4.9 Derivatives of Exponential Functions

Consider the function

$$f(x) = e^x.$$

The derivative is

$$\boxed{f'(x) = \frac{d}{dx}(e^x) = e^x.}$$

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1$$

$$= e^x.$$

FACT:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

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$$f(x) = a^x.$$

The derivative is

$$f'(x) = \frac{d}{dx}(a^x) = a^x \ln a.$$

$$\begin{aligned}
\frac{d}{dx}(a^x) &= \frac{d}{dx}\left(e^{\ln a^x}\right) \\
&= \frac{d}{dx}\left(e^{x \ln a}\right) \\
&= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \\
&= e^{\ln a^x} \cdot \ln a \\
&= \boxed{a^x \ln a}
\end{aligned}$$

Example: Find the derivative of

$$f(x) = e^{-\frac{x^2}{2}}.$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right)$$

$$= e^{-\frac{x^2}{2}} \cdot \frac{d}{dx}\left(-\frac{x^2}{2}\right)$$

$$= \boxed{e^{-\frac{x^2}{2}} \cdot (-x)}$$

$$= \boxed{-xe^{-\frac{x^2}{2}}}$$

Example: Find the derivative of

$$f(x) = 3^{\sqrt{x}}.$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left(3^{\sqrt{x}}\right)$$

$$= 3^{\sqrt{x}} \ln 3 \cdot \frac{d}{dx}\left(\sqrt{x}\right) \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$= \boxed{3^{\sqrt{x}} \ln 3 \cdot \frac{1}{2\sqrt{x}}} \qquad \frac{d}{dx}(3^x) = 3^x \ln 3$$

$$= \boxed{\frac{3^{\sqrt{x}} \ln 3}{2\sqrt{x}}}$$

Example: Find the

$f(x) =$ e

SOLUTION:

$f'(x) = \frac{d}{dx}$

$= \frac{d}{dx}($

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Example: Find the derivative of

$$f(x) = \frac{e^{-x}}{x}$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left(\frac{e^{-x}}{x}\right)$$

$$= \frac{\frac{d}{dx}(e^{-x})\cdot x - e^{-x}\,\frac{d}{dx}x}{x^2}$$

$$= \frac{\left(e^{-x}\overset{=-1}{\boxed{\frac{d}{dx}(-x)}}\right)\cdot x - e^{-x}\cdot 1}{x^2}$$

$$= \boxed{\frac{-e^{-x}x - e^{-x}}{x^2}}$$

Example: Find the derivative of

$$f(x) = e^{\sin\sqrt{x}}$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left(e^{\sin\sqrt{x}}\right)$$

$$= e^{\sin\sqrt{x}}\,\frac{d}{dx}\left(\sin\sqrt{x}\right)$$

$$= e^{\sin\sqrt{x}}\cos\sqrt{x}\,\frac{d}{dx}\sqrt{x}$$

$$= \boxed{e^{\sin\sqrt{x}}\cos\sqrt{x}\,\frac{1}{2\sqrt{x}}}$$

Example: Find the derivative of

$$f(x) = \frac{x - e^{-x}}{1 + xe^{-x}}$$

SOLUTION:

$$f'(x) = \frac{\boxed{\frac{d}{dx}(x - e^{-x})}(1 + xe^{-x}) - (x - e^{-x})\boxed{\frac{d}{dx}(1 + xe^{-x})}}{(1 + xe^{-x})^2}$$

