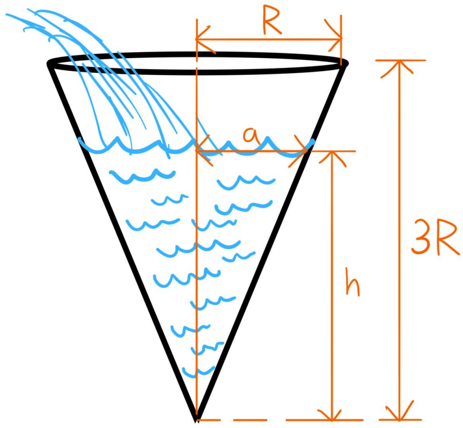
MATH 7A
Summer 2022
Discussion Quiz 4 Solution

Five (5) points on completion plus five (5) points on correctness, for a total of ten (10) points

The formula for the volume of the water in a cone-shaped cup in terms of the water's height h and the radius of the water's surface a is given by

$$V = \frac{1}{3}\pi a^2 h$$

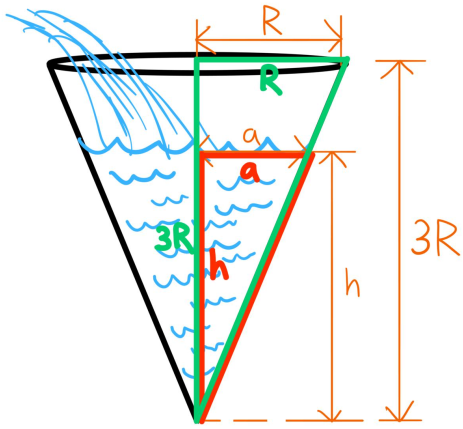
for $0 \leq h \leq 3R$ and $0 \leq a \leq R$.



Water is being filled in the cone-shaped cup at a rate of 800 cm^3/sec. If the maximal radius of the cup is 25 cm and the height of the cup is three times its maximal radius, what is the rate of the water rising when the height of the water is 60 cm?

Either express your final answer as an exact expression or approximate it to two decimal places. Don't forget units in your final answer! You may use a calculator for this quiz.

Solution. First, notice that we can eliminate the parameter a and re-express our volume formula in terms of h and R, which would be advantageous because R is a constant value.



Indeed, by using similar triangles, we have the relationship

$$\frac{a}{h} = \frac{R}{3R},$$

which implies

$$\begin{aligned} a &= \frac{hR}{3R} \\ &= \frac{h}{3}. \end{aligned}$$

So we can rewrite our formula for the volume of the water in the semi-spherical bowl as

$$\begin{aligned} V &= \frac{1}{3}\pi a^2 h \\ &= \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h \\ &= \frac{1}{3}\pi \frac{h^2}{9} h \\ &= \frac{1}{27}\pi h^3. \end{aligned}$$

The only variables that change with respect to time are V and h. By applying implicit differentiation with respect to time t, as well as the chain rule, we obtain

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt}\left(\frac{1}{27}\pi h^3\right) \\ &= \frac{1}{27}\pi \frac{d}{dt}(h^3) \\ &= \frac{1}{27}\pi \left(3h^2 \frac{dh}{dt}\right) \\ &= \frac{1}{9}\pi h^2 \frac{dh}{dt}. \end{aligned}$$

So the rate of which the height of the water in the bowl is rising is

$$\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}.$$

Finally, we substitute $h = 60$ cm and $\dfrac{dV}{dt} = 800$ cm^3/sec to obtain

$$\begin{aligned} \frac{dh}{dt} &= \frac{9}{\pi h^2}\frac{dV}{dt} \\ &= \frac{9}{\pi (60)^2} 800 \\ &= \boxed{\frac{2}{\pi} \text{ cm/sec}} \\ &\approx \boxed{0.64 \text{ cm/sec}}. \end{aligned}$$

□