3.4 Trigonometric Limits and the Sandwich Theorem

Know these limits

- $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = 0$

More generally, if a is a constant, then:

- $\lim\limits_{x\to 0} \frac{\sin(ax)}{ax} = 1$
- $\lim\limits_{x\to 0} \frac{1-\cos(ax)}{ax} = 0$.

Why? Let $u = ax$.
If $x \to 0$ and a is constant,
then $u = ax \to a \cdot 0 = 0$.

$$\lim_{x\to 0} \frac{\sin(ax)}{x} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1$$

Scanné avec CamScanner

Example: Prove $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, assuming we can use $\lim_{x\to 0} \frac{\sin x}{x} = 1$ without proof.

PROOF:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{1^2-(\cos x)^2}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$\boxed{\begin{aligned} \sin^2 x + \cancel{\cos^2 x} &= 1 \\ -\cancel{\cos^2 x} \quad &\; -\cos^2 x \\ \hline \sin^2 x &= 1 - \cos^2 x \end{aligned}}$$

$$= \lim_{x\to 0} \frac{\sin x \cdot \sin x}{x(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}\right)$$

$$= \boxed{\lim_{x\to 0} \frac{\sin x}{x}} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x}$$

$$= \boxed{1} \cdot \frac{\sin 0}{1+\cos 0}$$

$$= 1 \cdot \frac{0}{1+1}$$

$$= 1 \cdot \frac{0}{2}$$

$$= \boxed{0}$$

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Example: $\lim_{x\to 0} \frac{\sin(3x)}{5x}$

$$\lim_{x\to 0}\frac{\sin(3x)}{5x} = \lim_{x\to 0}\frac{\sin(3x)}{5x}\cdot\frac{3}{3}$$
$$= \lim_{x\to 0}\frac{\sin(3x)}{3x}\,\frac{3}{5}$$
$$= \frac{3}{5}\lim_{x\to 0}\frac{\sin(3x)}{3x}$$
$$= \frac{3}{5}\cdot 1$$
$$= \boxed{\frac{3}{5}}$$

Example: $\lim_{x\to 0}\frac{\sin^2 x}{x^2}$

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = \lim_{x\to 0}\frac{(\sin x)^2}{x^2}$$
$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$
$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{x}\right)$$
$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{x}$$
$$= 1\cdot 1$$
$$= \boxed{1}$$

Example: $\lim_{x\to 0}\frac{\sec x - 1}{x\sec x}$

$$\sec x = \frac{1}{\cos x}$$

$$\lim_{x\to 0}\frac{\sec x - 1}{x\sec x} = \lim_{x\to 0}\frac{\frac{1}{\cos x} - 1}{x\,\frac{1}{\cos x}}$$
$$= \lim_{x\to 0}\frac{\frac{1}{\cos x} - 1}{x\cdot\frac{1}{\cos x}}\cdot\frac{\cos x}{\cos x}$$
$$= \lim_{x\to 0}\frac{1-\cos x}{x}$$
$$= \boxed{0}$$

Sandwich Theorem (Squeeze Theorem)

If $f(x)$, $g(x)$, $h(x)$ are three functions that satisfy

$$f(x) \le g(x) \le h(x)$$

in an open interval that contains c,

and

$$\lim_{x \to c} f(x) = L$$

and $\lim_{x \to c} h(x) = L$,

then:

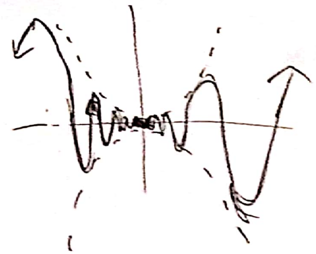
$$\lim_{x \to c} g(x) = L.$$

Example: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

For all real numbers u, we know:

$$-1 \le \sin u \le 1$$

and $-1 \le \cos u \le 1$



$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Here,
$f(x) = -x^2$
$g(x) = x^2 \sin\left(\frac{1}{x}\right)$
$h(x) = x^2$
and
$f(x) \le g(x) \le h(x)$

$$x^2 \cdot (-1) \le x^2 \cdot \sin\left(\frac{1}{x}\right) \le x^2 \cdot 1$$

$$\underbrace{\boxed{-x^2}}_{f(x)} \le \underbrace{\boxed{x^2 \sin\left(\frac{1}{x}\right)}}_{g(x)} \le \underbrace{\boxed{x^2}}_{h(x)}$$

$$\lim_{x \to 0}(-x^2) \le \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \le \lim_{x \to 0} x^2$$

$$\underbrace{\boxed{0}}_{\lim_{x \to 0} f(x)} \le \underbrace{\boxed{\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)}}_{\lim_{x \to 0} g(x)} \le 0$$

By the Sandwich Theorem, we conclude

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = \boxed{0}$$

Example: $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Case 1: $x \ge 0$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$x\cdot(-1) \le x\cdot\sin\left(\frac{1}{x}\right) \le x\cdot 1$$

$$-x \le x\sin\left(\frac{1}{x}\right) \le x$$

$$\lim_{x\to 0}(-x) \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x$$

$$0 \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le 0$$

By the Sandwich Theorem, we conclude

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \boxed{0}$$

Case 2: $x < 0$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$x\cdot(-1) \ge x\sin\left(\frac{1}{x}\right) \ge x\cdot 1$$

$$-x \ge x\sin\left(\frac{1}{x}\right) \ge x$$

$$\lim_{x\to 0}(-x) \ge \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \ge \lim_{x\to 0} x$$

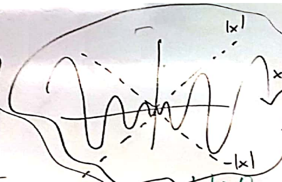
$$0 \ge \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \ge 0$$

