Mini-Quiz 6/23 solution

Apply the Sandwich Theorem to compute
$$\lim_{x\to 0} x^3\cos\left(\frac{1}{x}\right).$$

SOLUTION: $$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

$x \ge 0$

$$-x^3 \le x^3\cos\left(\frac{1}{x}\right) \le x^3$$
$$\lim_{x\to 0^+}(-x^3) \le \lim_{x\to 0^+} x^3\cos\left(\frac{1}{x}\right) \le \lim_{x\to 0^+} x^3$$
$$0 \le \lim_{x\to 0^+} x^3\cos\left(\frac{1}{x}\right) \le 0$$

By the Sandwich Theorem,
$$\lim_{x\to 0^+} x^3\cos\left(\frac{1}{x}\right) = 0$$

If $x<0$, then $x^3<0$

$x<0$

$$-x^3 \ge x^3\cos\left(\frac{1}{x}\right) \ge x^3$$
$$\lim_{x\to 0^-}(-x^3) \ge \lim_{x\to 0^-} x^3\cos\left(\frac{1}{x}\right) \ge \lim_{x\to 0^-} x^3$$
$$0 \ge \lim_{x\to 0^-} x^3\cos\left(\frac{1}{x}\right) \ge 0$$

By the Sandwich Theorem,
$$\lim_{x\to 0^-} x^3\cos\left(\frac{1}{x}\right) = 0.$$

Together, we conclude $\lim_{x\to 0} x^3\cos\left(\frac{1}{x}\right) = 0$.

Another solution:

$$-|x^3| \le x^3\cos\left(\frac{1}{x}\right) \le |x^3|$$
$$\lim_{x\to 0}(-|x^3|) \le \lim_{x\to 0} x^3\cos\left(\frac{1}{x}\right) \le \lim_{x\to 0}|x^3|$$
$$0 \le \lim_{x\to 0} x^3\cos\left(\frac{1}{x}\right) \le 0$$

By the Sandwich Theorem,
$$\lim_{x\to 0} x^3\cos\left(\frac{1}{x}\right) = 0.$$

Mini-Quiz 6/27 solution

Use the limit definition of the derivative to compute the derivative of $f(x) = (2x+1)^2$.

Solution

$$f(x) = (2x+1)^2 = 4x^2+4x+1$$

$$\begin{aligned} f(x+h) &= (2(x+h)+1)^2 \\ &= (2x+2h+1)^2 \\ &= (2x+2h)^2 + 2(2x+2h)\cdot 1 + 1^2 \\ &= (4x^2+8xh+4h^2)+(4x+4h)+1 \end{aligned}$$

$$\begin{aligned} f(x+h) &= 4(x+h)^2+4(x+h)+1 \\ &= 4(x^2+2xh+h^2)+4(x+h)+1 \\ &= 4x^2+8xh+4h^2+4x+4h+1 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{[4x^2+8xh+4h^2+4x+4h+1]-[4x^2+4x+1]}{h} \\ &= \frac{\cancel{4x^2}+8xh+4h^2+\cancel{4x}+4h+\cancel{1}-\cancel{4x^2}-\cancel{4x}-\cancel{1}}{h} \\ &= \frac{8xh+4h^2+4h}{h} \\ &= \frac{h(8x+4h+4)}{\cancel{h}} \\ &= 8x+4h+4 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} (8x+4h+4) \\ &= 8x+4\cdot 0+4 \\ &= \boxed{8x+4} \end{aligned}$$

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4.2 Properties of the Derivative

DIFFERENTIABILITY AND CONTINUITY.

Recall

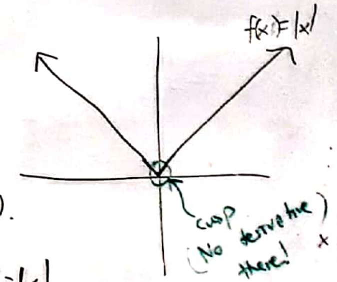
$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

This limit may or may not exist.

Theorem If f is differentiable at $x=c$, then f is also continuous at $x=c$.

Example: Let

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0. \end{cases}$$



Show that $f(x)$ is differentiable for $x \neq 0$ but $f(x)$ is NOT differentiable at $x=0$.

SOLUTION: Suppose $x \neq 0$. We want to show $f(x)=|x|$ is differentiable for all $x \neq 0$.

Case 1: $x > 0$

$f(x) = |x| = x$

$f(x+h) = x+h$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

So $f'(x)$ exists for all $x > 0$. Therefore, $f(x)$ is differentiable for all $x > 0$.

