Wrapping up Section 4.4

Theorem Let $f(x)$ and $g(x)$ be differentiable functions. Use the limit definition of the derivative to prove:

(a) the product rule
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

(b) the quotient rule
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$
assuming $g(x) \neq 0$.

PROOF of (a).

Let $p(x) = f(x)g(x)$.

Then $p(x+h) = f(x+h)\,g(x+h)$. So

$$\frac{p(x+h) - p(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

add and subtract $f(x+h)g(x)$

$$= \frac{f(x+h)g(x+h) \boxed{- f(x+h)g(x) + f(x+h)g(x)} - f(x)g(x)}{h}$$

$$= \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

factor $f(x+h)$ factor $g(x)$

$$= \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

$\rightarrow = f$

$= f$

$=$

$$= \frac{f(x+h)[g(x+h)-g(x)]}{h} + \frac{g(x)[f(x+h)-f(x)]}{h}$$

$$= \boxed{f(x+h)\,\frac{g(x+h)-g(x)}{h}} + \boxed{g(x)\,\frac{f(x+h)-f(x)}{h}}$$

$$= \frac{f(x+h)-f(x)}{h}\,g(x) + f(x+h)\,\frac{g(x+h)-g(x)}{h}$$

Therefore,

$$(f(x)g(x))' = p'(x) = \lim_{h\to 0} \frac{p(x+h)-p(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\,g(x) + f(x+h)\,\frac{g(x+h)-g(x)}{h}\right]$$

$$= \left(\overbrace{\boxed{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}}^{f'(x)}\right) g(x) + \left(\lim_{h\to 0} f(x+h)\right)\left(\overbrace{\boxed{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}}^{g'(x)}\right)$$

$$= f'(x)\,g(x) + f(x+0)\,g'(x)$$

$$= f'(x)\,g(x) + f(x)\,g'(x).$$

Wrapping up Section 4.4

Theorem Let $f(x)$ and $g(x)$ be differentiable functions. Use the limit definition of the derivative to prove:

(a) the product rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

(b) the quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

assuming $g(x) \neq 0$.

PROOF of (b):

Let $p(x) = \frac{f(x)}{g(x)}$

Then $p(x+h) = \frac{f(x+h)}{g(x+h)}$. So

$$\frac{p(x+h)-p(x)}{h} = \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]\frac{g(x)g(x+h)}{g(x)g(x+h)}$$

$$= \frac{1}{h}\cdot\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$

$$= \frac{1}{h}\cdot\frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{g(x)g(x+h)}$$

$$= \frac{1}{h}\,\boxed{f(x+}$$

$$= \frac{1}{h}\,g(x)[$$

$$= \frac{1}{g(x)g(x+h)}$$

$$= \frac{1}{g(x)g(x+h)}$$

$$= \frac{1}{g(x)g(x+}$$

$$= \frac{1}{h}\,\frac{\boxed{f(x+h)g(x) - f(x)g(x)} - f(x)g(x+h) + f(x)g(x)}{g(x)g(x+h)}$$

factor $g(x)$ factor $-f(x)$

$$= \frac{1}{h}\,\frac{g(x)[f(x+h)-f(x)] - f(x)[g(x+h)-g(x)]}{g(x)g(x+h)}$$

swap

$$= \frac{1}{g(x)g(x+h)}\,\frac{g(x)[f(x+h)-f(x)] - f(x)[g(x+h)-g(x)]}{h}$$

$$= \frac{1}{g(x)g(x+h)}\left[g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}\right]$$

$$= \frac{1}{g(x)g(x+h)}\left[\frac{f(x+h)-f(x)}{h}g(x) - f(x)\frac{g(x+h)-g(x)}{h}\right]$$

