a closed curve with a self-intersecting point, then, as we traverse (counterclockwise) on $\gamma$, the argument of $f\circ\gamma$ has increased by $4\pi$. So we have*

$$\begin{aligned}N_{0,f,\gamma} - N_{\infty,f,\gamma} &= \frac{\textit{Total increment of the argument}}{2\pi}\\ &= \frac{4\pi}{2\pi}\\ &= 2.\end{aligned}$$

*That is, there are two "zeros-poles" inside $\gamma$.*

**Corollary** (Rouché's Theorem)**.** *Let $f, g : \Omega \to \mathbb{C}$ be meromorphic in a simple, closed, rectifiable curve contractible in $\Omega$ that misses the zeros and poles of $f$. If $f$ and $g$ satisfy*

$$|g| < |f| + |f+g|$$

*on the curve $\gamma$, then we have*

$$N_{0,f+g,\gamma} - N_{\infty,f+g,\gamma} = N_{0,f,\gamma} - N_{\infty,f,\gamma}.$$

**Remark.** *We make the following remarks.*

- *Most commonly in the literature, Rouché's Theorem is reported with the hypothesis $|g| < |f|$ on $\gamma$. The modified hypothesis $|g| < |f| + |f+g|$ is considered to be an improvement because it not necessarily require $f$ to be bigger than $g$, though both $f$ and $g$ should be going to a common limit for large values of $z \in \mathbb{C}$. (If, for instance, $f \to +\infty$ and $g \to -\infty$ as $|z| \to \infty$, then $|g| < |f| + |f+g|$ will probably not be satisfied.)*
- *The zeros depend continuously in $\mathbb{C}$ on $f$.*
- *Rouché's Theorem holds only on $\mathbb{C}$, not always on $\mathbb{R}$.*

*Proof of Rouché's Theorem.* In view of the Argument Principle, it suffices to establish

$$\begin{aligned}\oint_{f\circ\gamma} \frac{dw}{w} &= \int_a^b \frac{\frac{d}{ds}(f\circ\gamma(s))}{f\circ\gamma(s)}\dot{\gamma}(s)\,ds \\ &= \int_a^b \frac{(f+tg)'\circ\gamma(s)}{(f+tg)\circ\gamma(s)}\dot{\gamma}(s)\,ds.\end{aligned}$$

We would like to view the map $(t,s) \mapsto (f+tg)\circ\gamma(s)$ as a homotopy on $\mathbb{C}\setminus\{0\}$. So we really only need to show that $(f+tg)\circ\gamma(s)$ never vanishes. By the reverse triangle inequality, we have the following two inequalities:

$$\begin{aligned}|f+tg| &\geq |f| - t|g|, \\ |f+tg| &= |(f+g) - (1-t)g| \geq |f+g| - (1-t)|g|.\end{aligned}$$

Taking averages, we have

$$\begin{aligned}|f+tg| &= \frac{1}{2}(|f+tg| + |f+tg|) \\ &\geq \frac{1}{2}((|f| - t|g|) + (|f+g| - (1-t)|g|)) \\ &\geq \frac{1}{2}(|f| + |f+g| - |g|) \\ &\geq \frac{1}{2}(|f| - |g|) \\ &> \frac{1}{2}\cdot 0 \\ &= 0,\end{aligned}$$

thereby showing that $f+tg$ (and in particular $(f+tg)\circ\gamma(s)$) never vanishes. □

One can prove this corollary using the argument principle instead of the Inverse Function Theorem.

**Proposition.** *If $f$ is holomorphic in a neighborhood of $z_0 \in \Omega$ and satisfies $f'(z_0) \neq 0$, then $f$ is a biholomorphism of a neighborhood of $z_0$ onto a neighborhood of $w_0 := f(z_0)$, and $f$ is inverible with a holomorphic invese. Moreover, we have*

$$(f^{-1})'(w) = [f'\circ f^{-1}(w)]^{-1}.$$

*Thus, $(f^{-1})'$ also has the structure of a complex number.*

*Proof.* We define

$$\begin{aligned}F(z) &:= f(z_0) + f'(z_0)(z-z_0), \\ G(z) &:= \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k.\end{aligned}$$

If $z$ satisfies $|z-z_0| < \delta$, then we would have

$$\begin{aligned}|G(z)| &\leq \frac{1}{3}|f'(z_0)|\delta \\ &=: \epsilon.\end{aligned}$$

By comparison, we have $|z - z_0| = \delta$ and $|w - w_0| < \epsilon$, and so we have

$$\begin{aligned}|F(z) - w| &= |(F(z) - w_0) + (w_0 - w)| \\ &= |F(z) - w_0| - |w - w_0| \\ &\geq |f'(z_0)|\delta - |w - w_0| \\ &\geq |f'(z_0)|\delta - \frac{1}{3}|f'(z_0)|\delta \\ &= \frac{2}{3}|f'(z_0)|\delta.\end{aligned}$$

By Rouché's Theorem, we conclude

$$|G(z)| \leq \frac{2}{3}|F(z) - w|$$

or $|z - z_0| = \delta$. And $F(z) - w$ has the same number of roots in $|z - z_0| < \delta$ as $F(z) + G(z) - w = f(z) - w$. So $F(z) = w$ if and only if we have

$$z - z_0 = \frac{w - w_0}{f'(z_0)}.$$

Notice that we have

$$\begin{aligned}|z - z_0| &= \frac{|w - w_0|}{|f'(z_0)|} \\ &< \frac{\epsilon}{|f'(z_0)|} \\ &= \frac{\delta}{3}.\end{aligned}$$

Thus, the equation $F(z) = w$ has exactly one root in our ball. So we have shown, for all $w$ satisfying $|w - w_0| < \epsilon$, that there is exactly one $z$ satisfying $|z - z_0| < \delta$ such that $f(z) = w$. So $f$ is a bijection of some neighborhood of $z_0$ onto $|w - w_0| < \epsilon$.

