MATH 7A
Summer 2022
Group Activity 3 Solution

Find the equation of the tangent line to the curve

$$f(x) = \left(\frac{x^2 \sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}} \right)^9$$

at $x = 1$.

Solution. By applying the chain rule, we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2 \sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}} \right)^9 \\ &= 9 \left(\frac{x^2 \sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}} \right)^8 \frac{d}{dx} \left(\frac{x^2 \sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}} \right). \end{aligned}$$

Furthermore, the quotient rule gives us

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2 \sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}} \right) &= \frac{\frac{d}{dx}[x^2 \sqrt[3]{2x^2+3}][\sqrt[4]{3x^4+2}] - [x^2 \sqrt[3]{2x^2+3}]\frac{d}{dx}[\sqrt[4]{3x^4+2}]}{[\sqrt[4]{3x^4+2}]^2} \\ &= \frac{\frac{d}{dx}[x^2 \sqrt[3]{2x^2+3}][\sqrt[4]{3x^4+2}] - [x^2 \sqrt[3]{2x^2+3}]\frac{d}{dx}[\sqrt[4]{3x^4+2}]}{\sqrt{3x^4+2}} \\ &= \frac{\frac{d}{dx}[x^2 \sqrt[3]{2x^2+3}][\sqrt[4]{3x^4+2}]}{\sqrt{3x^4+2}} - \frac{[x^2 \sqrt[3]{2x^2+3}]\frac{d}{dx}[\sqrt[4]{3x^4+2}]}{\sqrt{3x^4+2}} \\ &= \frac{\frac{d}{dx}[x^2 \sqrt[3]{2x^2+3}]}{\sqrt[4]{3x^4+2}} - \frac{[x^2 \sqrt[3]{2x^2+3}]\frac{d}{dx}[\sqrt[4]{3x^4+2}]}{\sqrt{3x^4+2}}. \end{aligned}$$

By the product rule and the chain rule, we obtain

$$\begin{aligned} \frac{d}{dx}[x^2 \sqrt[3]{2x^2+3}] &= \frac{d}{dx}(x^2)\sqrt[3]{2x^2+3} + x^2 \frac{d}{dx}\sqrt[3]{2x^2+3} \\ &= 2x\sqrt[3]{2x^2+3} + x^2 \frac{d}{dx}(2x^2+3)^{\frac{1}{3}} \\ &= 2x\sqrt[3]{2x^2+3} + x^2 \left[\frac{1}{3}(2x^2+3)^{-\frac{2}{3}} \frac{d}{dx}(2x^2+3) \right] \\ &= 2x\sqrt[3]{2x^2+3} + x^2 \left[\frac{1}{3}(2x^2+3)^{-\frac{2}{3}}(4x) \right] \\ &= 2x\sqrt[3]{2x^2+3} + \frac{4x^3}{3\sqrt[3]{(2x^2+3)^2}}. \end{aligned}$$

