4.3 continued

Theorem Use the limit definition of the derivative to prove the power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for all positive integers n.

Recall the Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2 b^{n-2} + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

for all positive integers n.

The binomial coefficient is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$
and $k = 0, 1, 2, 3, \ldots, n$.

Example: $4! = 4\cdot 3\cdot 2\cdot 1 = 24$
$3! = 3\cdot 2\cdot 1$
$2! = 2\cdot 1$
$1! = 1$
$0! = 1$

PROOF OF OUR THEOREM

Let $f(x) = x^n$.

Then $f(x+h) = (x+h)^n$

$$= \binom{n}{0}x^n h^0 + \binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}x^1h^{n-1} + \binom{n}{n}x^0h^n$$

$$= 1\cdot x^n\cdot 1 + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n$$

$$= x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n$$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{\cancel{n!}}{1\cdot\cancel{n!}} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n\cancel{(n-1)!}}{1\cancel{(n-1)!}} = n$$

$$\boxed{n! = n\underbrace{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}_{(n-1)!} = n(n-1)!}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[\cancel{x^n} + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n\right] - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n}{h}$$

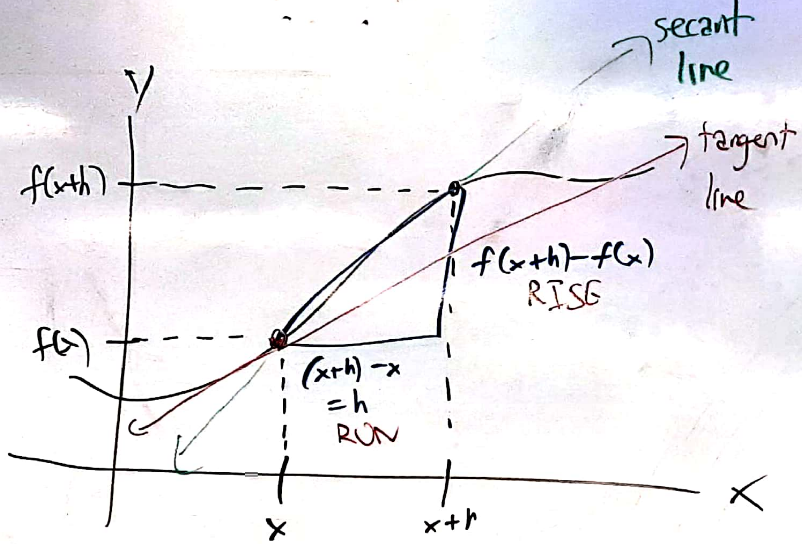
$$= \lim_{h \to 0} \frac{\cancel{h}\left[nx^{n-1} + \binom{n}{2}x^{n-2}h^1 + \cdots + \binom{n}{n-2}x^2h^{n-3} + \binom{n}{n-1}xh^{n-2} + \binom{n}{n}h^{n-1}\right]}{\cancel{h}}$$

$$= nx^{n-1} + \binom{n}{2}x^{n-2}0^1 + \cdots + \binom{n}{n-2}x^2 0^{n-3} + \binom{n}{n-1}x0^{n-2} + \binom{n}{n}0^{n-1}$$

$$= nx^{n-1} + 0 + \cdots + 0 + 0 + 0$$

$$= nx^{n-1}$$

Scanné avec CamScanner



SLOPE OF THE SECANT LINE $= \dfrac{\text{RISE}}{\text{RUN}}$

$$= \frac{f(x+h) - f(x)}{h}$$ difference quotient

SLOPE OF THE TANGENT LINE $= f'(x)$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Scanné avec CamScanner

Example: Differentiate the following:

(a) $y = \sqrt{x}$

$$y' = \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$
$$= \frac{1}{2}x^{\frac{1}{2}-1}$$
$$= \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2}\,\frac{1}{x^{\frac{1}{2}}}$$
$$= \boxed{\frac{1}{2\sqrt{x}}}$$

(b) $y = \sqrt[3]{x}$

$$y' = \frac{d}{dx}\left(\sqrt[3]{x}\right)$$
$$= \frac{d}{dx}\left(x^{\frac{1}{3}}\right)$$
$$= \frac{1}{3}x^{\frac{1}{3}-1}$$
$$= \frac{1}{3}x^{-\frac{2}{3}}$$
$$= \frac{1}{3}\,\frac{1}{x^{\frac{2}{3}}}$$
$$= \boxed{\frac{1}{3x^{\frac{2}{3}}}}$$

(c) $y = \frac{1}{\sqrt[5]{x}}$

$$y' = \frac{d}{dx}\left(\frac{1}{\sqrt[5]{x}}\right)$$
$$= \frac{d}{dx}\left(\frac{1}{x^{\frac{1}{5}}}\right)$$
$$= \frac{d}{dx}\left(x^{-\frac{1}{5}}\right)$$
$$= -\frac{1}{5}x^{-\frac{1}{5}-1}$$
$$= -\frac{1}{5}x^{-\frac{6}{5}}$$
$$= \boxed{-\frac{1}{5x^{\frac{6}{5}}}}$$

