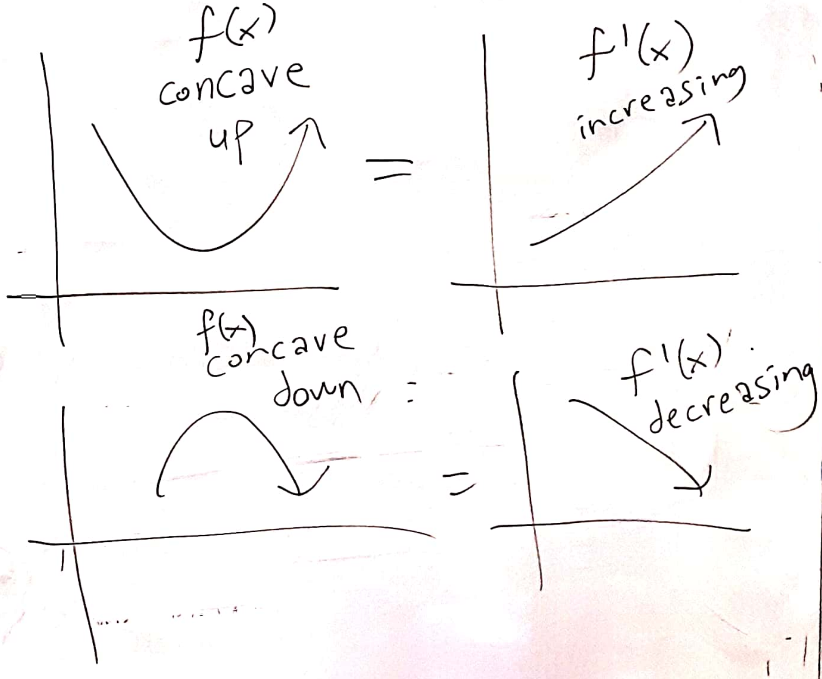
5.2 continued: Concavity

Definition A differentiable function $f(x)$ is concave up on $[a,b]$ if the first derivative $f'(x)$ is increasing on $[a,b]$, and concave down on $[a,b]$ if the first derivative $f'(x)$ is decreasing on $[a,b]$.



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Second Derivative Test for Concavity

Suppose that f is twice differentiable on (a,b).

- If $f''(x) > 0$ for all $x \in (a,b)$, then f is concave up on (a,b).
- If $f''(x) < 0$ for all $x \in (a,b)$, then f is concave down on (a,b).

Example: Show that $f(x) = x^2$ is concave up on $\mathbb{R}$.

SOLUTION: $f(x) = x^2$

$$f'(x) = \frac{d}{dx} x^2 = 2x$$

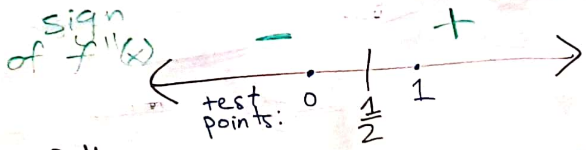
$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}(2x) = 2 > 0$$

Since $f''(x) > 0$ for all real #'s x, we conclude that $f(x) = x^2$ is concave up on $\mathbb{R}$.

Example: Determine, for all x in $\mathbb{R}$, where

$$f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3$$

is concave up and where it is concave down.



$$f''(0) = 6 \cdot 0 - 3 = -3 < 0 \quad -$$

$$f''(1) = 6 \cdot 1 - 3 = 3 > 0 \quad +$$

SOLUTION:

$$f'(x) = \frac{d}{dx}\left(x^3 - \frac{3}{2}x^2 - 6x + 3\right)$$
$$= 3x^2 - 3x - 6$$
$$f''(x) = \frac{d}{dx} f'(x)$$
$$= \frac{d}{dx}(3x^2 - 3x - 6)$$
$$= 6x - 3$$

Set $f''(x) = 0$

$$6x - 3 = 0$$
$$6x = 3$$
$$x = \frac{3}{6}$$
$$x = \frac{1}{2}$$

$f''(x)$ is concave down on $\left(-\infty, \frac{1}{2}\right)$ and $f''(x)$ is concave up on $\left(\frac{1}{2}, \infty\right)$