By the chain rule, we obtain

$$\begin{aligned}\frac{d}{dx}[\sqrt[4]{3x^4+2}] &= \frac{d}{dx}(3x^4+2)^{\frac{1}{4}}\\ &= \frac{1}{4}(3x^4+2)^{-\frac{3}{4}}\frac{d}{dx}(3x^4+2)\\ &= \frac{1}{4}(3x^4+2)^{-\frac{3}{4}}(12x^3)\\ &= \frac{3x^3}{(3x^4+2)^{\frac{3}{4}}}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\right) &= \frac{\frac{d}{dx}[x^2\sqrt[3]{2x^2+3}]}{\sqrt[4]{3x^4+2}} - \frac{[x^2\sqrt[3]{2x^2+3}]\frac{d}{dx}[\sqrt[4]{3x^4+2}]}{\sqrt{3x^4+2}}\\ &= \frac{\frac{d}{dx}[x^2\sqrt[3]{2x^2+3}]}{\sqrt[4]{3x^4+2}} - \frac{[x^2\sqrt[3]{2x^2+3}]\frac{d}{dx}[\sqrt[4]{3x^4+2}]}{\sqrt{3x^4+2}}\\ &= \frac{1}{\sqrt[4]{3x^4+2}}\left(2x\sqrt[3]{2x^2+3} + \frac{4x^3}{3\sqrt[3]{(2x^2+3)^2}}\right) - \frac{[x^2\sqrt[3]{2x^2+3}]}{\sqrt{3x^4+2}}\frac{3x^3}{(3x^4+2)^{\frac{3}{4}}}\\ &= \frac{1}{\sqrt[4]{3x^4+2}}\left(2x\sqrt[3]{2x^2+3} + \frac{4x^3}{3\sqrt[3]{(2x^2+3)^2}}\right) - \frac{3x^5\sqrt[3]{2x^2+3}}{(3x^4+2)^{\frac{5}{4}}}\\ &= \frac{1}{\sqrt[4]{3x^4+2}}\left(2x\sqrt[3]{2x^2+3} + \frac{4x^3}{3\sqrt[3]{(2x^2+3)^2}} - \frac{3x^5\sqrt[3]{2x^2+3}}{3x^4+2}\right)\\ &= \frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\frac{1}{x^2}\left(2x + \frac{4x^3}{3(2x^2+3)} - \frac{3x^5}{3x^4+2}\right)\\ &= \frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\left(\frac{18}{x} + \frac{12x}{2x^2+3} - \frac{27x^3}{3x^4+2}\right),\end{aligned}$$

and so our first derivative is

$$\begin{aligned}f'(x) &= 9\left(\frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\right)^8\frac{d}{dx}\left(\frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\right)\\ &= 9\left(\frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\right)^8\frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\left(\frac{18}{x} + \frac{12x}{2x^2+3} - \frac{27x^3}{3x^4+2}\right)\\ &= \left(\frac{x^2\sqrt[3]{2x^2+3}}{\sqrt[4]{3x^4+2}}\right)^9\left(\frac{18}{x} + \frac{12x}{2x^2+3} - \frac{27x^3}{3x^4+2}\right)\\ &= f(x)\left(\frac{18}{x} + \frac{12x}{2x^2+3} - \frac{27x^3}{3x^4+2}\right).\end{aligned}$$

At $x = 1$, we have

$$\begin{aligned}
f(1) &= \left(\frac{(1)^2 \sqrt[3]{2(1)^2 + 3}}{\sqrt[4]{3(1)^4 + 2}} \right)^9 \\
&= \left(\frac{\sqrt[3]{5}}{\sqrt[4]{5}} \right)^9 \\
&= (5^{\frac{1}{3} - \frac{1}{4}})^9 \\
&= (5^{\frac{1}{12}})^9 \\
&= 5^{\frac{3}{4}}
\end{aligned}$$

and

$$\begin{aligned}
f'(1) &= f(1) \left(\frac{18}{1} + \frac{12(1)}{2(1)^2 + 3} - \frac{27(1)^3}{3(1)^4 + 2} \right) \\
&= 5^{\frac{3}{4}} \left(18 + \frac{12}{5} - \frac{27}{5} \right) \\
&= 5^{\frac{3}{4}} \frac{90 + 12 - 27}{5} \\
&= 5^{\frac{3}{4}} \frac{75}{5} \\
&= 5^{\frac{3}{4}} \cdot 15.
\end{aligned}$$

Now, we know that the point-slope formula for a linear equation is

$$y - y_1 = m(x - x_1).$$

With $x_1 = 1$, $y_1 = f(1) = 5^{\frac{3}{4}}$, and $m = f'(1) = 5^{\frac{3}{4}} \cdot 15$, we have

$$y - 5^{\frac{3}{4}} = 5^{\frac{3}{4}} \cdot 15(x - 1),$$

which is algebraically equivalent to

$$\boxed{y = (5^{\frac{3}{4}} \cdot 15)x - 5^{\frac{3}{4}} \cdot 14}.$$

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