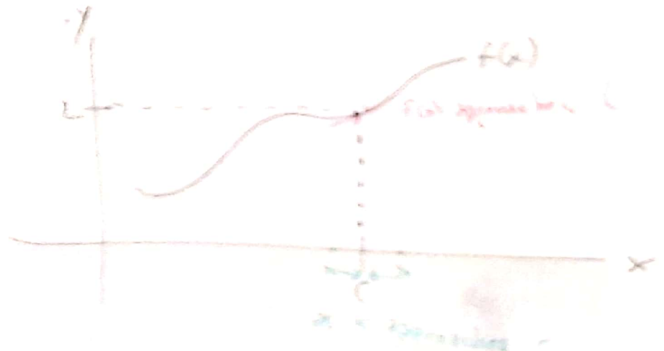
MATH 7A SUMMER 2022



3.1 Limits

Definition The limit of $f(x)$, as x approaches c, is equal to L:

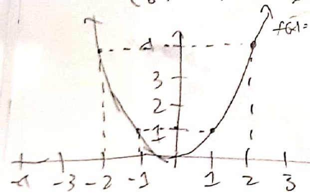
$f(x)$ is close to L
if x is close to c

NOTATION:

$$\lim_{x \to c} f(x) = L$$

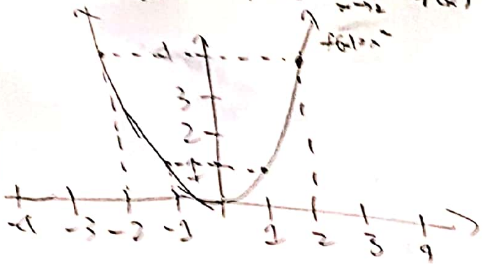
Example 1: De

(a) Find lim



Example 1: Define $f(x)=x^2$.

(a) Find $\lim_{x\to 2} f(x)$



x	$f(x)$
1.9	3.6100
1.99	3.9601
1.999	3.9960
1.9999	3.9996

we rounded off to the nearest fourth decimal place.

x approaches 2 from the left   f(x) approaches 4

x	$f(x)$
2.1	4.4100
2.01	4.0400
2.001	4.0040
2.0001	4.0004

x approaches 2 from the right   f(x) approaches 4

$$\lim_{x\to 2} x^2 = \boxed{4}$$

(b) Find $\lim_{x\to 1} f(x)$

x	$f(x)$
0.9	0.8100
0.99	0.9801
0.999	0.9980
0.9999	0.9998

x	$f(x)$
1.1	1.21
1.01	1.0201
1.001	1.0020
1.0001	1.0001

$$\lim_{x\to 1} x^2 = 1$$

Example 2: (a) Find $\lim_{x \to 0} \frac{x^2-9}{x-3}$

$$\lim_{x \to 0} \frac{x^2-9}{x-3} = \frac{0^2-9}{0-3} = \frac{-9}{-3} = \boxed{3}$$

(b) Find $\lim_{x \to 3} \frac{x^2-9}{x-3}$

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \frac{3^2-9}{3-3} = \frac{9-9}{3-3} = \frac{0}{0}$$ indeterminate

Try this instead:

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim_{x \to 3} (x+3) = 3+3 = \boxed{6}$$

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Example: Find $\lim_{x \to 0} e^{-|x|}$

$$e^{-|x|} = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{-(-x)} & \text{for } x < 0 \end{cases}$$

$$= \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{x} & \text{for } x < 0. \end{cases}$$

ONE SIDED LIMITS

Left-hand limit

$$\lim_{x \to 0^-} e^{-|x|} = \lim_{x \to 0^-} e^{x} = e^{0} = 1$$

minus sign for left-hand limits

Right-hand limit

$$\lim_{x \to 0^+} e^{-|x|} = \lim_{x \to 0^+} e^{-x} = e^{-0} = 1$$

The left-hand limit and the right-hand limit are the same.

Therefore,

$$\lim_{x \to 0} e^{-|x|} = \boxed{1}$$

NOTE: Otherwise, if the left-hand and right-hand limits are different, then the two-sided limit DOES NOT EXIST (D.N.E.).

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Example: (a) Find $\lim_{x\to 0} \frac{1}{x}$

x	$\frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
-0.0001	-10000

x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10000

getting very large in "−" the direction

getting very large in "+" the direction

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\qquad$ $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

Two-sided limit

$\lim_{x\to 0} \frac{1}{x} = \boxed{D.N.E.}$

does not exist

(b) Find $\lim_{x\to 0} \frac{1}{x^2}$

x	$\frac{1}{x^2}$
-0.1	100
-0.01	10 000
-0.001	1000000
-0.0001	100000000

x	$\frac{1}{x^2}$
0.1	100
0.01	10000
0.001	1000000
0.0001	100 000 000

$\lim_{x\to 0^-} \frac{1}{x} = +\infty$ $\qquad$ $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\lim_{x\to 0} \frac{1}{x} = \boxed{+\infty}$

Divergent limits

$\lim_{x\to c} f(x) = \pm\infty$

Convergent limits

$\lim_{x\to c} f(x) = L$

for $-\infty < L < \infty$.