WRITE THIS the other way:

$$0 \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le 0$$

By the Sandwich Theorem, we conclude

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \boxed{0}$$



$|x| \ge 0$ inequalities will not flip

ALTERNATE SOLUTION

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Observe from this graph

$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$

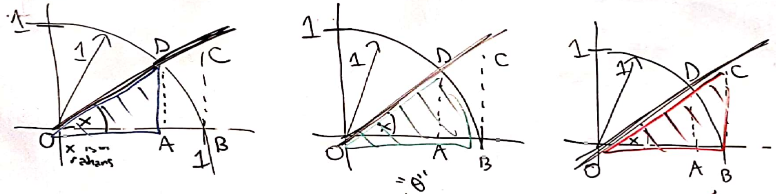
$$\lim_{x\to 0}(-|x|) \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} |x|$$

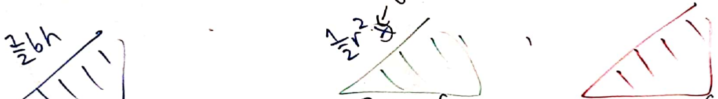
$$0 \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le 0$$

By the Sandwich Theorem,

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

Example: Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$





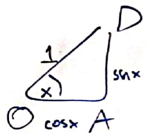
area of triangle OAD $\le$ area of sector OBD $\le$ area of triangle OBC

$$\frac{1}{2}\cdot\overline{OA}\cdot\overline{AD} \le \frac{1}{2}\left(\overline{OB}\right)^2\cdot x \le \frac{1}{2}\cdot\overline{OB}\cdot\overline{BC}$$

$$\frac{1}{2}\cdot\cos x\cdot\sin x \le \frac{1}{2}\cdot 1^2 x \le \frac{1}{2}\cdot 1\cdot\frac{\sin x}{\cos x}$$

$$2\left(\frac{1}{2}\cos x\sin x\right) \le 2\left(\frac{1}{2}x\right) \le 2\left(\frac{1}{2}\frac{\sin x}{\cos x}\right)$$

$$\cos x\sin x \le x \le \frac{\sin x}{\cos x}$$

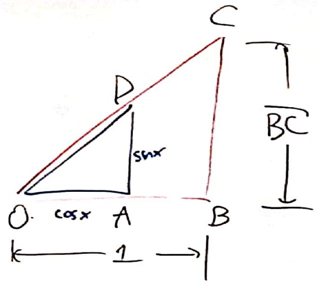


$\overline{OA} = \cos x$

$\overline{AD} = \sin x$

$\overline{OB} = 1$

$\overline{BC} = \frac{\sin x}{\cos x}$



Similar triangles

$$\frac{\text{OPP}}{\text{ADJ}} = \frac{\sin x}{\cos x} = \frac{\overline{BC}}{1}$$

$$\overline{BC} = \frac{\sin x}{\cos x}$$

$$\frac{\cos x \sin x}{\sin x} \le \frac{x}{\sin x} \le \frac{\frac{\sin x}{\cos x}}{\sin x}$$

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$\frac{\cos x}{1} \le \frac{x}{\sin x}$$

Cross-multiply:

$$\cos x \sin x \le x \cdot 1$$

$$\frac{\cos x \sin x}{\cos x} \le \frac{x}{\cos x}$$

$$\frac{\sin x}{x} \le \frac{\frac{x}{\cos x}}{x}$$

$$\frac{\sin x}{x} \le \frac{1}{\cos x}$$

$$\frac{x}{\sin x} \le \frac{1}{\cos x}$$

Cross multiply:

$$x \cdot \cos x \le 1 \cdot \sin x$$

$$\frac{x \cdot \cos x}{x} \le \frac{\sin x}{x}$$

$$\cos x \le \frac{\sin x}{x}$$

Altogether,

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}$$

$$\lim_{x\to 0} \cos x \le \lim_{x\to 0} \frac{\sin x}{x} \le \lim_{x\to 0} \frac{1}{\cos x}$$

$$1 \le \lim_{x\to 0} \frac{\sin x}{x} \le 1$$

By the Sandwich Theorem, we conclude

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

This concludes our argument. :)

$$\lim_{x\to 0} \cos x = \cos 0 = 1$$

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

Mini-Quiz 6/23

Use the Sandwich Theorem to compute

$$\lim_{x \to 0} x^3 \cos\left(\frac{1}{x}\right)$$

SOLUTION:

Case 1: $x \geq 0$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^3 \leq x^3 \cos\left(\frac{1}{x}\right) \leq x^3$$

$$\lim_{x \to 0^+}(-x^3) \leq \lim_{x \to 0^+} x^3 \cos\left(\frac{1}{x}\right) \leq \lim_{x \to 0^+} x^3$$

$$0 \leq \lim_{x \to 0^+} x^3 \cos\left(\frac{1}{x}\right) \leq 0$$

By the Sandwich Theorem,

$$\lim_{x \to 0^+} x^3 \cos\left(\frac{1}{x}\right) = 0$$

Case 2: $x < 0$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^3 \geq x^3 \cos\left(\frac{1}{x}\right) \geq x^3$$

$$\lim_{x \to 0^-}(-x^3) \geq \lim_{x \to 0^-} x^3 \cos\left(\frac{1}{x}\right) \geq \lim_{x \to 0^-} x^3$$

$$0 \geq \lim_{x \to 0^-} x^3 \cos\left(\frac{1}{x}\right) \geq 0$$

By the Sandwich Theorem,

$$\lim_{x \to 0^-} x^3 \cos\left(\frac{1}{x}\right) = 0.$$

Altogether, we conclude

$$\lim_{x \to 0} x^3 \cos\left(\frac{1}{x}\right) = 0$$