9.10 continued

Example: Let $f(x) = 2x + e^x$ for $x \in \mathbb{R}$.
Find $\frac{d}{dx} f^{-1}(x) \Big|_{x=1}$

SOLUTION: ① Find $f^{-1}(1)$

$$f(x) = 1$$
$$2x + e^x = 1$$

Guess $x=0$: $2 \cdot 0 + e^0 = 1$
$$0 + 1 = 1$$
$$1 = 1$$

$x=0$, $f(x) = 1$
$$f(0) = 1$$
$$f^{-1}(f(0)) = f^{-1}(1)$$
$$0 = f^{-1}(1)$$

② Find $f'(x)$

$$f(x) = 2x + e^x$$
$$f'(x) = \frac{d}{dx}(2x + e^x)$$
$$= 2 + e^x.$$

③ Use $\frac{d}{dx} f^{-1}(x) = \frac{1}{f ...}$

$\frac{d}{dx} f^{-1}(x)\Big|_{x=1} = \frac{1}{f'(f ...}$

$= \frac{1}{f'(0 ...}$

$= \boxed{\frac{1}{3}}$

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③ Use $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=1} = \frac{1}{f'(f^{-1}(1))}$$

$$= \frac{1}{f'(0)}$$

$$= \boxed{\frac{1}{3}}$$

$$f'(x) = 2 + e^x$$

$$f'(0) = 2 + e^0$$
$$= 2 + 1$$
$$= 3$$

## Logarithmic Differentiation

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Recall: $\ln x = \log_e x$

Why is this true?

Let $y = \ln x$

$$e^y = e^{\ln x}$$

$$e^y = x$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Also, $\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$

To see why, use the change-of-base formula

$$\log_a x = \frac{\ln x}{\ln a}.$$

So

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a}$$

$$= \frac{1}{\ln a}\frac{d}{dx}\ln x$$

$$= \frac{1}{\ln a}\frac{1}{x}$$

$$= \frac{1}{(\ln a)x}$$

Example. Find the derivative of

$$y = \ln(3x)$$

SOLUTION:

$$y' = \frac{d}{dx}\ln(3x)$$

chain rule $= \dfrac{1}{3x}\cdot\dfrac{d}{dx}(3x)$

$$= \frac{1}{\cancel{3}x}\cdot\cancel{3}$$

$$= \frac{1}{x}$$

ALTERNATE SOLUTION

Use $\ln(ab) = \ln a + \ln b$

$$y' = \frac{d}{dx}\ln(3x)$$

$$= \frac{d}{dx}\left[\ln 3 + \ln x\right]$$

$$= 0 + \frac{1}{x}$$

$$= \frac{1}{x}$$

Example Find the derivative of

$$y = \ln(x^2+1)$$

SOLUTION.

$$y' = \frac{d}{dx}\ln(x^2+1)$$

$$= \frac{1}{x^2+1}\,\frac{d}{dx}(x^2+1)$$

$$= \frac{1}{x^2+1}\cdot 2x$$

$$= \boxed{\frac{2x}{x^2+1}}$$

Example Differentiate

$$y = \ln(\sin x)$$

SOLUTION.

$$y' = \frac{d}{dx}\ln(\sin x)$$

$$= \frac{1}{\sin x}\,\frac{d}{dx}(\sin x)$$

$$= \frac{1}{\sin x}\cos x$$

$$= \frac{\cos x}{\sin x}$$

$$= \boxed{\cot x}$$

Example Differentiate

SOLUTION

$$= \boxed{\ln x}$$

Example: Differentiate

$$y = \ln(\sin x)$$

SOLUTION:

$$y' = \frac{d}{dx}\ln(\sin x)$$

$$= \frac{1}{\sin x}\,\frac{d}{dx}(\sin x)$$

$$= \frac{1}{\sin x}\cdot\cos x$$

$$= \frac{\cos x}{\sin x}$$

$$= \boxed{\cot x}$$

Example: Differentiate

$$y = x\ln x - x$$

SOLUTION:

