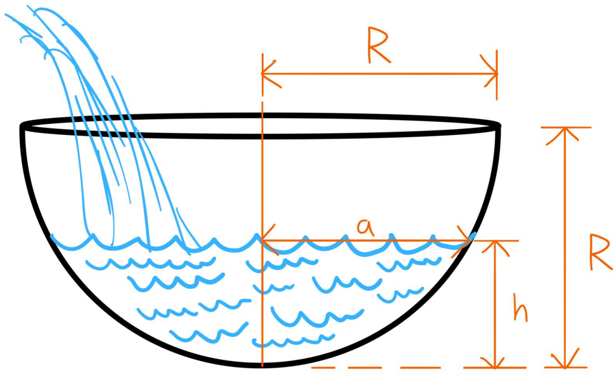
MATH 7A
Summer 2022
Group Activity 4 Solution

The formula for the volume of the water in a semi-spherical bowl in terms of the water's height h and the radius of the water's surface a is given by

$$V = \frac{1}{6}\pi h(3a^2 + h^2)$$

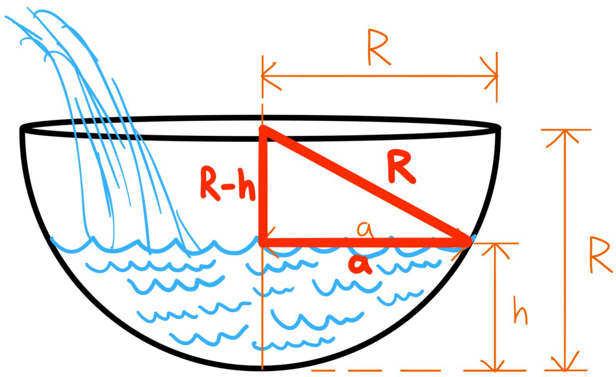
for $0 \leq h \leq R$ and $0 \leq a \leq R$.



Water is being filled in the semi-spherical bowl at a rate of 1,200 cm^3/sec. If the maximal radius of the bowl is 40 cm, what is the rate of the water rising when the height of the water is 20 cm?

Either express your final answer as an exact expression or approximate it to two decimal places. Don't forget units in your final answer! You may use a calculator for this activity.

Solution. First, notice that we can eliminate the parameter a and re-express our volume formula in terms of h and R, which would be advantageous because R is a constant value.



Indeed, by the Pythagorean Theorem for the red triangle in our figure above, we have

$$a^2 + (R-h)^2 = R^2,$$

which implies

$$\begin{aligned}
a^2 &= R^2 - (R-h)^2 \\
&= R^2 - (R^2 - 2Rh + h^2) \\
&= R^2 - R^2 + 2Rh - h^2 \\
&= 2Rh - h^2.
\end{aligned}$$

So we can rewrite our formula for the volume of the water in the semi-spherical bowl as

$$\begin{aligned}
V &= \frac{1}{6}\pi h(3a^2 + h^2) \\
&= \frac{1}{6}\pi h(3(2Rh - h^2) + h^2) \\
&= \frac{1}{6}\pi h(6Rh - 3h^2 + h^2) \\
&= \frac{1}{6}\pi h(6Rh - 2h^2) \\
&= \frac{1}{6}\pi h(2h(3R - h)) \\
&= \frac{1}{3}\pi h^2(3R - h).
\end{aligned}$$

As we said before, R is a constant. So the only variables that change with respect to time are V and h. By applying implicit differentiation with respect to time t, as well as the product rule and chain rule, we obtain

$$\begin{aligned}
\frac{dV}{dt} &= \frac{d}{dt}\left(\frac{1}{3}\pi h^2(3R - h)\right) \\
&= \frac{1}{3}\pi \frac{d}{dt}(h^2(3R - h)) \\
&= \frac{1}{3}\pi \left(\frac{d}{dt}(h^2)(3R - h) + h^2\frac{d}{dt}(3R - h)\right) \\
&= \frac{1}{3}\pi \left(\left(2h\frac{dh}{dt}\right)(3R - h) + h^2\frac{d}{dt}(3R - h)\right) \\
&= \frac{1}{3}\pi \left(2h\frac{dh}{dt}(3R - h) + h^2\left(0 - \frac{dh}{dt}\right)\right) \\
&= \frac{1}{3}\pi \left(2h\frac{dh}{dt}(3R - h) - h^2\frac{dh}{dt}\right) \\
&= \frac{1}{3}\pi h\frac{dh}{dt}(2(3R - h) - h) \\
&= \frac{1}{3}\pi h\frac{dh}{dt}(6R - 2h - h) \\
&= \frac{1}{3}\pi h\frac{dh}{dt}(6R - 3h) \\
&= \frac{1}{\cancel{3}}\pi h\frac{dh}{dt}\cancel{3}(2R - h) \\
&= \pi h\frac{dh}{dt}(2R - h) \\
&= \pi h(2R - h)\frac{dh}{dt}.
\end{aligned}$$

So the rate of which the height of the water in the bowl is rising is

$$\frac{dh}{dt} = \frac{1}{\pi h(2R - h)} \frac{dV}{dt}.$$

Finally, we substitute $R = 40$ cm, $h = 20$ cm, and $\frac{dV}{dt} = 1200\ \text{cm}^3/\text{sec}$ to obtain

$$\begin{aligned}
\frac{dh}{dt} &= \frac{1}{\pi h(2R - h)} \frac{dV}{dt} \\
&= \frac{1}{\pi(20)(2(40) - 20)} 1200 \\
&= \frac{1}{1200\pi} 1200 \\
&= \boxed{\frac{1}{\pi}\ \text{cm/sec}} \\
&\approx \boxed{0.32\ \text{cm/sec}}.
\end{aligned}$$

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