To show that $f^{-1}$ is holomorphic, we use the Argument Principle: If $|w - w_0| < \epsilon$, then we have

$$f^{-1}(w) = \oint_{|z-z_0|=\delta} z\frac{f'(z)}{f(z) - w}\, dz.$$

And $f(z) - w$ has a simple pole at $z_0$ because $f(z_0) = w$. □

**Remark.** *Our last formula can be very useful. We may be able to solve $f(z) = w$. We do have an integral formula not only for $z$, but also even for functions of $z$:*

$$g \circ f^{-1}(w) = \oint_{|z-z_0|=\delta} g(z)\frac{f'(z)}{f(z) - w}\, dz.$$

**Proposition.** *Let $f$ be holomorphic in a neighborhood of $z_0$, with*

$$f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0$$

*but $f^{(n)}(z_0) \neq 0$. Then there is a biholomorphism $\varphi$ from a neighborhood of $z_0$ to a neighborhood of $0$, so that $f$ satisfies*

$$f(z) = f(z_0) + (\varphi(z))^n.$$

*Coloquially speaking, $f(z) - f(z_0)$ looks like $z^n$. In particular, for some $\epsilon > 0$, we have $0 < |w - f(z_0)| < \epsilon$. Then $f(z) = w$ has exactly n solutions in this neighborhood, while $f(z) - f(z_0)$ has one solution of order n.*

*Proof.* Write

$$f(z) - f(z_0) = z^n h(z),$$

where $h$ is holomorphic and satisfies $h(z_0) = \frac{f^{(n)}(z_0)}{n!}$ /–0. From the homework, we know that $h$ admits a holomorphic logarithm $F(z)$ in a neighborhood of $z_0$ (that is, a primitize $F(z)$ of $\frac{h'(z)}{h(z)}$). Now, just choose

$$\varphi(z) = (z - z_0)e^{\frac{F(z)}{n}},$$

noting that $\varphi'(z_0) = e^{\frac{F(z_0)}{n}}$. □

If $f$ is holomorphic near $z_0$, then there is a biholomorphic change of variables $\varphi$ defined in a neighborhood of $z_0$, say $\varphi(z_0) = 0$, that satisfy

$$\begin{aligned}f \circ \varphi^{-1}(w) &\equiv \text{const.}, \\ f \circ \varphi^{-1}(w) &= f(z_0) + w, \\ f \circ \varphi^{-1}(w) &= f(z_0) + w^n.\end{aligned}$$

**Corollary** (Open Mapping Theorem)**.** *If $f : \Omega \to \mathbb{C}$ is holomorphic and nowhere constant (i.e. not constant on any connected component of $\Omega$), then $f$ is open (that is, $f$ maps open sets to open sets).*

*Proof.* For all integers $n \geq 1$, the map $z \mapsto z^n$ is open (as it is nonconstant), and so this map takes open sets to open sets. Then we can apply the biholomorphism $\varphi$ to the map $z \mapsto z^n$ to conclude that $f$ given by $f(z) := \varphi(z^n)$ also maps open sets to open sets, making $f$ an open map as well. □

**Corollary** (Maximum Modulus Principle; i.e. Strong Maximum Principle)**.** *If $\Omega \subset \mathbb{C}$ is connected and $f : \Omega \to \mathbb{C}$ is holomorphic and satisfies*

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)|$$

*for some $z_0 \in \Omega$, then $f$ is constant.*

*Proof.* Suppose to the contrary that $f$ is not constant. As $\Omega$ contains a neighborhood around $z_0$, it follows that $f(\Omega)$ would contain a neighborhood around $f(z_0)$. But this would automatically violate the assumption $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$. □

**Corollary** (Weak Maximum Principle)**.** *If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic, and if $O$ is open and its closure $\overline{O} \subseteq \Omega$ is compact, then $f$ achieves its maximum over $\overline{O}$ on the boundary $\partial O$, and*

$$\sup_{z \in \overline{O}} |f(z)| = \sup_{z \in \partial O} |f(z)|.$$

**Theorem** (Schwarz Lemma)**.** *Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be a unit disk. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, satisfies $f(0) = 0$ and $|f(z)| \leq 1$, then we have the following:*

*(a) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$,*

*(b) If there exists $z_0 \in \mathbb{D} \setminus \{0\}$ that satisfies $|f(z_0)| = |z_0|$, then $f(z) = e^{i\varphi} z$ for some $\varphi \in [0, 2\pi)$.*

*(c) If $|f'(0)| = 1$, then $f(z) = e^{i\varphi} z$ for some $\varphi \in [0, 2\pi)$.*

*Proof.* Define $g : \mathbb{D} \to \mathbb{C}$ by $g(z) := \frac{f(z)}{z}$. Then, as we have $f(0) = 0$, it follows that $g$ has a removable singularity at $z = 0$. So we can regard $g$ to be holomorphic. For any $0 < r < 1$, we can use the Maximum Modulus Principle (Strong Maximum Principle) to obtain

$$\begin{aligned} \sup_{|z| \leq r} |g(z)| &= \sup_{|z|=r} |g(z)| \\ &\leq \frac{1}{r}. \end{aligned}$$

Sending $r \to 1^-$, we deduce

$$\sup_{|z| \leq 1} |g(z)| \leq 1,$$

which is equivalent to saying $|g(z)| \leq 1$, or $|f(z)| \leq |z|$, for all $z \in \mathbb{D}$.

If $|g(z_0)| = 1$, then, by the Maximum Principle, $g(z) \equiv g(z_0)$, which is equivalent to saying $f(z) = g(z_0)z$. The only complex numbers satisfying a modulus of 1 is $e^{i\varphi}$ for some angle $\varphi \in [0, 2\pi)$; this means we require $g(z) = g(z_0) = e^{i\varphi}$, proving (b).