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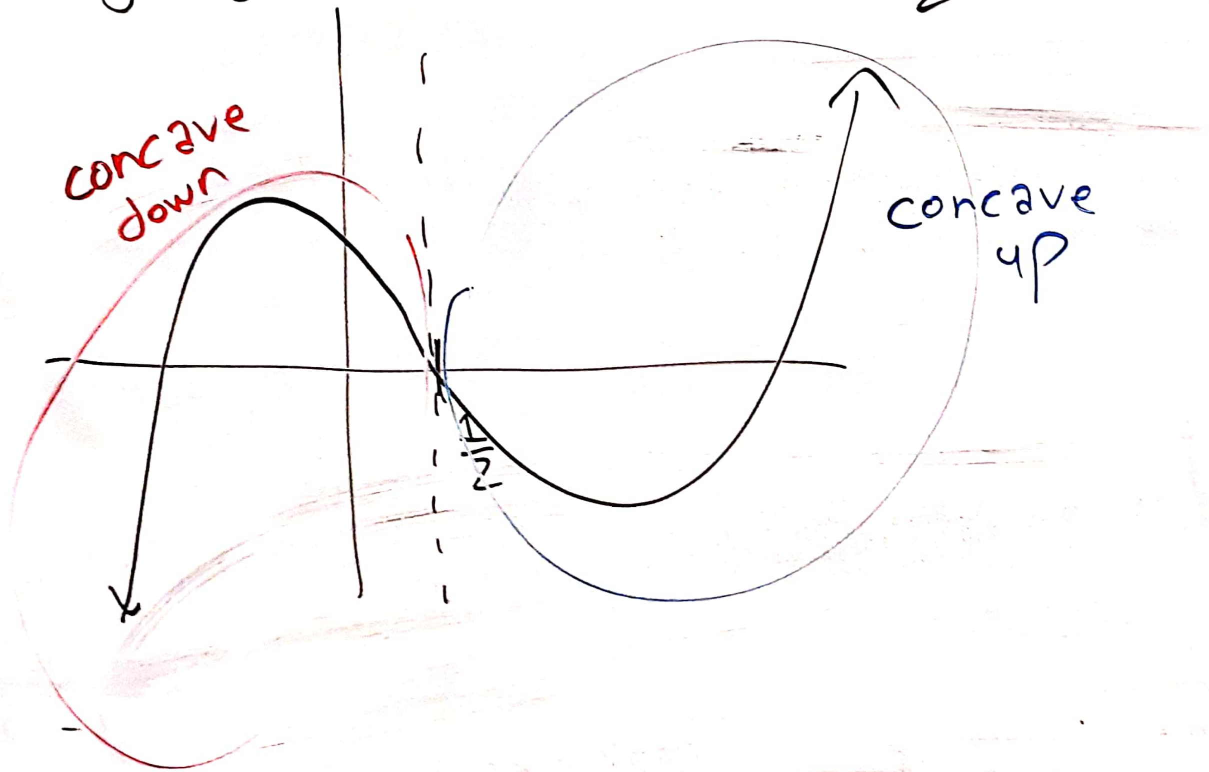
rough graph of $f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3$



5.3 Extrema and Inflection Points

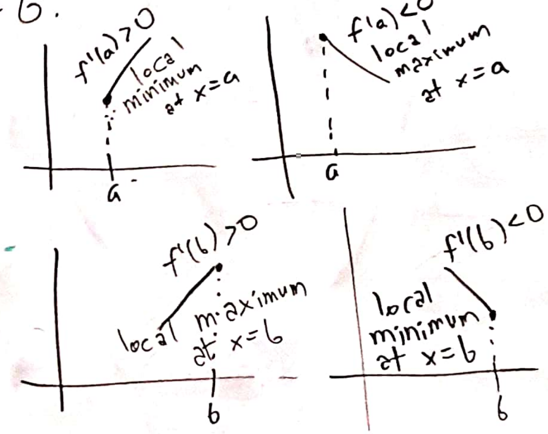
The First Derivative Test for Local Extrema at Endpoints

Suppose f is differentiable on $[a, b]$ and its derivative is continuous at $x=a$ and $x=b$.

