#1 Formal Definition of the Derivative

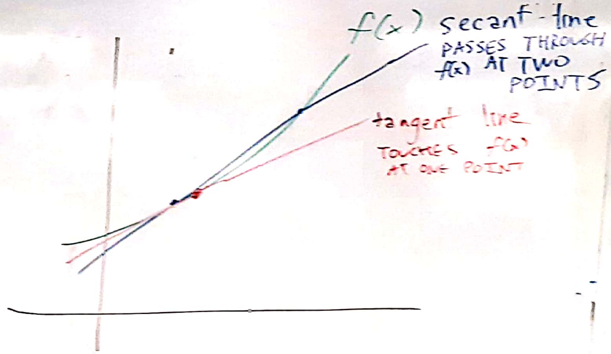
Definition The derivative of a function f at x, denoted $f'(x)$, is

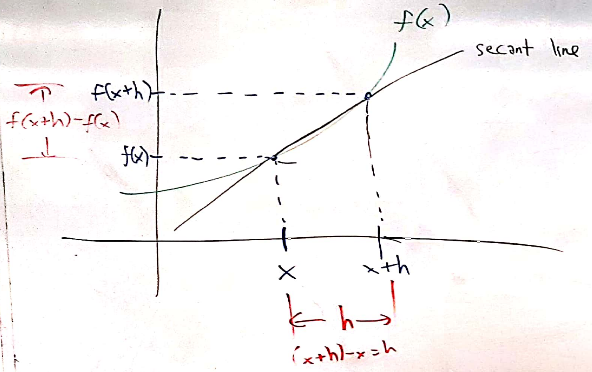
"f prime of x"

difference quotient

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists.



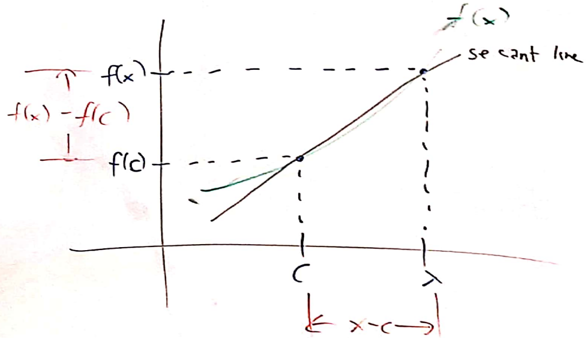


As $h \to 0$, the secant line becomes the tangent line.

$f'(x)$ denotes the slope of the tangent line

If we want to compute the derivative at $x=c$, we can also write

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$



Notations for the derivative of $y = f(x)$:

$$y' = \frac{dy}{dx} = f'(x) = \boxed{\frac{df}{dx}} = \frac{d}{dx} f(x)$$

Leibniz notation

Example 2: Use the limit definition of the [illegible] to compute the derivative of

$$f(x) = x^2$$

[illegible]

$$f(x) = x^2$$

$$f(x+h) = (x+h)^2$$

Difference quotient

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{(\cancel{x^2} + 2hx + h^2) - \cancel{x^2}}{h}$$

$$= \frac{2hx + h^2}{h}$$

$$= \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 2x + h$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} (2x + h)$$

$$= 2x + 0$$

$$= \boxed{2x}$$

The derivative of x^2 is $2x$.

(b) What is the slope of the tangent line of $y=x^2$ at:

(i) $x=0$

(ii) $x=1$

(iii) $x=2$

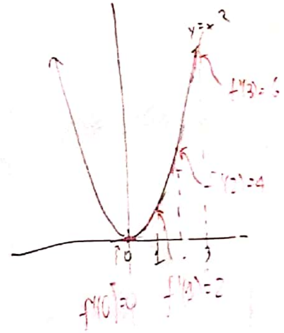
(iv) $x=3$

(i) $f'(0)=2\cdot 0 = \boxed{0}$

(ii) $f'(1)=2\cdot 1 = \boxed{2}$

(iii) $f'(2)=2\cdot 2 = 4$

$f'(3)=2\cdot 3 = 6$



Example: Find the derivative of

$f(x) = 4x + 7$

using the limit definition.

slope: 4

SOLUTION:

