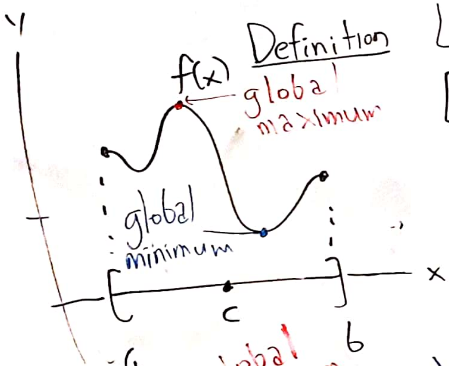
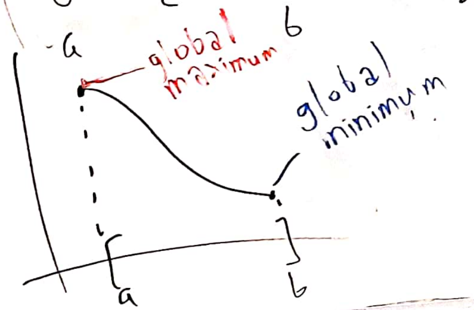
5.1 Extrema and the Mean-Value Theorem



Definition Let f be a function defined on an interval $[a,b]$ that contains the number c.

Then:

- f has a global maximum at $x=c$ if:
  $$f(c) \geq f(x) \quad \text{for all } x \in [a,b]$$
- f has a global minimum at $x=c$ if:
  $$f(c) \leq f(x) \quad \text{for all } x \in [a,b]$$



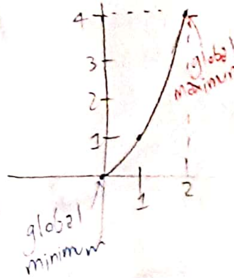
Extreme Value Theorem

If f is continuous on a closed interval $[a,b]$, then f has:

- a global maximum, and
- a global minimum

on $[a,b]$.

Example: Let $f(x)=x^2$ o…



Then f… global … on [0… and f… global … on [0…

This is because of the … Value Theorem, since … is continuous on [0,…

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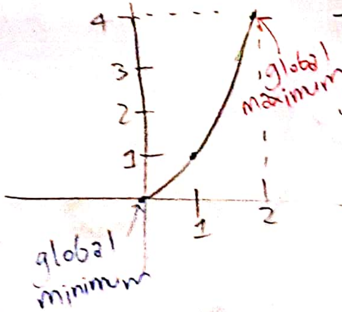
Example: Let $f(x)=x^2$ on $[0,2]$.



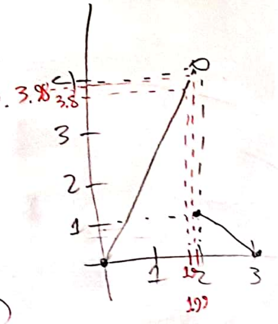
Then f has a global maximum on $[0,2]$ at $x=2$, and f has a global minimum on $[0,2]$ at $x=0$.

This is because of the Extreme Value Theorem, since $f(x)=x^2$ is continuous on $[0,2]$.

Example: Show that $f(x)$ defined by

$$f(x)=\begin{cases} 2x & \text{if } 0\le x<2 \\ 3-x & \text{if } 2\le x<3 \end{cases}$$

does not have a global maximum.



EXPLANATION.

$$\lim_{x\to 2^-} f(x)=4$$

But $f(2)=1$.

So you can reach values of $f(x)$ close to $y=4$ but NOT actually $y=4$.

In other words, $f(x) < 4$ strictly less than

So f has no global maximum.

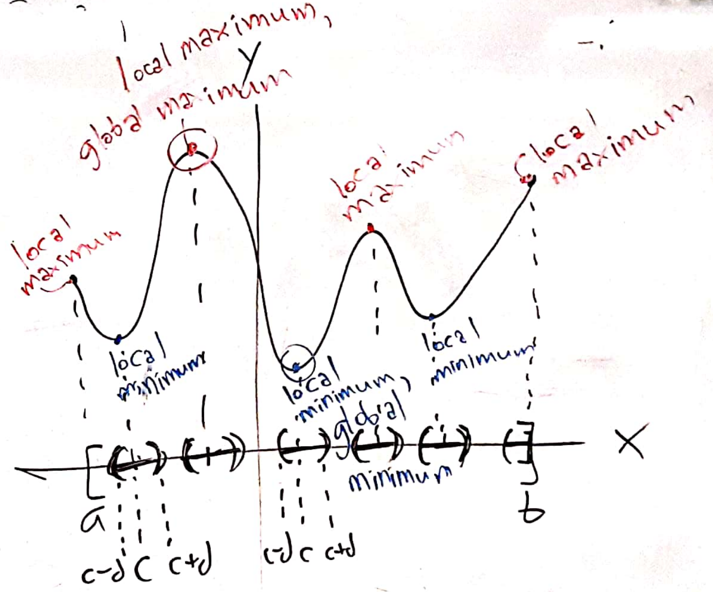
Local Extrema

Definition A function f on $[a,b]$ has a local maximum if there exists a really small $\delta > 0$ such that

