MATH 7A
Summer 2022
Discussion Quiz 2 Solution

Five (5) points on completion plus five (5) points on correctness, for a total of ten (10) points

Use the limit definition of the derivative to compute $f'(x)$ given

$$f(x) = \frac{3}{x^5}.$$

Hint: In order to compute $f(x+h)$, you will need to expand $(x+h)^5$ in the denominator of $f(x+h)$, which means you will need to apply the Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \cdots + \binom{n}{n-1}a^{n-1}b^1 + \binom{n}{n}a^0b^n,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is called the binomial coefficient and

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

denotes the factorial of n.

Solution. Our binomial coefficients are

$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = 1,$$
$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = 5,$$
$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10,$$
$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10,$$
$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = 5,$$
$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = 1.$$

The Binomial Theorem allows us to write

$$\begin{aligned} f(x+h) &= \frac{3}{(x+h)^5} \\ &= \frac{3}{\binom{5}{0}x^5h^0 + \binom{5}{1}x^4h^1 + \binom{5}{2}x^3h^2 + \binom{5}{3}x^2h^3 + \binom{5}{4}x^1h^4 + \binom{5}{5}x^0h^5} \\ &= \frac{3}{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}. \end{aligned}$$

So the difference quotient is

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\frac{3}{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5}-\frac{3}{x^5}}{h} \\
&= \frac{\frac{3x^5-3(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)}{x^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)}}{h} \\
&= \frac{\frac{3\cancel{x^5}-3\cancel{x^5}-15x^4h-30x^3h^2-30x^2h^3-15xh^4-3h^5}{x^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)}}{h} \\
&= \frac{\frac{-15x^4h-30x^3h^2-30x^2h^3-15xh^4-3h^5}{x^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)}}{h} \\
&= \frac{-15x^4h-30x^3h^2-30x^2h^3-15xh^4-3h^5}{hx^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)} \\
&= \frac{\cancel{h}(-15x^4-30x^3h-30x^2h^2-15xh^3-3h^4)}{\cancel{h}x^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)} \\
&= \frac{-15x^4-30x^3h-30x^2h^2-15xh^3-3h^4}{x^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)}.
\end{aligned}$$

Therefore, the derivative of $f(x)$ is

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{-15x^4-30x^3h-30x^2h^2-15xh^3-3h^4}{x^5(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)} \\
&= \frac{-15x^4-30x^3 0-30x^2 0^2-15x0^3-3\cdot 0^4}{x^5(x^5+5x^4 0+10x^3 0^2+10x^2 0^3+5x0^4+0^5)} \\
&= \frac{-15x^4}{x^5(x^5)} \\
&= -\frac{15x^4}{x^{10}} \\
&= \boxed{-\frac{15}{x^6}}.
\end{aligned}$$

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