(L is a real number.)

Example $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$.

x	$\sin\left(\frac{\pi}{x}\right)$
$\frac{2}{11}$	-1
$\frac{2}{101}$	1
$\frac{2}{1001}$	1
$\frac{2}{10001}$	1
approaches $x = 0$	-1, -1, 1, -1, -1, 1, 1, ...

The values of $\sin\left(\frac{\pi}{x}\right)$ oscillate from -1 to 1, 1 to -1

y-values do NOT approach any value (they oscillate)

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = \boxed{DNE}$$

A limit is called divergent if:

- The limit is $\pm\infty$
- The limit does not exist.

Example $\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2}$

$$\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} = \lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+16})^2-4^2}{x^2(\sqrt{x^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{x^2+\cancel{16}-\cancel{16}}{x^2(\sqrt{x^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}(\sqrt{x^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+16}+4}$$

$$= \frac{1}{\sqrt{0^2+16}+4}$$

$$= \boxed{\frac{1}{8}}$$

Limit Laws for functions

Let a be a constant, and suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then the following rules hold.

1. $\lim_{x\to c} (af(x)) = a \lim_{x\to c} f(x)$
2. $\lim_{x\to c} (f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
3. $\lim_{x\to c} (f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$

Example: $\lim_{x \to 2} (x^3 + 4x - 1)$

$$\lim_{x \to 2} x^3 = \lim_{x \to 2} (x \cdot x \cdot x)$$
$$= \lim_{x \to 2} x \cdot \lim_{x \to 2} x \cdot \lim_{x \to 2} x$$
$$= 2 \cdot 2 \cdot 2$$
$$= 2^3$$
$$= 8$$

$$\lim_{x \to 2} 4x = 4 \lim_{x \to 2} x$$
$$= 4 \cdot 2$$
$$= 8$$

$$\lim_{x \to 2} 1 = 1$$

Therefore

$$\lim_{x \to 2} (x^3 + 4x - 1) = \lim_{x \to 2} x^3 + \lim_{x \to 2} 4x - \lim_{x \to 2} 1$$
$$= 8 + 8 - 1$$
$$= \boxed{15}$$

Example: $\lim_{x \to 4} \frac{x^2+1}{x-3}$

$$\lim_{x \to 4} \frac{x^2+1}{x-3} = \frac{\lim_{x \to 4} (x^2+1)}{\lim_{x \to 4} (x-3)}$$

$$= \frac{\lim_{x \to 4} x^2 + \lim_{x \to 4} 1}{\lim_{x \to 4} x - \lim_{x \to 4} 3}$$

$$= \frac{16 + 1}{4 - 3}$$

$$= \boxed{17}$$

$\lim_{x \to 4} x^2 = 4^2 = 16$

$\lim_{x \to 4} 1 = 1$

$\lim_{x \to 4} x = 4$

$\lim_{x \to 4} 3 = 3$

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Example: $\lim_{x\to\infty} \frac{2x^2+3x+5}{4x^2+x+7}$

$$\lim_{x\to\infty} \frac{2x^2+3x+5}{4x^2+x+7} = \lim_{x\to\infty} \frac{2x^2\cdot\frac{1}{x^2}+3x\cdot\frac{1}{x^2}+5\cdot\frac{1}{x^2}}{4x^2\cdot\frac{1}{x^2}+x\cdot\frac{1}{x^2}+7\cdot\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{5}{x^2}}{4+\frac{1}{x}+\frac{7}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{3}{x} + \lim_{x\to\infty}\frac{5}{x^2}}{\lim_{x\to\infty} 4 + \lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{7}{x^2}}$$

$$= \frac{2+0+0}{4+0+0}$$

$$= \frac{2}{4} = \boxed{\frac{1}{2}}$$

3.2 Continuity

__Definition:__ We say a function f is CONTINUOUS AT $x=c$ if

$$\lim_{x\to c} f(x) = f(c).$$

CONDITIONS FOR $f(x)$ to be continuous at $x=c$

1. $f(x)$ is defined at $x=c$ (that is, $f(c)$ exists)
2. $\lim_{x\to c} f(x)$ exists (Equivalently, $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and are equal to each other.)
3. $\lim_{x\to c} f(x) = f(c)$.

If any one of these conditions fails, then we say $f(x)$ is DISCONTINUOUS at $x=c$.

Example: Show that $f(x) = 2x-3$ is continuous at $x=1$.

1. $f(1) = 2\cdot 1 - 3$
$= 2-3$
$= -1$

So $f(1)$ exists, or $f(x)$ is defined at $x=1$.