Then:

- If $f'(a) \gtrless 0$, then f has a: local minimum at $x=a$.
  maximum

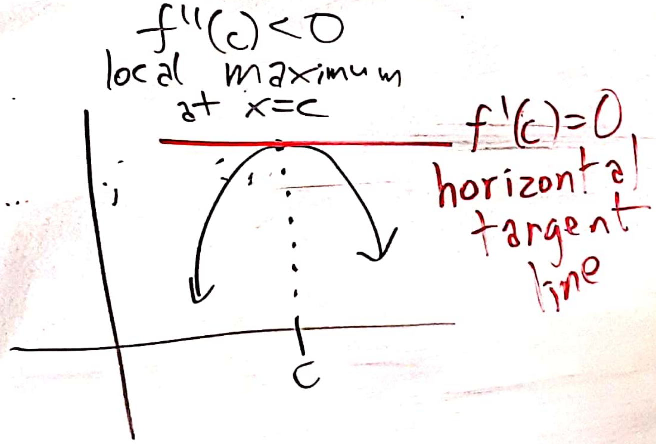
- If $f'(b) \gtrless 0$, then f has a local maximum at $x=b$.
  minimum

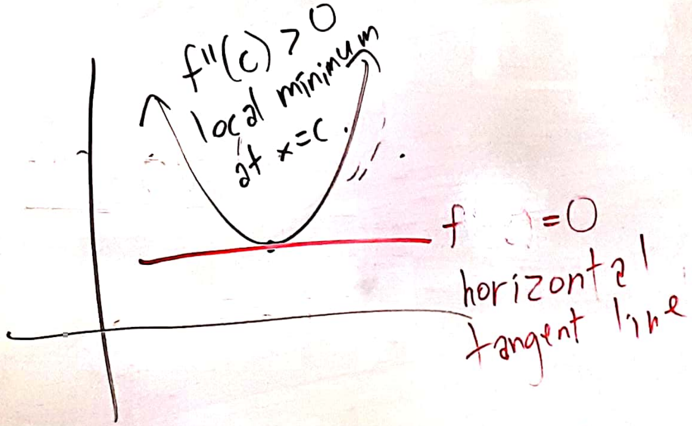


Second Derivative Test for Local Extrema

Suppose that f is twice differentiable on an open interval containing c.

- If $f'(c)=0$ and $f''(c) \underset{>}{<} 0$, then f has a local maximum at $x=c$.
  minimum

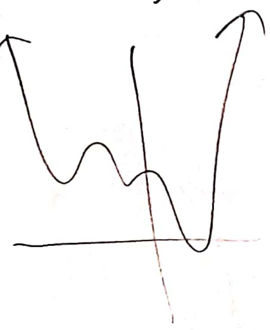




Example: Find all local and global extrema of

$$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2$$

for all $x \in \mathbb{R}$.



SOLUTION:
$$f'(x) = \frac{d}{dx}\left(\frac{3}{2}x^4 - 2x^3 - 6x^2 + 2\right)$$
$$= 6x^3 - 6x^2 - 12x$$

$$f''(x) = \frac{d}{dx} f'(x)$$
$$= \frac{d}{dx}\left(6x^3 - 6x^2 - 12x\right)$$
$$= 18x^2 - 12x - 12$$

Set $f'(x) = 0$

$$6x^3 - 6x^2 - 12x = 0$$
$$6x(x^2 - x - 2) = 0$$
$$6x(x-2)(x+1) = 0$$

$6x = 0 \quad x - 2 = 0 \quad x + 1 = 0$

Critical points: $x = 0 \quad x = 2 \quad x = -1$

To find global minimum and global maximum, plug into $f(x)$.

LOCAL MAXIMA

$f(0) = 2$

Since $\lim_{x \to \pm\infty} f(x) = +\infty$

No global maximum

LOCAL MINIMA

$f(-1) = 18$

$f(2) = -14$

Apply the Second Derivative Test

$$f''(0) = 18 \cdot 0^2 - 12 \cdot 0 - 12$$
$$= -12$$
$$< 0$$

Local maximum at $x = 0$ by the Second Derivative Test

$$f''(2) = 18 \cdot 2^2 - 12 \cdot 2 - 12$$
$$= 36$$
$$> 0$$

Local minimum at $x = 2$ by the Second Derivative Test

$$f''(-1) = 18 \cdot (-1)^2 - 12 \cdot (-1) - 12$$
$$= 18$$
$$> 0$$

Local minimum at $x = -1$ by the Second Derivative Test

Global minimum at $x = 2$

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Example: Show that the function

$$f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x + 1$$

has an inflection point at $x = 1$

An inflection point is a point at which the concavity of the function changes

Set $f''(x) = 0$.