Finally, we observe

$$\begin{aligned} g(0) &= \lim_{z \to 0} \frac{f(z)}{z} \\ &= \lim_{z \to 0} \frac{f(z) - 0}{z - 0} \\ &= \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} \\ &= f'(0), \end{aligned}$$

and so we see that (c) is a simple consequence of (b). □

**Corollary.** *If $f : \mathbb{D} \to \mathbb{D}$ is a biholomorphism (onto $\mathbb{D}$), then we have*

$$f(z) = e^{i\varphi} \frac{z - a}{1 - \bar{a} z}$$

*for some $a \in \mathbb{D}$ and $\varphi \in [0, 2\pi)$.*

*Proof.* Define $g : \mathbb{D} \to \mathbb{D}$ by

$$g(z) := \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)}.$$

Then $g$ is holomorphic, invertible, and satisfies $g(0) = 0$. By the Schwarz Lemma applied to $g$, we get $|g(z)| \leq |z|$. By the Schwarz Lemma applied to $g^{-1}$, we get $|z| = |g^{-1}(g(z))| \leq |g(z)|$. The two inequalities force the equality $|g(z)| = z$. By part (b) of Schwarz Lemma, we must have $g(z) = e^{i\varphi}z$. Setting our two expressions of $g(z)$ together, we conclude

$$\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)} = e^{i\varphi}z.$$

Solving for $f(z)$, we get

$$f(z) = e^{i\varphi}\frac{z - (-f(0)e^{-i\varphi})}{1 - ((-f(0)e^{-i\varphi}))z}$$

establsihing the corollary with $a := -f(0)e^{-i\varphi} \in \mathbb{D}$. □

**Theorem** (Schwarz Reflection Principle)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, with $\Omega \cap \mathbb{R} \neq \emptyset$. Then:*

*(a) If $f : \Omega \to \mathbb{C}$ is holomorphic when $f(\mathbb{R} \cap \Omega) \subseteq \mathbb{R}$, then $f(\overline{z}) = \overline{f(z)}$ whenever both sides are defined.*

*(b) On the set $\Omega_+ := \{z \in \Omega : \operatorname{Im} z \geq 0\}$, if $g : \Omega_+ \to \mathbb{C}$ is holomorphic on the interior of $\Omega_+$ and continuous on all of $\Omega_+$, and $g(x) \in \mathbb{R}$ for all $x \in \mathbb{R} \cap \Omega_+$, then $G$ deifned by*

$$G(z) := \begin{cases} g(z) & \text{if } z \in \Omega+, \\ \overline{g(\overline{z})} & \text{if } z \in \Omega_+ \end{cases}$$

*is holomorphic in $\Omega_+ \cup \Omega_-$, where $\Omega_- := \{\overline{z} : z \in \Omega_+\}$.*

*Proof.* First, assuming (b), we will prove (a). By part (b), we can build a curve $\gamma$ via reflection that agrees with $f$ on $\operatorname{int}(\Omega_+)$. By uniqueness and bcause $\Omega$ is connected, we have $f \equiv a$, wherever both $f$ and $a$ are defined. This implies in particular $f(\bar{z}) = \overline{f(z)}$, which is (a).

Next, we will prove (b). We must show that, for every closed, solid triangle $T \subseteq \Omega_+ \cup \Omega_-$, we have

$$\int_{\partial T} G(z)\, dz = 0.$$

For triagnles that are entirely contained in either $\Omega_+$ or $\Omega_-$, we can use Green's Theorem as soon as we observe the following:

- $z \mapsto \overline{g(\bar{z})}$ obeys the Cauchy-Riemann equations.
- If the triangle "sits on the axis".

For triangles that sit on the axis, we can use continuity and approximation. Consider the "$\epsilon$-lifted" triangle $T_\epsilon$ (a triangle entirely contained in either $\Omega_+$ or $\Omega_-$ with one of its sides being parallel and very close to but not touching the axis) which satisfies

$$\oint_{\partial T_\epsilon} G(z)\, dz = 0.$$

Send $\epsilon \to 0^+$ accordingly to conclude (a) for all triangles that touch the axis. Now, for triangles that cross the real axis, remove the thin strip of the triangle that contains the axis to form into an "$\epsilon$-lifted" triangle in (say) $\Omega_+$ and an "$\epsilon$-lifted" trapezoid. Then partition the "$\epsilon$-lifted" triangle in (say) $\Omega_-$ into smaller triangles in (say) $\Omega_-$ that partition the trapezoid. Then apply Green's Theorem to each of the subdivided triangles to assert part (a) for each of these triangles. Then send $\epsilon \to 0^+$ accordingly to conclude (a) for the triangle that crosses the real axis. □

**Example.** *For any $z \in \mathbb{C}$, its reflection about the unit circle $|z| = 1$ is*

$$\frac{1}{\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

*This is conformal in <u>any</u> dimension, but orientation preserving.*

**Theorem** (Painlevé Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, let $\gamma : [a, b] \to \mathbb{C}$ be rectifiable, and let $f : \Omega \to \mathbb{C}$ be continuous on all of $\Omega$ and holomorphic on $\Omega \setminus \mathbb{C} \subset \Omega$. Then $f$ is holomorphic on all of $\Omega$.*

*Proof.* We seek to apply Morera's Theorem. With that in mind, let $T \subseteq \Omega$ be a solid, closed triangle. If $T$ satisfies $\operatorname{Area}(T) = 0$, then we would have

$$\oint_{\partial T} f(z)\, dz = 0$$

for <u>any</u> continous function. As $f$ is continuous and $T$ is compact, we can subdivide $T$ into congruent triangles $T_k$, similar to $T$, so that $z, w \in T_k$ implies $|f(z) - f(w)| < \epsilon$ and $\operatorname{diameter}(T_k) < \operatorname{diameter}(\gamma)$. Now, by Goursat's Theorem, we have

$$\begin{aligned} \oint_{\partial T} f(z)\, dz &= \sum_k \oint_{\partial T_k} f(z)\, dz \\ &= \sum_{k \in \{k : T_k \cap \gamma \neq \emptyset\}} \oint_{\partial T_k} f(z)\, dz \\ &\leq (\text{number of } \{T_k : T_k \cap \gamma \neq \emptyset\})\epsilon \cdot 3 \operatorname{diameter}(T_k). \end{aligned}$$

(Note that $T_k \cap \gamma \neq \varnothing$ represents only those triangles in $\Omega$ that intersect the curve $\gamma$ in some way.) Now, we will be done once we can prove

$$(\text{number of } \{T_k : T_k \cap \gamma \neq \varnothing\}) \lesssim_{\gamma,T} \frac{1}{\text{diameter}(T_k)}.$$

Consider now $4T_k$ to denote the triangle that is concentric and similar to $T_k$ but four times bigger. Then we have diameter$(\gamma) > 100\,$diameter$(T_k)$. If $\gamma \cap T_k = \varnothing$, then it must cross over $4T_k$ and satisfy

$$\text{length}(\gamma \cap 4T_k) \geq C_T \,\text{diameter}(T_k),$$

where $C_T$ is a small constant (least angle) depending on $T$. Finally, we have

$$\begin{aligned}\text{length}(\gamma) &\gtrsim \sum_k \text{length}(\gamma \cap 4T_k) \\ &\geq C_T \,\text{diameter}(T_k) \cdot (\text{number of } \{T_k : T_k \cap \gamma \neq \varnothing\}),\end{aligned}$$

which completes the proof. □

The remainder of these notes will focus on the theory of harmonic functions.