2. $\lim_{x\to 1} f(x) = \lim_{x\to 1} (2x-3)$
$= \lim_{x\to 1} (2x) - \lim_{x\to 1} 3$
$= 2\lim_{x\to 1} x - \lim_{x\to 1} 3$
$= 2\cdot 1 - 3$
$= 2-3$
$= -1$

3. $\lim_{x\to 1} f(x) = -1$
$f(1) = -1$

So $\lim_{x\to 1} f(x) = f(1)$.

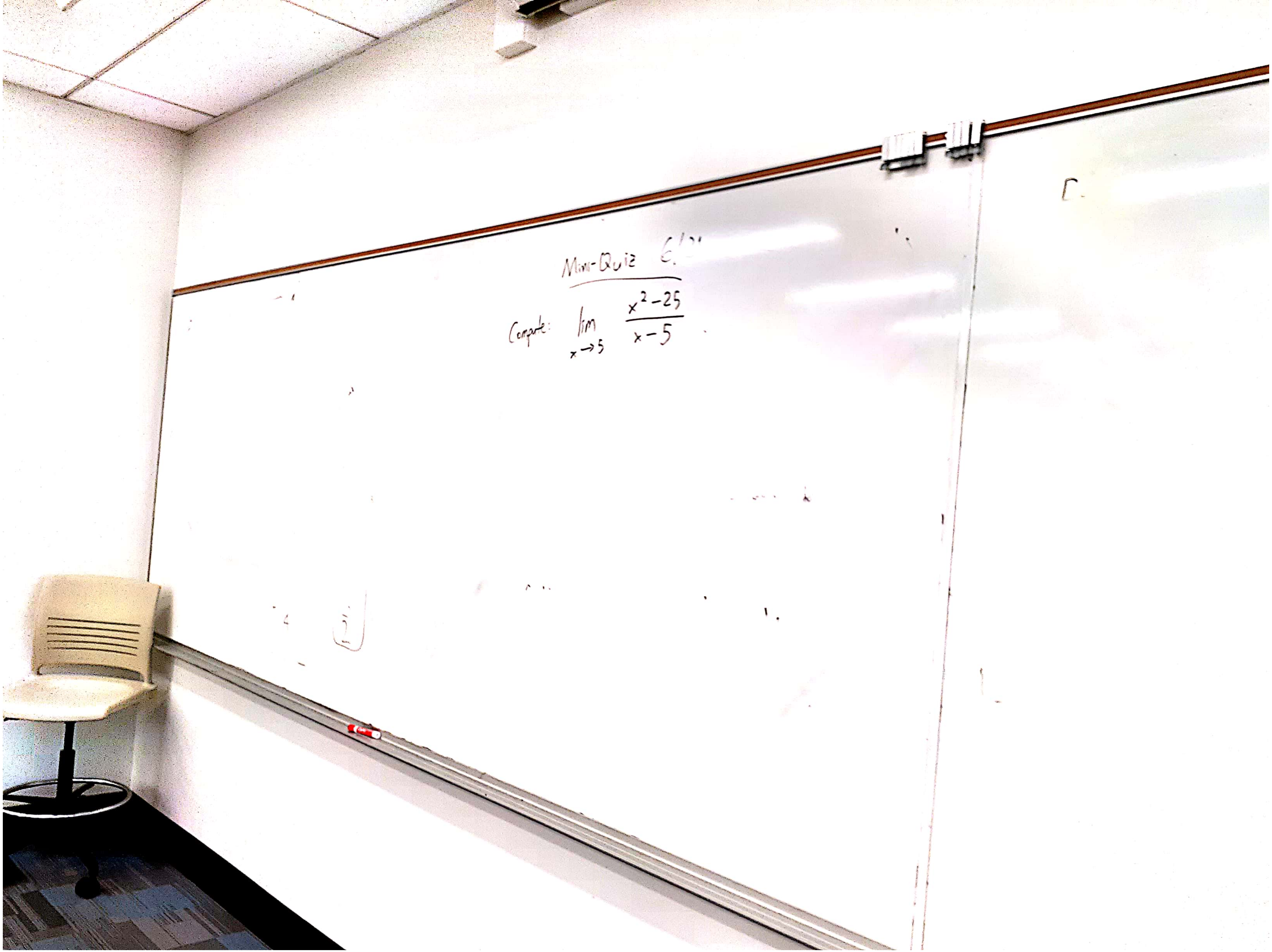
Example: Consider $f(x) = \begin{cases} \frac{x^2-x-6}{x-3} & \text{for } x \neq 3 \\ a & \text{for } x = 3 \end{cases}$

find a so that $f(x)$ is continuous at $x=3$

SOLUTION: $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-x-6}{x-3}$
$= \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3}$
$= \lim_{x\to 3} x+2$
$= 3+2$
$= 5$

We want
$\lim_{x\to 3} f(x) = f(3)$
$\lim_{x\to 3} f(x) = a$
$5 = a$

Choose $a = \boxed{5}$.


Mini-Quiz
Compute: $\lim_{x \to 5} \frac{x^2-25}{x-5}$