$$f'(x) = \frac{d}{dx}\left(\frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x + 1\right)$$
$$= \frac{3}{2}x^2 - 3x + 2$$

$$f''(x) = \frac{d}{dx}\left(\frac{3}{2}x^2 - 3x + 2\right)$$
$$= 3x - 3$$

Set $f''(x) = 0$

$$3x - 3 = 0$$
$$3x = 3$$
$$x = 1$$

$\underline{x=1}$

$$f(1) = \frac{1}{2}\cdot 1^3 - \frac{3}{2}\cdot 1^2 + 2\cdot 1 + 1$$
$$= \frac{1}{2} - \frac{3}{2} + 2 + 1$$
$$= 2$$

$(1,2)$ is the inflection point of $f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x + 1$

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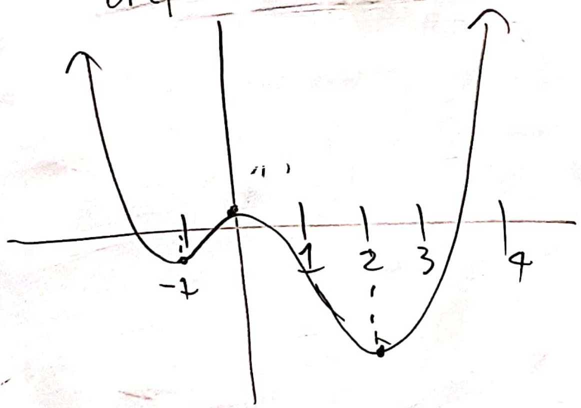
Example: Find all local and global extrema of

$$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2$$

for all $x \in \mathbb{R}$.

SOLUTION:

Graph of $f(x)$



Local maxima

$x = 0$

$f(0) = 2$

NO global maximum

b/c $\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

Local minima

$x = -1, \; x = 2$

$f(-1) = -\frac{1}{2}, \quad f(2) = -14$ low val

So global minimum occurs at $x = 2$

Class Example

Find all local and global extrema of

$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2$

$f'(x) = 6x^3 - 6x^2 - 12x$

$f''(x) = 18x^2 - 12x - 12$

set $f'(x)=0$: $6x^3 - 6x^2 - 12x = 0$

$6x(x^2 - x - 2) = 0$

$6x(x-2)(x+1) = 0$

critical points $x = -2, 0, 1$

$f''(-1) = 18(-1)^2 - 12(-1) - 12$
$= 18 \cdot 1 + 12 - 12$
$= 18$
> 0

By the Second Derivative Test, $f(x)$ is CONCAVE UP at $x=-1$.

$\boxed{\text{So } f(x) \text{ has a local minimum at } x=-1.}$

$f''(0) = 18 \cdot 0^2 - 12 \cdot 0 - 12$
$= 0 - 0 - 12$
$= -12$
< 0

By the Second Derivative Test, $f(x)$ is CONCAVE DOWN at $x=0$.

$\boxed{\text{So } f(x) \text{ has a local maximum at } x=0.}$

$f''(2) = 18 \cdot 2^2 - 12 \cdot 2 - 12$
$= 18 \cdot 4 - 24 - 12$
$= 72 - 24 - 12$
$= 36$
> 0

By the Second Derivative Test, $f(x)$ is CONCAVE UP at $x=2$.