$f(x)g(x)$

Therefore,

$$\left(\frac{f(x)}{g(x)}\right)' = p'(x) = \lim_{h\to 0}\frac{p(x+h)-p(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{1}{g(x)g(x+h)}\left[\frac{f(x+h)-f(x)}{h}g(x) - f(x)\frac{g(x+h)-g(x)}{h}\right]\right)$$

$$= \left(\lim_{h\to 0}\frac{1}{g(x)g(x+h)}\right)\cdot\left(\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x) - f(x)\frac{g(x+h)-g(x)}{h}\right]\right)$$

$$= \frac{1}{g(x)\lim_{h\to 0} g(x+h)}\left(\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)g(x) - f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$

$$= \frac{1}{g(x)g(x+0)}\left(f'(x)g(x) - f(x)g'(x)\right)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

4.5 The Chain Rule

$(f \circ g)(x) = f(g(x))$

Chain rule If g is differentiable at x and f is differentiable at $u = g(x)$, then

$$(f \circ g)'(x) = f'[g(x)]\, g'(x).$$

LEIBNIZ NOTATION:

$$\frac{d}{dx}[(f \circ g)(x)] = \frac{df}{du}\frac{du}{dx},$$

where $u = g(x)$.

Example: Find the derivative of $y = (3x^2-1)^2$.

SOLUTION

$$\begin{aligned} y' &= \frac{d}{dx}\left((3x^2-1)^2\right) \\ &= 2(3x^2-1)^{2-1} \cdot \frac{d}{dx}(3x^2-1) \\ &= \boxed{2(3x^2-1)^1 \cdot 6x} \\ &= \boxed{12x(3x^2-1)} \end{aligned}$$

Set $f(x) = x^2$ and $g(x) = 3x^2-1$.

Then $f'(x) = 2x$ and $g'(x) = 6x$.

So

$$\begin{aligned} (f \circ g)'(x) &= y' \\ &= \underbrace{\boxed{2(3x^2-1)}}_{f'(3x^2-1)} \cdot \underbrace{\boxed{6x}}_{g'(x)} \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

Example:

SOLUTION:

y'

Example: find the derivative of

$$y=(2x+1)^3$$

SOLUTION:

$$y' = \frac{d}{dx}\left((2x+1)^3\right)$$

$$= 3(2x+1)^2 \cdot \frac{d}{dx}(2x+1)$$

$$= 3(2x+1)^2 \cdot 2$$

$$= \boxed{6(2x+1)^2}$$

Example: Find the derivative of

$$y=\sqrt{x^2+1}$$

SOLUTION:

$$y' = \frac{d}{dx}\left(\sqrt{x^2+1}\right)$$

$$= \frac{d}{dx}\left((x^2+1)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2+1)$$

$$= \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}} \cdot \cancel{2}x$$

$$= x(x^2+1)^{-\frac{1}{2}}$$

$$= \boxed{\frac{x}{\sqrt{x^2+1}}}$$

Example: Find the derivative of

$$y=\sqrt[7]{2x^2+3x}$$

SOLUTION:

$$y' = \frac{d}{dx}\left(\sqrt[7]{2x^2+3x}\right)$$

$$= \frac{d}{dx}\left((2x^2+3x)^{\frac{1}{7}}\right)$$

$$= \frac{1}{7}(2x^2+3x)^{-\frac{6}{7}} \cdot \frac{d}{dx}(2x^2+3x)$$

$$= \frac{1}{7}(2x^2+3x)^{-\frac{6}{7}} \cdot (4x+3)$$

$$= \boxed{\frac{4x+3}{7(2x^2+3x)^{\frac{6}{7}}}}$$

Example: Find the derivative of

$$y=\left(\frac{x}{x+1}\right)^2.$$

SOLUTION:

$$y' = \frac{d}{dx}\left(\left(\frac{x}{x+1}\right)^2\right)$$

$$= 2\frac{x}{x+1} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$= \frac{2x}{x+1}\ \frac{\frac{d}{dx}(x)(x+1) - x\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{2x}{x+1}\ \frac{1(x+1) - x \cdot 1}{(x+1)^2}$$