$f(x) = 4x + 7$

$f(x+h) = 4(x+h) + 7$

Difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{\underbrace{(4(x+h)+7)}_{f(x+h)} - \underbrace{(4x+7)}_{f(x)}}{h}$$

$$= \frac{\cancel{4x} + 4h + \cancel{7} - \cancel{4x} - \cancel{7}}{h}$$

$$= \frac{4\cancel{h}}{\cancel{h}}$$

$$= 4$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} 4$$

$$= \boxed{4}$$

Example: Find the derivative of $f(x)=x^3-x$ using the limit definition.

$$\begin{array}{ccccccccc} & & & 1 & & & & n=0 \\ & & 1 & & 1 & & & n=1 \\ & 1 & & 2 & & 1 & & n=2 \\ 1 & & 3 & & 3 & & 1 & n=3 \end{array}$$

SOLUTION:

$5!=5\cdot4\cdot3\cdot2\cdot1$
$4!=4\cdot3\cdot2\cdot1$
$3!=3\cdot2\cdot1$
$2!=2\cdot1$
$1!=1$
$0!=1$

$\binom{3}{0}=\frac{3!}{0!(3-0)!}=1$

$\binom{3}{1}=\frac{3!}{1!(3-1)!}=3$

$\binom{3}{2}=\frac{3!}{2!(3-2)!}=3$

$\binom{3}{3}=\frac{3!}{3!(3-3)!}=1$

$f(x)=x^3-x$

$$\begin{aligned} f(x+h) &= (x+h)^3-(x+h) \\ &= (x^3+3x^2h+3xh^2+h^3)-(x+h) \\ &= x^3+3x^2h+3xh^2+h^3-x-h \end{aligned}$$

BINOMIAL THEOREM FOR $(a+b)^3$

$$\begin{aligned} (a+b)^3 &= \binom{3}{0}a^3b^0+\binom{3}{1}a^2b^1+\binom{3}{2}a^1b^2+\binom{3}{3}a^0b^3 \\ &= 1a^3b^0+3a^2b^1+3a^1b^2+1a^0b^3 \\ &= a^3+3a^2b+3ab^2+b^3 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\overbrace{[x^3+3x^2h+3xh^2+h^3-x-h]}^{f(x+h)}-\overbrace{(x^3-x)}^{-f(x)}}{h} \\ &= \frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{-x}-h\cancel{-x^3}\cancel{+x}}{h} \\ &= \frac{3x^2h+3xh^2+h^3-h}{h} \\ &= \frac{\cancel{h}(3x^2+3xh+h^2-1)}{\cancel{h}} \\ &= 3x^2+3xh+h^2-1 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to0}(3x^2+3xh+h^2-1) \\ &= 3x^2+3x\cdot0+0^2-1 \\ &= \boxed{3x^2-1} \end{aligned}$$

Example: Use the limit definition to find the derivative of

$$f(x) = \frac{1}{x}.$$

SOLUTION:

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

Difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{-h}{x(x+h)}}{h}$$

$$\frac{1}{x+h} - \frac{1}{x} = \frac{1}{x+h}\cdot\frac{x}{x} - \frac{1}{x}\cdot\frac{x+h}{x+h}$$

$$= \frac{x - (x+h)}{x(x+h)}$$

$$= \frac{\cancel{x} - \cancel{x} - h}{x(x+h)}$$

$$= \frac{-h}{x(x+h)}$$

$$= -\frac{\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}}$$

$$= -\frac{1}{x(x+h)}$$

$$\frac{\frac{3}{5}}{2} = \frac{3}{5}\cdot\frac{1}{2}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\left[-\frac{1}{x(x+h)}\right]$$

$$= -\frac{1}{x(x+0)}$$

$$= \boxed{-\frac{1}{x^2}}$$

Example: Use the limit definition to find the derivative of

$$f(x) = \sqrt{x}$$

SOLUTION

$$f(x) = \sqrt{x}$$

$$f(x+h) = \sqrt{x+h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

Example: Find the equation of the tangent line of

$$f(x) = \sqrt{x}$$

at $x = 4$.

SOLUTION:

$$f(4) = \sqrt{4} = 2$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

Equation of the tangent line

$$y - y_1 = m(x - x_1)$$

$$y - f(4) = f'(4)(x - 4)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y - 2 = \frac{1}{4}x - 1$$

$$+2 \qquad\qquad +2$$

$$\boxed{y = \frac{1}{4}x + 1}$$

Scanné avec CamScanner

Mini-Quiz 6/27

Use the limit definition of the derivative to compute the derivative of

$$f(x) = (2x+1)^2$$