Case 2 $x < 0$

$f(x) = |x| = -x$

$f(x+h) = -(x+h) = -x-h$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(-x-h) - (-x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-x} - h + \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h}$$

$$= \lim_{h\to 0} -1$$

$$= -1$$

So $f'(x)$ exists for all $x < 0$.

Therefore, $f(x)$ is differentiable for all $x < 0$.

Cases 1 & 2 together:

We conclude that $f(x) = |x|$ is differentiable for all $x \neq 0$.

$f(x) = |x|$ ~ NOT differentiable at $x = 0$.

$$f(h) = |h| = \begin{cases} h & \text{for } h \geq 0 \\ -h & \text{for } h < 0 \end{cases}$$

$f(0) = |0| = 0$

$h \geq 0$

$$f'(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{h - 0}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h}$$

$$= \lim_{h\to 0^+} 1$$

$$= 1$$

$h < 0$

$$f'(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{-h - 0}{h}$$

$$= \lim_{h\to 0^-} -\frac{h}{h}$$

$$= \lim_{h\to 0^-} -1$$

$$= -1$$

$$1 \neq -1$$

In summary,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = 1$$

and

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = -1.$$

Therefore,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist.

Therefore, $f(x) = |x|$ is NOT differentiable at $x = 0$.

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4.3 The Power Rule

Power Rule Let a be a constant. If

$$f(x) = x^a,$$

then

$$f'(x) = ax^{a-1}$$

In other words (using Leibniz' notation),

$$\frac{d}{dx}\left(x^a\right) = ax^{a-1}$$

Example: (a) $y = x^3$

$$y' = \frac{d}{dx}(x^3) = 3x^{3-1} = \boxed{3x^2}$$

(b) $y = x^{300}$

$$y' = \frac{d}{dx}(x^{300}) = 300x^{300-1} = \boxed{300x^{299}}$$

(c) $y = x^2$

$$y' = \frac{d}{dx}(x^2) = 2x^{2-1} = \boxed{2x}$$

(d) $y = x$

$$y' = \frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1x^{1-1} = 1x^0 = 1 \cdot 1 = \boxed{1}$$

(e) $y = 6$

$$y' = \frac{d}{dx}(6)$$
$$= 0$$

(f) $y = \frac{1}{x}$

$$y' = \frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{d}{dx}(x^{-1})$$
$$= -1x^{-1-1}$$
$$= -x^{-2}$$
$$= \boxed{-\frac{1}{x^2}}$$

(g) $y = \frac{1}{x^3}$

$$y' = \frac{d}{dx}\left(\frac{1}{x^3}\right)$$
$$= \frac{d}{dx}(x^{-3})$$
$$= -3x^{-4}$$
$$= \boxed{-\frac{3}{x^4}}$$

"Sum and multiplication rules

Suppose a is constant, and $f(x)$ and $g(x)$ are differentiable at x. Then the following relationships hold:

$$\frac{d}{dx}(af(x)) = a\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Example: Differentiate

$$y = 2x^4 - 3x^3 + x - 7.$$

SOLUTION:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^4 - 3x^3 + x - 7)$$
$$= \frac{d}{dx}(2x^4) - \frac{d}{dx}(3x^3) + \frac{d}{dx}x -$$
$$= 2\frac{d}{dx}(x^4) - 3\frac{d}{dx}(x^3) + \frac{d}{dx}x -$$
$$= 2(4x^3) - 3(3x^2) + 1 - 0$$
$$= \boxed{8x^3 - 9x^2 + 1}$$

Example: Differentiate

$$y = \frac{7}{x} + \frac{8}{x^2}$$

SOLUTION:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{7}{x} + \frac{8}{x^2}\right)$$

$$= \frac{d}{dx}\left(\frac{7}{x}\right) + \frac{d}{dx}\left(\frac{8}{x^2}\right)$$

$$= 7\frac{d}{dx}\left(\frac{1}{x}\right) + 8\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= 7\frac{d}{dx}(x^{-1}) + 8\frac{d}{dx}(x^{-2})$$

$$= 7(-1x^{-2}) + 8(-2x^{-3})$$

$$= 7\left(-\frac{1}{x^2}\right) + 8\left(-\frac{2}{x^3}\right)$$

$$= \boxed{-\frac{7}{x^2} - \frac{16}{x^3}}$$

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Differentiate

$$y = 2x^4 + \frac{7}{x^3}$$