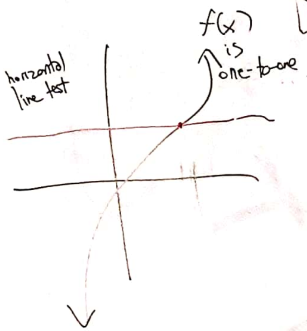
$$= \boxed{\frac{(1 + e^{-x})(1 + xe^{-x}) - (x - e^{-x})(e^{-x} - xe^{-x})}{(1 + xe^{-x})^2}}$$

$$\frac{d}{dx}(x - e^{-x}) = \frac{d}{dx}x - \frac{d}{dx}e^{-x}$$
$$= 1 - \left(e^{-x} \cdot \frac{d}{dx}(-x)\right)$$
$$= 1 - \left(e^{-x} \cdot (-1)\right)$$
$$= 1 - (-e^{-x})$$
$$= 1 + e^{-x}$$

$$\frac{d}{dx}(1 + xe^{-x}) = \frac{d}{dx}1 + \frac{d}{dx}(xe^{-x})$$
$$= 0 + \left[\left(\frac{d}{dx}x\right)e^{-x} + x\frac{d}{dx}e^{-x}\right]$$
$$= 1e^{-x} + x(-e^{-x})$$
$$= e^{-x} - xe^{-x}$$

4.10 Inverse Functions and Logarithms

Derivative of an Inverse Function:



If $f(x)$ is one-to-one and differentiable with inverse function $f^{-1}(x)$ and $f'[f^{-1}(x)] \neq 0$, then $f^{-1}(x)$ is differentiable and

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'[f^{-1}(x)]}$$

Recall the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Let $g(x) = f^{-1}(x)$

$$\frac{d}{dx} f(f^{-1}(x)) = f'(f^{-1}(x))(f^{-1})'(x)$$

$$\frac{d}{dx} x = f'(f^{-1}(x))(f^{-1})'(x)$$

$$\frac{1}{f'(f^{-1}(x))} = \frac{\cancel{f'(f^{-1}(x))}(f^{-1})'(x)}{\cancel{f'(f^{-1}(x))}}$$

$$\frac{1}{f'(f^{-1}(x))} = (f^{-1})'(x)$$

Example: L

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Example: Let $f(x) = \frac{x}{1+x}$ for $x \geq 0$.

Find $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=\frac{1}{3}}$.

① Find $f^{-1}(\frac{1}{3})$

Set $f(x) = \frac{1}{3}$

$$\frac{x}{1+x} = \frac{1}{3}$$

$$3x = 1+x$$

$$2x = 1$$

$$x = \frac{1}{2}$$

So $f^{-1}(\frac{1}{3}) = \frac{1}{2}$

② Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x}\right)$$

$$= \frac{\frac{d}{dx}x\,(1+x) - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{1(1+x) - x\cdot 1}{(1+x)^2}$$

$$= \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

③ Use $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=\frac{1}{3}} = \frac{1}{f'(f^{-1}(\frac{1}{3}))}$$

$$= \frac{1}{f'(\frac{1}{2})}$$

$$= \frac{1}{\frac{1}{(1+\frac{1}{2})^2}}$$

$$= \boxed{\frac{9}{4}}$$

Example: Find the derivative of $\tan^{-1} x$

$\arctan x$

SOLUTION: Set $y = \tan^{-1} x$

$\tan y = \tan(\tan^{-1} x)$

$\tan y = x$

$\frac{d}{dx}(\tan y) = \frac{d}{dx} x$

$\sec^2 y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} x) = \boxed{\frac{1}{1+x^2}}$$

$$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

$$\tan^2 y + 1 = \sec^2 y$$

$\tan y = x$

$(\tan y)^2 = x^2$

$\tan^2 y = x^2$

Example: Find

SOLUTION: Set

$\frac{d}{dx}(s$

$\cos y$

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Example: Find the derivative of $\sin^{-1} x$

arcsin x.

SOLUTION: Set $y = \sin^{-1} x$

$$\sin y = x$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx} x$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cos^2 y}}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\sin y = x$$

$$\sin^2 y = x^2$$

$$\frac{d}{dx}(\sin^{-1} x) = \boxed{\frac{1}{\sqrt{1 - x^2}}}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

Mini-Quiz 7/12

Find the derivative of

$$y = xe^{-x}.$$