**Definition.** *A function $f \in C^2(\mathbb{R}^d)$ is called* harmonic *if it satisfies*

$$\Delta f = 0,$$

*where $\Delta f := \sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}$ is the Laplacian of $f$.*

**Definition.** *A continuous function is called* distributionally harmonic *if we have*

$$\int_{\mathbb{R}^d} f \Delta\varphi \, dvol = 0$$

*for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. An integration by parts shows that (classically) harmonic implies distributionally harmonic.*

As we will see later, the Weyl lemma shows that distributionally harmonic implies (locally) real analytic.

**Proposition.** *Let $\Omega \subset \mathbb{C} \cong \mathbb{R}^2$.*

*(a) If $f : \Omega \to \mathbb{C}$ is holomorphic, then* Re $f$, Im $f$, *and $f$ are all harmonic*

*Proof.* Write $f = u + iv$, which means $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$. Since $f$ is holomorphic, Cauchy-Riemann equations hold: $u_x = v_y$ and $u_y = -v_x$. So we have

$$\begin{aligned}\Delta(\operatorname{Re} f) &= \Delta u \\ &= u_{xx} + u_{yy} \\ &= (u_x)_x + (u_y)_y \\ &= (v_y)_x + (-v_x)_y \\ &= v_{yx} - v_{xy} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\Delta(\operatorname{Im} f) &= \Delta v \\ &= v_{xx} + v_{yy} \\ &= (v_x)_x + (v_y)_y \\ &= (u_y)_x + (-u_x)_y \\ &= u_{yx} - u_{xy} \\ &= 0,\end{aligned}$$

meaning that Re $f$ and Im $f$ are harmonic. Consequently, we have

$$\begin{aligned}\Delta f &= \Delta(\operatorname{Re} f + i \operatorname{Im} f) \\ &= \Delta \operatorname{Re} f + i\Delta \operatorname{Im} f \\ &= 0 + i0 \\ &= 0,\end{aligned}$$

meaning that $f$ is harmonic. □

*(b) If $h : \mathbb{C} \to \mathbb{R}$ is harmonic and $f : \Omega \to \mathbb{C}$ is holomorphic, then $h \circ f$ is also harmonic.*

*Proof.* As in part (a), let $f = u + iv \equiv (u, v)$. Then, for each $j = 1, \dots, d$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial x_j^2}(h \circ f) &= \frac{\partial}{\partial x_j}\left(\left(\left(\frac{\partial h}{\partial u} \circ f\right)\left(\frac{\partial u}{\partial x_j}\right) + \left(\frac{\partial h}{\partial v}\right)\left(\frac{\partial v}{\partial x_j}\right)\right)\right) \\
&= \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\left(\frac{\partial u}{\partial x_j}\right)^2 + \left(\frac{\partial h}{\partial u} \circ f\right)\left(\frac{\partial^2 u}{\partial x_j^2}\right) \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right)\left(\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial v}{\partial x_j}\right) + \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\left(\frac{\partial v}{\partial x_j}\right)^2 \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right)\left(\frac{\partial v}{\partial x_j}\right)\left(\frac{\partial u}{\partial x_j}\right) + \left(\frac{\partial h}{\partial v} \circ f\right)\left(\frac{\partial^2 v}{\partial x_j^2}\right).
\end{aligned}$$

As we assumed $h$ is harmonic, we have $\Delta h = 0$. Also, the Cauchy-Riemann equations imply $\|\nabla u\| = \|\nabla v\|$ and $\nabla u \cdot \nabla v = 0 = \nabla v \cdot \nabla u$. So we have

$$\begin{aligned}
\Delta(h \circ f) &= \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}(h \circ f) \\
&= \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2 + \left(\frac{\partial h}{\partial u} \circ f\right)\sum_{j=1}^{d}\left(\frac{\partial^2 u}{\partial x_j^2}\right) \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right)\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial v}{\partial x_j}\right) + \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\sum_{j=1}^{d}\left(\frac{\partial v}{\partial x_j}\right)^2 \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right)\sum_{j=1}^{d}\left(\frac{\partial v}{\partial x_j}\right)\left(\frac{\partial u}{\partial x_j}\right) + \left(\frac{\partial h}{\partial v} \circ f\right)\sum_{j=1}^{d}\left(\frac{\partial^2 v}{\partial x_j^2}\right) \\
&= \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\|\nabla u\|^2 + \left(\frac{\partial h}{\partial u} \circ f\right)\Delta u \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right)\nabla u \cdot \nabla v + \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\|\nabla v\|^2 \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right)\nabla v \cdot \nabla u + \left(\frac{\partial h}{\partial v} \circ f\right)\Delta v \\
&= \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\|\nabla u\|^2 + \left(\frac{\partial h}{\partial u} \circ f\right) \cdot 0 \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right) \cdot 0 + \left(\frac{\partial^2 h}{\partial u^2} \circ f\right)\|\nabla u\|^2 \\
&\quad + \left(\frac{\partial^2 h}{\partial v \partial u} \circ f\right) \cdot 0 + \left(\frac{\partial h}{\partial v} \circ f\right) \cdot 0 \\
&= \left(\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2}\right)\|\nabla u\|^2 \\
&= \Delta h \|\nabla u\|^2 \\
&= 0 \cdot \|\nabla u\|^2 \\
&= 0,
\end{aligned}$$

which means $h \circ f$ is harmonic. □

**Remark.** *We make the following remarks.*

- *Ultimately, we only used that $f$ is conformal. A harmonic function composed with, e.g., translation, rotation, reflection, and sphere inversion is still harmonic.*
- *The preceding calculation shows that, for every $h$,*

$$\Delta(h \circ f) = [(\Delta h) \circ f] \cdot |f'|^2.$$

- *The Cauchy-Riemann equations say*

$$\begin{aligned}\nabla u &= \begin{bmatrix} u_x \\ u_y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla v,\end{aligned}$$