(d) $y = x^{\frac{2}{3}}$

$$y' = \frac{d}{dx}\left(x^{\frac{2}{3}}\right)$$
$$= \frac{2}{3}x^{\frac{2}{3}-1}$$
$$= \frac{2}{3}x^{-\frac{1}{3}}$$
$$= \frac{2}{3x^{\frac{1}{3}}}$$
$$= \boxed{\frac{2}{3\sqrt[3]{x}}}$$

(e) $y = \sqrt[5]{x^3}$

$$y' = \frac{d}{dx}\left(\sqrt[5]{x^3}\right)$$
$$= \frac{d}{dx}\left(x^{\frac{3}{5}}\right)$$
$$= \frac{3}{5}x^{\frac{3}{5}-1}$$
$$= \frac{3}{5}x^{-\frac{2}{5}}$$
$$= \frac{3}{5}\,\frac{1}{x^{\frac{2}{5}}}$$
$$= \boxed{\frac{3}{5x^{\frac{2}{5}}}}$$

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4.4 Product and Quotient Rules

Product Rule: If $f(x)$ and $g(x)$ are differentiable functions, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

LEIBNIZ' NOTATION:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x))\, g(x) + f(x)\frac{d}{dx}(g(x)).$$

Example: Differentiate

$$f(x) = (3x+1)(2x^2-5)$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left[(3x+1)(2x^2-5)\right]$$

$$= \left[\frac{d}{dx}(3x+1)\right](2x^2-5) + (3x+1)\frac{d}{dx}(2x^2-5)$$

$$= \boxed{3(2x^2-5) + (3x+1)\cdot(4x)}$$

Example: Differen

$f(x) =$

SOLUTION:

$f'(x) =$

$=$

$=$

$=$

$=$

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Example: Differentiate

$$f(x) = (3x^3-2x)^2.$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left[(3x^3-2x)^2\right]$$

$$= \frac{d}{dx}\left[(3x^3-2x)(3x^3-2x)\right]$$

$$= \frac{d}{dx}(3x^3-2x)\cdot(3x^3-2x) + (3x^3-2x)\frac{d}{dx}(3x^3-2x)$$

$$= \underset{a}{(9x^2-2)}\cdot\underset{b}{(3x^3-2x)} \underset{+}{+} \underset{b}{(3x^3-2x)}\cdot\underset{a}{(9x^2-2)} = 2\,a\cdot b$$

$$= \boxed{2(9x^2-2)(3x^3-2x)}$$

$$2 \quad a \cdot b$$

Example: Differentiate

$$y = (2x+1)(x+1)(3x-4)$$

SOLUTION:

$$y' = \frac{d}{dx}\left[(2x+1)(x+1)(3x-4)\right]$$

$$= \frac{d}{dx}(2x+1)\left[(x+1)(3x-4)\right] + (2x+1)\frac{d}{dx}\left[(x+1)(3x-4)\right]$$

$$= \frac{d}{dx}(2x+1)\left[(x+1)(3x-4)\right] + (2x+1)\left[\frac{d}{dx}(x+1)\cdot(3x-4) + (x+1)\frac{d}{dx}(3x-4)\right]$$

$$= \frac{d}{dx}(2x+1)(x+1)(3x-4) + (2x+1)\frac{d}{dx}(x+1)(3x-4) + (2x+1)(x+1)\frac{d}{dx}(3x-4)$$

$$= \boxed{2(x+1)(3x-4) + (2x+1)\cdot 1\cdot(3x-4) + (2x+1)(x+1)\cdot 3}$$

Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions and $g(x) \neq 0$, then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

LEIBNIZ NOTATION

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x))\,g(x) - f(x)\,\frac{d}{dx}(g'(x))}{[g(x)]^2}$$

$HI = f(x)$

$LO = g(x)$

" $\dfrac{LO \cdot D(HI) - HI \cdot D(LO)}{[LO]^2}$ "

Example: Differentiate

$$y = \frac{x^3 - 3x + 2}{x^2 + 1}$$

SOLUTION:

$$y' = \frac{d}{dx}\left(\frac{x^3 - 3x + 2}{x^2 + 1}\right)$$

$$= \frac{\frac{d}{dx}(x^3 - 3x + 2)(x^2+1) - (x^3 - 3x + 2)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \boxed{\frac{(3x^2-3)(x^2+1) - (x^3-3x+2)(2x)}{(x^2+1)^2}}$$

optional beyond this point

$$= \frac{(3x^4 + \cancel{3x^2} - \cancel{3x^2} - 3) - (2x^4 - 6x^2 + 4x)}{(x^2+1)^2}$$

$$= \frac{3x^4 - 3 - 2x^4 + 6x^2 - 4x}{(x^2+1)^2}$$

$$= \boxed{\frac{x^4 + 6x^2 - 4x - 3}{(x^2+1)^2}}$$

fully simplified

Mini-Quiz 6/29

Differentiate

$$y = \sqrt{2x}\,(3x+1)$$