$$y' = \frac{d}{dx}(x\ln x - x)$$

$$= \frac{d}{dx}(x\ln x) - \frac{d}{dx}x$$

$$= \left[\left(\frac{d}{dx}x\right)\ln x + x\frac{d}{dx}\ln x\right] - 1$$

$$= \left[1\cdot\ln x + x\cdot\frac{1}{x}\right] - 1$$

$$= \ln x + \cancel{1 - 1}$$

$$= \boxed{\ln x}$$

Example: Differentiate

$$y = \log(2x^3 - 1).$$

Here, $\log = \log_{10}$

$$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$$

SOLUTION:

$$y' = \frac{d}{dx}\log(2x^3 - 1)$$

$$= \frac{1}{(\ln 10)(2x^3-1)}\cdot\frac{d}{dx}(2x^3 - 1)$$

$$= \frac{1}{(\ln 10)(2x^3-1)}\cdot 6x^2$$

$$= \boxed{\frac{6x^2}{(\ln 10)(2x^3-1)}}$$

Example: Find the derivative of

$$y = x^x.$$

SOLUTION:

$$y = x^x$$

$$\ln(y) = \ln(x^x)$$

$$\ln(y) = x \ln x$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{d}{dx}x\right)\ln x + x\frac{d}{dx}\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

Properties of Logarithms

$$\log_a(x^k) = k \log_a x$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$y\left(\frac{1}{y}\frac{dy}{dx}\right) = y(\ln x + 1)$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = \boxed{x^x(\ln x + 1)}$$

Example: Differentiate

$$y = \frac{e^x x^{\frac{3}{2}}\sqrt{1+\ldots}}{(x^2+3)^4(3x-\ldots}$$

SOLUTION:

$$\ln y = \ln\left(\frac{e^x x^{\frac{3}{2}}\ldots}{(x^2+3)^{\ldots}}\right.$$

$$\ln y = \ln(e^x) + \ldots$$

$$- \ln(x\ldots$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\Big[x + \frac{3}{2}\ldots$$

$$- 4\ln\ldots$$

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Example: Differentiate

$$y=\frac{e^x x^{\frac{3}{2}}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}$$

SOLUTION:

$$\ln y=\ln\left(\frac{e^x x^{\frac{3}{2}}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}\right)$$

$\rightarrow \ln(1+x)^{\frac{1}{2}}$

$$\ln y=\ln(e^x)+\ln\left(x^{\frac{3}{2}}\right)+\ln\sqrt{1+x}$$
$$-\ln(x^2+3)^4-\ln(3x-2)^3$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left[x+\frac{3}{2}\ln x+\frac{1}{2}\ln(1+x)\right.$$
$$\left.-4\ln(x^2+3)-3\ln(3x-2)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=1+\frac{3}{2}\cdot\frac{1}{x}+\frac{1}{2}\frac{1}{1+x}$$
$$-4\cdot\left(\frac{1}{x^2+3}\cdot 2x\right)-3\cdot\left(\frac{1}{3x-2}\cdot 3\right)$$

$$\frac{dy}{dx}=y\left(1+\frac{3}{2x}+\frac{1}{2(1+x)}-\frac{8x}{x^2+3}-\frac{9}{3x-2}\right)$$

$$\boxed{=\frac{e^x x^{\frac{3}{2}}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}\left[1+\frac{3}{2}x+\frac{1}{2(1+x)}-\frac{8x}{x^2+3}-\frac{9}{3x-2}\right]}$$

Power Rule Let $y = x^a$, where $a$ is any real number.

Then $$\frac{dy}{dx} = ax^{a-1}.$$

In other words, $$\frac{d}{dx}(x^a) = ax^{a-1}.$$

PROOF: Set $y = x^a$. Then

$$\ln y = \ln(x^a)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(a \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{a}{x}$$

$$\frac{dy}{dx} = y \cdot \frac{a}{x}$$

$$\frac{dy}{dx} = x^a \cdot \frac{a}{x}$$

$$\frac{dy}{dx} = x^a x^{-1} a$$

$$\frac{dy}{dx} = x^{a-1} a$$

$$\frac{dy}{dx} = ax^{a-1}$$

Therefore, $$\frac{d}{dx}(x^a) = ax^{a-1}$$

M

Let

Find

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# Mini-Quiz 7/13

Let $f(x) = 3x^2 + 2x - 3$.

Find $\frac{d}{dx} f^{-1}(x) \Big|_{x=2}$.