  *where the* $2 \times 2$ *matrix appearing the equation represents rotation of a vector by* $90°$*. So* $u$ *and* $v$ *are both harmonic.*
- *If* $f$ *does not vanish, then* $\log|f|$ *is harmonic; that is,*

$$(x, y) \mapsto \log(\sqrt{x^2 + y^2}) = \operatorname{Re}\log(x + iy)$$

  *is harmonic.*

**Definition.** *Let* $\Omega \subset \mathbb{R}^d$*. We say that a continuous function* $u : \Omega \to \mathbb{R}$ *is* subharmonic *if it satisfies*

$$\int_\Omega u \Delta\varphi \, dx \geq 0$$

*for all* $\varphi \in C_c^\infty(\Omega)$*. If* $u : \Omega \to \mathbb{R}$ *is* $C^2$*, then* $u$ *is subharmonic if and only if* $\Delta u \geq 0$*.*

**Remark.** *Many authors allow* $u$ *to merely be upper semi-continuous:*

$$\limsup_{\bar{x} \to \bar{x}_0} u(\bar{x}) \leq u(\bar{x}_0).$$

*For instance,* $\log(\sqrt{x^2 + y^2})$ *is subharmonic in* $\mathbb{C}$*, with* $\log 0 = -\infty$*.*

**Theorem** ((Sub-)mean value property)**.** *If* $u$ *is distributionally subharmonic on* $\Omega$ *and* $B := B(x_0, R) \subseteq \Omega$*, then:*

*(a)* $u(x_0) \leq \frac{1}{\operatorname{vol}(B)} \int_B u(y) \, dvol(y)$*.*

*(b)* $u(x_0) \leq \frac{1}{\operatorname{vol}(\partial B)} \int_{\partial B} u(y) \, darea(y)$*.*

*Proof.* Part (a) follows from part (b) by using polar coordinates: $d\mathrm{vol} = \int_\Omega r^{d-1} \, dr$.

To prove part (b), we first mollify

$$u_\epsilon(x) := \int \epsilon^{-d} \varphi\left(\frac{x - y}{\epsilon}\right) u(y) \, dy,$$

where $\varphi \in C_c^\infty$ satisfies $\int_B \varphi = 1$. Note $u_\epsilon \in C^\infty$. Also, we have

$$\begin{aligned}\Delta u_\epsilon(x) &= \Delta \int \epsilon^{-d} \varphi\left(\frac{x - y}{\epsilon}\right) u(y) \, dy \\ &= \int \epsilon^{-d} \Delta_x \varphi\left(\frac{x - y}{\epsilon}\right) u(y) \, dy \\ &\geq 0.\end{aligned}$$

This reduces the proof to the case where $u$ is clasically subharmonic, once we take $\epsilon \to 0^+$. For simplicity, assume without loss of generality that $x_0 = 0$ and $R = 1$. And also consider just $d = 2$. Then we have

$$g(x) = \frac{1}{2\pi} \log\left(\frac{1}{|x|}\right).$$

Let us apply Green's second identity on the annulus $\epsilon \leq |x| \leq 1$:

$$\begin{aligned}\int_{\{x : \epsilon \leq |x| \leq 1\}} [(\Delta g)u - g\Delta u] \, d\mathrm{area} &= \int_{\{x : \epsilon \leq |x| \leq 1\}} [(\nabla \cdot \nabla g)u - g\nabla \cdot \nabla u] \, d\mathrm{area} \\ &= \int_{\{x : \epsilon \leq |x| \leq 1\}} \nabla \cdot [(\nabla g)u - g\nabla u] \, d\mathrm{area} \\ &= \int_{\partial(\{x : \epsilon \leq |x| \leq 1\})} [(\nabla g)u - g\nabla u] \cdot \vec{n} \, d\mathrm{length},\end{aligned}$$

where $\vec{n}$ is the outward normal. As $\Delta g = 0$, we have

$$\begin{aligned}0 &\geq \int_{\{x : \epsilon \leq |x| \leq 1\}} -g\Delta w \, d\mathrm{area} \\ &= \int_{\partial(\{x : \epsilon \leq |x| \leq 1\})} \left(-\frac{1}{\sqrt{\pi}} u - (g = 0)\right) d\mathrm{length} + \int_{\partial(\{x : \epsilon \leq |x| \leq 1\})} \frac{1}{2\pi\epsilon} u - \frac{1}{2\pi} \log\left(\frac{1}{\epsilon}\right) \vec{h} \cdot \nabla u \, d\mathrm{length}.\end{aligned}$$

That is,

$$
\begin{aligned}
&\frac{1}{\text{perimeter}(\partial(\{x : \epsilon \le |x| \le 1\}))} \int_{\partial(\{x:\epsilon\le|x|\le 1\})} u \, d\text{length} \\
&\quad \ge \frac{1}{\text{perimeter}(\partial(\{x : \epsilon \le |x| \le \frac{1}{2}\}))} \int_{\partial(\{x:\epsilon\le|x|\le\frac{1}{2}\})} u \, d\text{length} - \frac{1}{2\pi} \int_{\partial(\{x:\epsilon\le|x|\le 1\})} \log\left(\frac{1}{\epsilon}\right) (\vec{v} \cdot \nabla u) \, d\text{length} \\
&\quad \ge \frac{1}{\text{perimeter}(\partial(\{x : \epsilon \le |x| \le \frac{1}{2}\}))} \int_{\partial(\{x:\epsilon\le|x|\le\frac{1}{2}\})} u \, d\text{length} - \frac{1}{2\pi} \int_{\partial(\{x:\epsilon\le|x|\le 1\})} O(\epsilon \log(\epsilon))(\vec{v} \cdot \nabla u) \, d\text{length} \\
&\quad \to \frac{1}{\text{perimeter}(\partial(\{x : \epsilon \le |x| \le \frac{1}{2}\}))} \int_{\partial(\{x:\epsilon\le|x|\le\frac{1}{2}\})} u \, d\text{length} - 0 \\
&\quad = \frac{1}{\text{perimeter}(\partial(\{x : \epsilon \le |x| \le \frac{1}{2}\}))} \int_{\partial(\{x:\epsilon\le|x|\le\frac{1}{2}\})} u \, d\text{length},
\end{aligned}
$$