$$f(c) \geq f(x) \quad \text{for all } x \in [a,b] \cap (c-\delta, c+\delta)$$

and has a local minimum if there exists a really small $\delta > 0$ such that

$$f(c) \leq f(x) \quad \text{for all } x \in [a,b] \cap (c-\delta, c+\delta)$$



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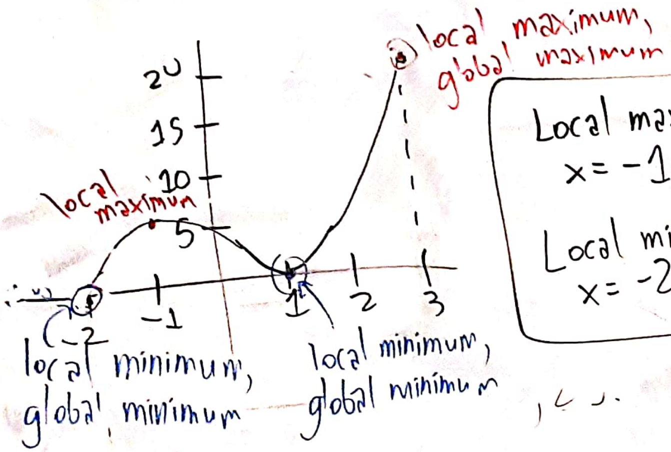
Example: Let $f(x) = (x-1)^2(x+2)$ for $-2 \le x \le 3$.

Graph the function over this interval, and find all local and global extrema.



Local maxima at: $x = -1$, $x = 3$

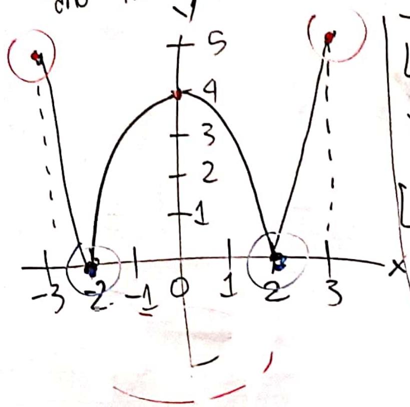
Global maxima at: $x = 3$

Local mimima at: $x = -2$, $x = 1$

Global minima at: $x = -2$, $x = 1$

Example: Let $f(x) = |x^2 - 4|$ for $-3 \le x \le 3$.

Graph the function over this interval, and find all local and global extrema.



Local maxima at: $x = -3$, $x = 0$, $x = 3$

Global maxima at: $x = -3$, $x = 3$

Local minima at: $x = -2$, $x = 2$

Global minima at: $x = -2$, $x = 2$

Fermat's Theorem

If f has a local extremum at an interior point c and $f'(c)$ exists, then

$$f'(c) = 0.$$

This theorem implies:

If $f'(c) \neq 0$, then f does NOT have an extrema at $x = c$.

Example: Explain why $f(x) = \tan x$ does not have a local maximum or minimum at $x = 0$.

SOLUTION:

$$f'(x) = \frac{d}{dx}\tan x = \sec^2 x.$$

At $c = 0$:

$$f'(c) = f'(0) = \sec^2 0 = \frac{1}{\cos^2 0} = \frac{1}{1^2} = 1 \neq 0$$

By Fermat's Theorem, $f(x) = \tan x$ does NOT have an extremum at $x = 0$.

☺

Here are the guidelines for finding local extrema of a function f:

1. Find all points c where $f'(c)=0$.
2. Find all points c where $f'(c)$ does not exist.
3. Find the endpoints of the domain of f.

Set $f(x)=0$

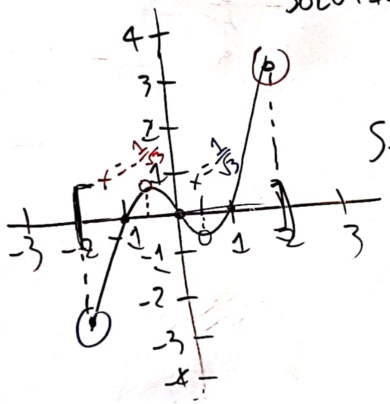
$$x^3-x=0$$

$$x(x^2-1)=0$$

$$x(x+1)(x-1)=0$$

x-ints: $x=0$, $x+1=0$, $x-1=0$

$x=-1$, $x=1$



Example: Find all local and global extrema of

$$f(x)=x^3-x$$

on the interval $[-2, 2]$

SOLUTION: $f'(x)=\frac{d}{dx}(x^3-x)$

$$=3x^2-1$$

Set $f'(x)=0$

$$3x^2-1=0$$

$$3x^2=1$$

$$x^2=\frac{1}{3}$$

$$x=\pm\frac{1}{\sqrt{3}}$$

Local maxima at: $x=-\frac{1}{\sqrt{3}}$, $x=2$	Global maxima at: $x=2$
Local mimima at: $x=+\frac{1}{\sqrt{3}}$, $x=-2$	Global minima at: $x=-2$

Mini-Quiz 7/18

Find all local and global extrema of the function

$$f(x) = x^3 - 4x$$

on the interval $[-3, 3]$. Also graph the function.