MATH 7A
Summer 2022
Discussion Quiz 3 Solution

Five (5) points on completion plus five (5) points on correctness, for a total of ten (10) points

Find the equation of the tangent line to the curve

$$f(x) = \left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right)^2$$

at $x = -1$.

*Solution.* By applying the chain rule, we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right)^2 \\ &= 2\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\frac{d}{dx}\left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right). \end{aligned}$$

Furthermore, the quotient rule gives us

$$\begin{aligned} \frac{d}{dx}\left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right) &= \frac{\frac{d}{dx}[(x+2)^2(3x+4)^4](5x+6)^6 - [(x+2)^2(3x+4)^4]\frac{d}{dx}(5x+6)^6}{((5x+6)^6)^2} \\ &= \frac{\frac{d}{dx}[(x+2)^2(3x+4)^4](5x+6)^6 - [(x+2)^2(3x+4)^4]\frac{d}{dx}(5x+6)^6}{(5x+6)^{12}} \\ &= \frac{\frac{d}{dx}[(x+2)^2(3x+4)^4](5x+6)^6}{(5x+6)^{12}} - \frac{[(x+2)^2(3x+4)^4]\frac{d}{dx}(5x+6)^6}{(5x+6)^{12}} \\ &= \frac{\frac{d}{dx}[(x+2)^2(3x+4)^4]}{(5x+6)^6} - \frac{[(x+2)^2(3x+4)^4]\frac{d}{dx}(5x+6)^6}{(5x+6)^{12}}. \end{aligned}$$

By the product rule and the chain rule, we obtain

$$\begin{aligned} \frac{d}{dx}[(x+2)^2(3x+4)^4] &= \frac{d}{dx}(x+2)^2(3x+4)^4 + (x+2)^2\frac{d}{dx}(3x+4)^4 \\ &= \left[2(x+2)\frac{d}{dx}(x+2)\right](3x+4)^4 + (x+2)^2\left[4(3x+4)^3\frac{d}{dx}(3x+4)\right] \\ &= [2(x+2)\cdot 1](3x+4)^4 + (x+2)^2[4(3x+4)^3\cdot 3] \\ &= 2(x+2)(3x+4)^4 + 12(x+2)^2(3x+4)^3. \end{aligned}$$

By the chain rule, we obtain

$$\begin{aligned} \frac{d}{dx}(5x+6)^6 &= 6(5x+6)^5\frac{d}{dx}(5x+6) \\ &= 6(5x+6)^5\cdot 5 \\ &= 30(5x+6)^5. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right) &= \frac{\frac{d}{dx}[(x+2)^2(3x+4)^4]}{(5x+6)^6} - \frac{[(x+2)^2(3x+4)^4]\frac{d}{dx}(5x+6)^6}{(5x+6)^{12}} \\
&= \frac{2(x+2)(3x+4)^4 + 12(x+2)^2(3x+4)^3}{(5x+6)^6} - \frac{[(x+2)^2(3x+4)^4][30(5x+6)^5]}{(5x+6)^{12}} \\
&= \frac{2(x+2)(3x+4)^4}{(5x+6)^6} + \frac{12(x+2)^2(3x+4)^3}{(5x+6)^6} - \frac{30(x+2)^2(3x+4)^4}{(5x+6)^7} \\
&= \frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\left(\frac{2}{x+2} + \frac{12}{3x+4} - \frac{30}{5x+6}\right),
\end{aligned}$$

and so our first derivative is

$$\begin{aligned}
f'(x) &= 2\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\frac{d}{dx}\left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right) \\
&= 2\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\left(\frac{2}{x+2} + \frac{12}{3x+4} - \frac{30}{5x+6}\right) \\
&= 2\left(\frac{(x+2)^2(3x+4)^4}{(5x+6)^6}\right)^2\left(\frac{2}{x+2} + \frac{12}{3x+4} - \frac{30}{5x+6}\right) \\
&= 2f(x)\left(\frac{2}{x+2} + \frac{12}{3x+4} - \frac{30}{5x+6}\right).
\end{aligned}$$

At $x = -1$, we have

$$\begin{aligned}
f(-1) &= \left(\frac{(-1+2)^2(3(-1)+4)^9}{(5(-1)+6)^6}\right)^2 \\
&= \left(\frac{1^2 \cdot 1^9}{1^6}\right)^2 \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
f'(-1) &= 2f(-1)\left(\frac{2}{-1+2} + \frac{12}{3(-1)+4} - \frac{30}{5(-1)+6}\right) \\
&= 2 \cdot 1 \cdot \left(\frac{2}{1} + \frac{12}{1} - \frac{30}{1}\right) \\
&= 2 \cdot 1 \cdot (-16) \\
&= -32.
\end{aligned}$$

Now, we know that the point-slope formula for a linear equation is

$$y - y_1 = m(x - x_1).$$

With $x_1 = -1$, $y_1 = f(-1) = 1$, and $m = f'(-1) = -32$, we have

$$y - 1 = -32(x - (-1)),$$

which is algebraically equivalent to

$$\boxed{y = -32x - 31}.$$

□