$$= \frac{2x}{x+1}\ \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2}$$

$$= \frac{2x}{x+1}\ \frac{1}{(x+1)^2}$$

$$= \boxed{\frac{2x}{(x+1)^3}}$$

Example: Find the derivative of

$$y = \left(\sqrt{x^2+1} + 1\right)^2$$

SOLUTION:

$$y' = \frac{d}{dx}\left(\left(\sqrt{x^2+1} + 1\right)^2\right)$$

$$= 2\left(\sqrt{x^2+1} + 1\right)^1 \frac{d}{dx}\left(\sqrt{x^2+1} + 1\right)$$

$$= 2\left(\sqrt{x^2+1} + 1\right)\left(\frac{d}{dx}\left(\sqrt{x^2+1}\right) + \frac{d}{dx} 1\right)$$

$$= 2\left(\sqrt{x^2+1} + 1\right)\left(\frac{d}{dx}\left(\left(x^2+1\right)^{\frac{1}{2}}\right) + 0\right)$$

$$= \cancel{2}\left(\sqrt{x^2+1} + 1\right)\left(\frac{1}{\cancel{2}}\left(x^2+1\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(x^2+1\right)\right)$$

$$= \left(\sqrt{x^2+1} + 1\right) \frac{1}{\sqrt{x^2+1}} \cdot 2x$$

$$= \boxed{\frac{2x\left(\sqrt{x^2+1} + 1\right)}{\sqrt{x^2+1}}}$$

Example: Find the derivative of

$$y = \left(2x^3 - \sqrt{3x^4 - 2}\right)^3$$

SOLUTION:

$$y' = \frac{d}{dx}\left(\left(2x^3 - \sqrt{3x^4-2}\right)^3\right)$$

$$= 3\left(2x^3 - \sqrt{3x^4-2}\right)^2 \cdot \frac{d}{dx}\left(2x^3 - \sqrt{3x^4-2}\right)$$

$$= 3\left(2x^3 - \sqrt{3x^4-2}\right)^2 \cdot \left(6x^2 - \frac{d}{dx}\left(\sqrt{3x^4-2}\right)\right)$$

$$= \boxed{3\left(2x^3 - \sqrt{3x^4-2}\right)^2\left(6x^2 - \frac{6x^3}{\sqrt{3x^4-2}}\right)}$$

$$\frac{d}{dx}\left(\sqrt{3x^4-2}\right) = \frac{d}{dx}\left(\left(3x^4-2\right)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(3x^4-2\right)^{-\frac{1}{2}} \frac{d}{dx}\left(3x^4-2\right)$$

$$= \frac{1}{2}\left(3x^4-2\right)^{-\frac{1}{2}} \cdot 12x^3$$

$$= 6x^3\left(3x^4-2\right)^{-\frac{1}{2}}$$

$$= \frac{6x^3}{\sqrt{3x^4-2}}$$

Theorem Let $f(x)$ and $g(x)$ be differentiable functions and assume $g(x) \neq 0$. Use the Product Rule and the Chain Rule to prove the Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$$

$$= \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right)$$

$$\underset{\text{Product Rule}}{=} \frac{d}{dx}(f(x))\cdot\frac{1}{g(x)} + f(x)\cdot\frac{d}{dx}\left(\frac{1}{g(x)}\right)$$

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{d}{dx}\left([g(x)]^{-1}\right)$$

$$\underset{\text{Chain Rule}}{=} -1\cdot[g(x)]^{-2}\cdot\frac{d}{dx}(g(x))$$

$$= -[g(x)]^{-2}\cdot g'(x)$$

$$= -\frac{g'(x)}{[g(x)]^2}$$

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(-\frac{g'(x)}{[g(x)]^2}\right)$$

$$= \frac{f'(x)}{g(x)}\cdot\frac{g(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x)}{[g(x)]^2} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Scanné avec CamScanner

Mini-Quiz 6/30

Find the derivative of

$$y = \frac{\sqrt{3x^4 + 2}}{x^3 - 6}.$$