MATH 7A
Summer 2022
Group Activity 2 Solution

Use the limit definition of the derivative to compute $f'(x)$ given

$$f(x) = \frac{1}{x^7}.$$

Hint: In order to compute $f(x+h)$, you will need to expand $(x+h)^7$ in the denominator of $f(x+h)$, which means you will need to apply the Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \cdots + \binom{n}{n-1}a^{n-1}b^1 + \binom{n}{n}a^0b^n,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is called the binomial coefficient and

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

denotes the factorial of n.

Solution. Our binomial coefficients are

$$\begin{aligned}
\binom{7}{0} &= \frac{7!}{0!(7-0)!} = 1,\\
\binom{7}{1} &= \frac{7!}{1!(7-1)!} = 7,\\
\binom{7}{2} &= \frac{7!}{2!(7-2)!} = 21,\\
\binom{7}{3} &= \frac{7!}{3!(7-3)!} = 35,\\
\binom{7}{4} &= \frac{7!}{4!(7-4)!} = 35,\\
\binom{7}{5} &= \frac{7!}{5!(7-5)!} = 21,\\
\binom{7}{6} &= \frac{7!}{6!(7-6)!} = 7,\\
\binom{7}{7} &= \frac{7!}{7!(7-7)!} = 1.
\end{aligned}$$

The Binomial Theorem allows us to write

$$\begin{aligned}
f(x+h) &= \frac{1}{(x+h)^7}\\
&= \frac{1}{\binom{7}{0}x^7h^0 + \binom{7}{1}x^6h^1 + \binom{7}{2}x^5h^2 + \binom{7}{3}x^4h^3 + \binom{7}{4}x^3h^4 + \binom{7}{5}x^2h^5 + \binom{7}{1}x^1h^6 + \binom{7}{0}x^0h^7}\\
&= \frac{1}{x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7}.
\end{aligned}$$

So the difference quotient is

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7}-\frac{1}{x^7}}{h} \\
&= \frac{\frac{x^7-(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)}{x^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)}}{h} \\
&= \frac{\frac{\cancel{x^7}-\cancel{x^7}-7x^6h-21x^5h^2-35x^4h^3-35x^3h^4-21x^2h^5-7xh^6-h^7}{x^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)}}{h} \\
&= \frac{\frac{-7x^6h-21x^5h^2-35x^4h^3-35x^3h^4-21x^2h^5-7xh^6-h^7}{x^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)}}{h} \\
&= \frac{-7x^6h-21x^5h^2-35x^4h^3-35x^3h^4-21x^2h^5-7xh^6-h^7}{hx^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)} \\
&= \frac{\cancel{h}(-7x^6-21x^5h-35x^4h^2-35x^3h^3-21x^2h^4-7xh^5-h^6)}{\cancel{h}x^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)} \\
&= \frac{-7x^6-21x^5h-35x^4h^2-35x^3h^3-21x^2h^4-7xh^5-h^6}{x^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)}.
\end{aligned}
$$

Therefore, the derivative of $f(x)$ is

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{-7x^6-21x^5h-35x^4h^2-35x^3h^3-21x^2h^4-7xh^5-h^6}{x^7(x^7+7x^6h+21x^5h^2+35x^4h^3+35x^3h^4+21x^2h^5+7xh^6+h^7)} \\
&= \frac{-7x^6-21x^50-35x^40^2-35x^30^3-21x^20^4-7x0^5-0^6}{x^7(x^7+7x^60+21x^50^2+35x^40^3+35x^30^4+21x^20^5+7x0^6+0^7)} \\
&= \frac{-7x^6}{x^7(x^7)} \\
&= -\frac{7x^6}{x^{14}} \\
&= \boxed{-\frac{7}{x^8}}.
\end{aligned}
$$

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