from which we can send $\epsilon \to 0^+$ both sides to complete the proof. □

**Remark.** *We make the following remarks.*

- *Harmonic functions appear with their known values.*
- *This $g$ is called Green's function:*

  $$
  \begin{aligned}
  \Delta g &= \delta_0 \quad \text{in } B, \\
  g &\equiv 0 \quad \text{on } \partial B.
  \end{aligned}
  $$

  *In dimensions $d \ge 3$, we have*

  $$g(x) := \frac{1}{d(d-2)|b||x|^{d-2}} - \frac{1}{d(d-2)|B|}.$$

**Proposition.** *If $u : \Omega \to \mathbb{R}$ is continuous and satisfies either of the two sub-mean-value properties, then $u$ is (distributionally) subharmonic.*

*Proof.* We know that, for all $B(x, \epsilon) \subseteq \Omega$, we have

$$\int_{B(0,\epsilon)} u(x-y) - u(x) \, dy \ge 0.$$

Now, for any $\varphi \in C_c^\infty$ and choosing $\epsilon < \text{dist}(\text{supp}(\varphi), \Omega^c)$, we have, as $\varphi \ge 0$, and employing the substitution $x' := x - y$,

$$
\begin{aligned}
0 &\le \frac{1}{\epsilon^{2+d}} \int_\Omega \left( \int_{|y|\le\epsilon} u(x-y) - u(x) \, dy \right) \varphi(x) \, dx \\
&= \frac{1}{\epsilon^{2+d}} \iint_{\Omega \times B_\epsilon} u(x')\varphi(x'-y') - u(x')\varphi(x') \, dx' \, dy' \\
&= \int_\Omega u(x') \left( \frac{1}{\epsilon^2} \int_{|y'|\le\epsilon} \varphi(x'-y') - \varphi(x') \, dy' \right) dx'
\end{aligned}
$$

Furthermore, from the Taylor expansion we have

$$\varphi(x'-y') - \varphi(x') = \nabla\varphi(x') \cdot (-y') + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} y_i' y_j' + O(|y|^3).$$

This completes the proof. (??) □

**Theorem** (Strong Maximum Principle)**.** *Let $u : \Omega \to \mathbb{R}$ be subharmonic, where $\Omega \subset \mathbb{R}^d$ is open and connected. If there exists $x_0 \in \Omega$ with $u(x_0) = \sup_\Omega u$, then $u$ is constant.*

**Remark.** *The Strong Maximum Principle implies the Weak Maximum Principle: If $\overline{\Omega'} \subseteq \Omega$ is compact, then*

$$\sup_{\Omega'} u = \sup_{\partial\Omega'} u.$$

*Proof of the Strong Maximum Principle.* Let $\overline{B(x_0, r)} \subseteq \Omega$. Then the hypothesis implies $u(x_0) - u(x) \ge 0$, and so $|u(x_0) - u(x)| = u(x_0) - u(x)$. Then, because $u$ is subharmonic (to justify the inequality below), we have

$$
\begin{aligned}
0 &\ge \fint_{B(x_0,r)} (u(x_0) - u(x)) \, dx \\
&= \iint |u(x_0) - u(x)| \, dx \, dy.
\end{aligned}
$$

Therefore, as $u$ is continuous, we see that $u \equiv u(x_0)$ on $B(x,r)$. Now, we can employ a "chain of balls" argument: Consider a chain of balls that intersect each other, and we can form a path tht passes through all the maximum points on each of the balls. Alternataively, observe that the set $\{x \in \Omega : u(x) = u(x_0)\} \subseteq \Omega$ is both closed (there exists an acccumulation of points is also in this set because $u$ is assumed to be continuous) and open (union of balls in $\Omega$), and also nonempty. So we have in fact $\{x \in \Omega : u(x) = u(x_0)\} = \Omega$, and so we conclude that $u$ is constant on all of $\Omega$. □

**Theorem** (Harnack's Inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and connected, and let $u : \Omega \to (0,\infty)$ be harmonic. Then, for any open, connected subset $\Omega'$ satisfying $\overline{\Omega'} \subseteq \Omega$, there exists a constant $C(\Omega,\Omega')$ that satisfies*

$$0 < \sup_{\overline{\Omega'}} u \leq C(\Omega,\Omega') \inf_{\overline{\Omega'}} u.$$

**Remark.** *It is necesssary to assume that $\Omega$ is connected. If $\Omega'$ were not connected, replace $\Omega'$ with a bigger set that is connected.*

*Proof of Harnack Inequality.* Giveen $a, b \in \overline{\Omega'}$, we join them by a path $\gamma$ in $\Omega'$. Our goal is to show

$$u(a) \leq C(\Omega,\Omega')u(b).$$

As $\overline{\Omega'}$ is compact, we can cover it by finitely many balls $B(x,r)$ with $x \in \Omega'$ and $B(x,4r) \subseteq \Omega$. Evidently, these balls cover $\gamma$. In a moment, we will show

$$\sup_{y \in \overline{B(x,r)}} u(y) \leq 3^d \inf_{y \in \overline{B(x,r)}} u(y).$$