$\boxed{\text{So } f(x) \text{ has a local minimum at } x=2}$

$$\underbrace{\begin{aligned} f(2) &= \tfrac{3}{2} \cdot 2^4 - 2 \cdot 2^3 - 6 \cdot 2^2 + 2 \\ &= \tfrac{3}{2} \cdot 16 - 2 \cdot 8 - 6 \cdot 4 + 2 \\ &= 24 - 16 - 24 + 2 \\ &= -14 \end{aligned} \qquad \begin{aligned} f(-1) &= \tfrac{3}{2}(-1)^4 - 2 \cdot (-1)^3 - 6(-1)^2 + 2 \\ &= \tfrac{3}{2} \cdot 1 - 2(-1) - 6 \cdot 1 + 2 \\ &= \tfrac{3}{2} + 2 - 6 + 2 \\ &= -\tfrac{1}{2} \end{aligned}}$$

Since $f(2) = -14$ is the LOWEST VALUE of $f(x)$,

$\boxed{f(x) \text{ has a global minimum at } x=2.}$

Since $\lim_{x \to \pm\infty} f(x) = +\infty$,

$\boxed{f(x) \text{ does } \underline{\text{not}} \text{ have any global maximum.}}$

Example

Find all local and global extrema of
$f(x) = x^4(1-x)$
on the interval $-1 \le x \le 1$.

$f(x) = x^4(1-x)$
$= x^4 - x^5$

$f'(x) = 4x^3 - 5x^4$

$f''(x) = 12x^2 - 20x^3$

Set $f'(x) = 0$: $4x^3 - 5x^4 = 0$

$x^3(4 - 5x) = 0$

$x^3 = 0 \qquad 4 - 5x = 0$

Critical points: $x = 0 \qquad x = \frac{4}{5}$

$$f''\left(\tfrac{4}{5}\right) = 12\cdot\left(\tfrac{4}{5}\right)^2 - 20\left(\tfrac{4}{5}\right)^3$$
$$= 12\cdot\frac{16}{25} - \cancel{20}^{4}\cdot\frac{64}{\cancel{125}_{25}}$$
$$= 12\cdot\frac{16}{25} - 4\cdot\frac{64}{25}$$
$$= \frac{192}{25} - \frac{256}{25}$$
$$= -\frac{64}{25}$$
$$< 0$$

By the Second Derivative Test, f is concave down at $x = \frac{4}{5}$

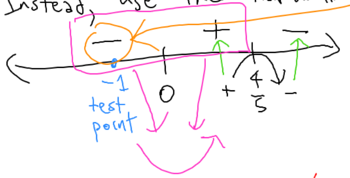
$\boxed{\text{So } f \text{ has a local maximum at } x = \tfrac{4}{5}.}$

$$f''(0) = 12\cdot 0^2 - 20\cdot 0^3$$
$$= 0 - 0$$
$$= 0$$

so we actually don't know whether f is concave up or concave down at $x=0$.

We cannot apply the Second Derivative Test here.

Instead, use the number line:



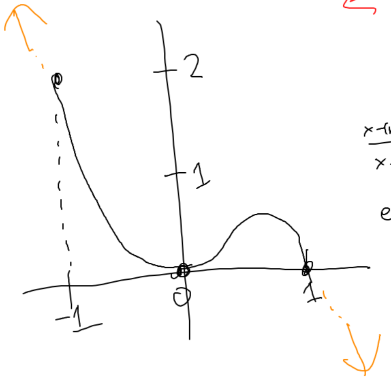
Test point

$f'(-1) = 4(-1)^3 - 5(-1)^4$
$= 4\cdot(-1) - 5\cdot 1$
$= -4 - 5$
$= -9$
< 0

So f is decreasing on $(-\infty, 0)$

Therefore, since f is decreasing on $(-\infty, 0)$ and increasing on $(0, \frac{4}{5})$, we conclude (from the number line) that f has a local minimum at $x=0$.

I think the easiest way to find GLOBAL extrema is to graph the function over the interval in question



x-intercepts
$x = 0,\ x = 1$

end behavior:
$-x^5$
since $f(x) = x^4 - x^5$ ← term with highest degree dominates for large x values

$\boxed{\text{Global maximum occurs at } x = -1}$
because $f(-1) = 2$ but f is decreasing at $x = -1$
(since $f'(-1) = -9 < 0$)

$\boxed{\text{Global minimum occurs at } x = 0 \text{ and } x = 1}$
because $y = 0$ is the lowest value of $f(x)$ over $-1 \le x \le 1$

We do not have to consider $\lim_{x\to\infty} f(x) = -\infty$ or $\lim_{x\to -\infty} f(x) = +\infty$ because our domain is $-1 \le x \le 1$, not the entire real line

Mini-Quiz 7/20

Find all local and global extrema of the function

$$f(x) = x^2$$

for all real numbers x.