MATH 7A
Summer 2022
Group Activity 5 Solution

Use l'Hôpital's Rule to compute

$$\lim_{x\to 0} \frac{\tan x - x}{x - x\cos x}.$$

Solution. We have

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x - x\cos x} &= \frac{\tan 0 - 0}{0 - 0\cos 0} \\ &= \frac{0-0}{0 - 0\cdot 1} \\ &= \frac{0}{0},\end{aligned}$$

which is indeterminate. So we apply l'Hôpital's Rule for the first time to obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x - x\cos x} &= \lim_{x\to 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(x - x\cos x)} \\ &= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - ((\frac{d}{dx}x)\cos x + x\frac{d}{dx}\cos x)} \\ &= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - (1\cos x + x(-\sin x))} \\ &= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x + x\sin x} \\ &= \frac{\sec^2 0 - 1}{1 - \cos 0 + 0\sin 0} \\ &= \frac{1^2 - 1}{1 - 1 + 0} \\ &= \frac{0}{0},\end{aligned}$$

which is indeterminate. So we apply l'Hôpital's Rule for the second time to obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x - x}{x - x\cos x} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{1-\cos x + x\sin x}\\
&= \lim_{x\to 0}\frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}(1-\cos x + x\sin x)}\\
&= \lim_{x\to 0}\frac{2\sec x\frac{d}{dx}\sec x - 0}{0-(-\sin x)+((\frac{d}{dx}x)\sin x + x\frac{d}{dx}(\sin x))}\\
&= \lim_{x\to 0}\frac{2\sec x(\sec x\tan x)}{\sin x + (1\sin x + x\cos x)}\\
&= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{2\sin x + x\cos x}\\
&= \frac{2\sec^2 0\tan 0}{2\sin 0 - 0\cos 0}\\
&= \frac{2\cdot 1^2\cdot 0}{2\cdot 0 - 0\cdot 1}\\
&= \frac{0}{0},
\end{aligned}$$

which is indeterminate. So we apply l'Hôpital's Rule for the third time to obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x - x}{x - x\cos x} &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{2\sin x + x\cos x}\\
&= \lim_{x\to 0}\frac{\frac{d}{dx}(2\sec^2 x\tan x)}{\frac{d}{dx}(2\sin x + x\cos x)}\\
&= \lim_{x\to 0}\frac{2((\frac{d}{dx}\sec^2 x)\tan x + \sec^2 x\frac{d}{dx}\tan x)}{2\cos x + ((\frac{d}{dx}x)\cos x + x\frac{d}{dx}\cos x)}\\
&= \lim_{x\to 0}\frac{2(2\sec x\frac{d}{dx}\sec x)\tan x + \sec^2 x\sec^2 x)}{2\cos x + (1\cos x + x(-\sin x))}\\
&= \lim_{x\to 0}\frac{2(2\sec x(\sec x\tan x))\tan x + \sec^4 x)}{3\cos x - x\sin x}\\
&= \lim_{x\to 0}\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{3\cos x - x\sin x}\\
&= \frac{4\sec^2 0\tan^2 0 + 2\sec^4 0)}{3\cos 0 - 0\sin 0}\\
&= \frac{4\cdot 1^2\cdot 0^2 + 2\cdot 1^4}{3\cdot 1 - 0\cdot 0}\\
&= \frac{0+2}{3-0}\\
&= \boxed{\frac{2}{3}}.
\end{aligned}$$

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