This would prove the result with the constant being $3^d \cdot$ (number of balls). Indeed, if these balls are in fact closed, one can choose the points on the boundary of $\Omega$ to represent the entrance and exit of the path through $\Omega$. By the Mean Valeu Property, we have

$$\begin{aligned} u(a) &= \frac{1}{|B(a,r)|} \int_{B(a,r)} \int_{B(a,r)} u(x)\,dx \\ &\leq \frac{1}{|B(a,r)|} \int_{B(a,2r)} u(x)\,dx \\ &\leq \frac{|B(b,3r)|}{|B(a,r)|} u(b) \\ &= 3^d u(b), \end{aligned}$$

as desired. □

**Theorem** (Poisson Integral Formula)**.** *Let $d = 2$, and let $\varphi : \partial B = \partial B(0,R) \to \mathbb{R}$ be continuous. Then:*

*(a) The formula*

$$u(x) := \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} \frac{\varphi(y)\,darea}{area(\partial B)}$$

*defines a harmonic function in $B(0,R)$.*

*(b) Moreover, if $x_n \in B(0,R) \to x_\infty \in \partial B(0,R)$, then $u(x_n) \to \varphi(x_\infty)$; tht is, $u$ extends continuously to $\partial B$ and agrees with $\varphi$ there. Lastly, $u$ is not a unique function with these properties.*

**Remark.** *We call $u$ the solution to the Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } B(0,R), \\ u &= \varphi \quad \text{on } \partial B(0,R) \end{aligned}$$

*on the ball with boundary values $\varphi$.*

*Proof of the Poisson Integral Formula.* First, we will prove (a). We observe that the Poisson kernel

$$x \mapsto \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d}$$

is a linear combination of harmonic functions. So we have

$$\Delta\left(\frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d}\right) = 0.$$

So we conclude

$$\begin{aligned} \Delta u(x) &= \Delta \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} \frac{\varphi(y)\,d\text{area}}{\text{area}(\partial B)} \\ &= \int_{\partial B} \Delta\left(\frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d}\right) \frac{\varphi(y)\,d\text{area}}{\text{area}(\partial B)} \\ &= \int_{\partial B} 0 \frac{\varphi(y)\,d\text{area}}{\text{area}(\partial B)} \\ &= 0, \end{aligned}$$

meaning that $u$, as given by the Poisson Integral Formula, is harmonic, proving (a).

Next, we will prove (b). We observe

$$\int_{B(0,R)} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} \frac{d\text{area}(y)}{\text{area}(y)} = 1.$$

So, by part (a), we have

$$\begin{aligned} u(x_n) - \varphi(x_\infty) &= \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} \frac{\varphi(y)\, d\text{area}}{\text{area}(\partial B)} - \varphi(x_\infty) \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} \frac{d\text{area}}{\text{area}(\partial B)} \\ &= \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} (\varphi(y) - \varphi(x_\infty)) \frac{d\text{area}}{\text{area}(\partial B)} \end{aligned}$$

and so we have

$$\begin{aligned} |u(x_n) - \varphi(x_\infty)| &= \left| \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} (\varphi(y) - \varphi(x_\infty)) \frac{d\text{area}}{\text{area}(\partial B)} \right| \\ &\le \int_{\partial B} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} |\varphi(y) - \varphi(x_\infty)| \frac{d\text{area}}{\text{area}(\partial B)} \\ &= \int_{|y-x_\infty| \le \delta} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} |\varphi(y) - \varphi(x_\infty)| \frac{d\text{area}}{\text{area}(\partial B)} + \int_{|y-x_\infty| \ge \delta} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} |\varphi(y) - \varphi(x_\infty)| \frac{d\text{area}}{\text{area}(\partial B)} \\ &< \int_{|y-x_\infty| \ge \delta} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} \epsilon \frac{d\text{area}}{\text{area}(\partial B)} + \int_{|y-x_\infty| \ge \delta} \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d} |\varphi(y) - \varphi(x_\infty)| \frac{d\text{area}}{\text{area}(\partial B)} \\ &\vdots \text{see Evans PDE p. 41, but that left the proof as an exercise} \\ &= \int_{\partial B} (\text{Poisson kernel}) \cdot \epsilon\, d\text{area} \\ &= \epsilon \end{aligned}$$

because $\varphi$ is continuous. □

**Lemma.** *Let $\Omega$ be open with compact closure $\overline{\Omega}$. If $u$ and $v$ are continuous on $\overline{\Omega}$ and harmonic on $\Omega$, and $u \equiv v$ on $\partial\Omega$, then $u \equiv v$ on $\overline{\Omega}$.*

*Proof.* We note that $u - v$ and $v - u$ are continuous and subharmonic, and that they vanish on the boundary $\partial\Omega$. Now choose some $x_n \in \Omega$, so that we have $(u - v)(x_n) \to \sup_{x_n \in \overline{\Omega}}(u - v)$. Passing to a subsequence, if necessary, we may assume the limit point $x_\infty \in \overline{\Omega}$.

Case 1: If $x_\infty \in \partial\Omega$, then we have

$$\begin{aligned} \sup_{x \in \overline{\Omega}}(u - v) &= \lim_{n\to\infty} (u - v)(x_n) \\ &= (u - v)(x_\infty) \\ &= 0. \end{aligned}$$

Case 2: If $x_\infty \in \Omega$, then the subharmonic function $u - v$ achieves its maximum at an interior point of $\overline{\Omega}$. By the Strong Maximum Principle, $u - v$ is constant, at least on the connected component of $\Omega$ containing $x_\infty$. We can find the constant by looking at the boundary of this component. Since we assumed $u \equiv v$, or equivalently $u - v \equiv 0$ on $\partial\Omega$, it follows that this constant is in fact zero. Therefore, we have $v \le u$ everywhere on $\overline{\Omega}$. By similar reasoning, we have $u \le v$ everywhere on $\overline{\Omega}$. Therefore, we conclude $u \equiv v$ on $\overline{\Omega}$. □

**Theorem** (Weyl's Lemma)**.** *Harmonic functions are (locally) real analytic.*

**Lemma** (Cauchy estimates)**.** *If $u$ is harmonic on $B(0, R)$, then we have*

$$\left| \frac{\partial^\alpha u}{\partial x^\alpha}(0) \right| \le \left( \frac{\alpha|\alpha|}{R} \right)^{|\alpha|} \sup_{x \in B} |u|,$$

*where $|\alpha|$ is the total number of derivatives we have taken with respect to the variables.*

For example, if we take one derivative with respect to $x$, oen with respect to $y$, and one with respect to $z$, then $|\alpha| = 3$.

**Corollary** (Liouville's Theorem for harmonic functions)**.** *If $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic and satisfies*

$$|u(x)| \le C(1 + |x|^2)^{\frac{n}{2}},$$

*where $n$ is an integer, then $u$ is a polynomial of degre less than or equival to $n$.*

For holomorphic functions, we saw that the derivative at 0 is 0, using Cauchy estimates. Every point also obeys

$$u(x) \leq C_1(1 + |x - y|^2)^{\frac{n}{2}}$$

for some $C_y$, so 0 is not special.

**Definition.** *Fix $\Omega \subset \mathbb{R}^d$ to be open. We define the* compact-open topology *on the set $\{u : \Omega \to \mathbb{R}, u$ is continuous$\}$ by prescribing the following* sub-basis*: For each compact set $K \subseteq \Omega$ and $O \subseteq \mathbb{R}$, we define the sub-basic open set*

$$\mathcal{U}_{K,O} := \{u : \Omega \to \mathbb{C} \text{ is continuous} : u(K) \subset O\}.$$

*Note that a* basis *is a collection of open sets such that every oepn set is a union of basic open sets.*

example, balls of lateral radius form a dense set of points in a metric space, because rational numbers are dense in $\mathbb{R}$. A sub-basis is a collection of open sets such that finite intersections form a basis.

This topology is metrizable: Let $K_n$, $n \in \mathbb{N}$, be an exhaustion of $\Omega$ (that is, $\bigcup_{n=1}^{\infty} K_n = \Omega$) of $\Omega$ by compact sets. For example,

$$K_n := \{x \in \Omega : |x| \leq n \text{ and } \operatorname{dist}(x, \Omega^c) \geq \frac{1}{n}\}.$$

The compact-open topology is given by the metric

$$d(u, v) := \sum_{n=1}^{\infty} 2^{-n} d_n(u, v),$$

where

$$d_n(u, v) := \frac{\sup\{|u(x) - v(x)| : x \in K_n\}}{1 + \sup\{|u(x) - v(x)| : x \in K_n\}}.$$

To find the Dirichlet Green's function for the upper half-space $\{x \in \mathbb{R}^d : x_1 > 0\}$, we use reflection. The Green's function for the whole space is given by

$$g(x) := \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } d = 2, \\ \frac{1}{(d-2)\operatorname{area}(\partial B_1)} |x|^{-(d-2)} & \text{if } d \geq 3. \end{cases}$$

Let $\tilde{x} := (-x_1, x_2, \ldots, x_n)$. Then

$$G(y, x) := g(y - x) - g(y - \tilde{x})$$

is the Dirichlet Green's function. It is harmonic in $\Omega \setminus \{x\}$, the difference $g(y - \tilde{x}) = G(y, x) - g(y - x)$ is continuous (in fact, smooth) in $\overline{\Omega}$, and, if $y \in \partial\Omega$, i.e. $y_1 = 0$, then $G(y, x) = 0$.

For $B(0, R) \subseteq \mathbb{R}^d$, there is a parallel construction:

$$G(y, x) := \begin{cases} g(y - x) - g(y - \tilde{x}) - \frac{1}{2\pi} \log(\frac{R}{|x|}) & \text{if } d - 2, |x| \neq 0, \\ g(y - x) - (\frac{R}{|x|})^{d-2} g(y - \tilde{x}) & \text{if } d \geq 3, |x| \neq 0, \end{cases}$$

where $\tilde{x} := \frac{R^2}{|x|} \frac{x}{|x|} = \frac{R^2}{|x|^2} x$. We notice (c.f. Homework 0) that we have

$$\{y \in \mathbb{R}^d : |y| = R\} = \{y \in \mathbb{R}^d : \operatorname{dist}(y, \tilde{x}) = \frac{|x|}{R} \operatorname{dist}(y, x)\}.$$

Therefore, $G(y, x)$ vanishes at $|y| = R$. Abnd, if $y \in \partial\Omega$, i.e. $y_1 = 0$, then $G(y, x) = 0$. In $d = 2$, the formula

$$G(z, \zeta) := -\frac{1}{2\pi} \log \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|$$

is valid for $\mathbb{D}$. Let $u$ be harmonic and bounded in a neighborhood of $\overline{B(x, R)}$. Then

$$\left| \frac{\partial u}{\partial x_i}(0) \right| \leq \left( \frac{d}{R} \right) \sup_B |u|.$$

(This is still true if $u$ is harmonic only in $B(0, R)$ by exhaustion.) We take $j = 1$. Then $\frac{\partial u}{\partial x_1}$ is harmonic, and we have, using the Mean

Value Property,

$$\frac{\partial u}{\partial x_1}(0) = \frac{1}{\mathrm{vol}(B(0,R))} \int_{\partial B(0,R)} \mathrm{div}\left(\begin{bmatrix} u \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) d\mathrm{vol}$$

$$= \frac{1}{\mathrm{vol}(B(0,R))} \int_{\partial B(0,R)} \mathrm{div} \begin{bmatrix} u \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \nu \, d\mathrm{area},$$

where $\nu$ is the unit outward normal. So we have

$$\left|\frac{\partial u}{\partial x_1}(0)\right| \leq \frac{1}{\mathrm{vol}(B(0,R))} \int_{\partial B(0,R)} \left\|\begin{bmatrix} u \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\| |\nu| \, d\mathrm{area}$$

$$\leq \frac{1}{\mathrm{vol}(B(0,R))} \int_{\partial B(0,R)} |u| \, d\mathrm{area}$$

$$= \frac{\mathrm{area}(\partial B(0,R))}{\mathrm{vol}(B(0,R))} \frac{1}{\mathrm{area}(\partial B(0,R))} \int_{\partial B(0,R)} |u| \, d\mathrm{area}$$

$$= \frac{d}{R} \frac{1}{\mathrm{area}(\partial B(0,R))} \int_{\partial B(0,R)} |u| \, d\mathrm{area}$$

$$\leq \frac{d}{R} \sup_{x \in B(0,R)} |u|.$$