MATH 7A
Summer 2022
Discussion Quiz 5 Solution

Five (5) points on completion plus five (5) points on correctness, for a total of ten (10) points

Use l'Hôpital's Rule to compute

$$\lim_{x\to 0} \frac{x - \sin x}{\tan x - x}.$$

Solution. We have

$$\begin{aligned}\lim_{x\to 0} \frac{x - \sin x}{\tan x - x} &= \frac{0 - \sin 0}{\tan 0 - 0} \\ &= \frac{0 - 0}{0 - 0} \\ &= \frac{0}{0},\end{aligned}$$

which is indeterminate. So we apply l'Hôpital's Rule for the first time to obtain

$$\begin{aligned}\lim_{x\to 0} \frac{x - \sin x}{\tan x - x} &= \lim_{x\to 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(\tan x - x)} \\ &= \lim_{x\to 0} \frac{1 - \cos x}{\sec^2 x - 1} \\ &= \frac{1 - \cos 0}{\sec^2 0 - 1} \\ &= \frac{1 - 1}{1 - 1} \\ &= \frac{0}{0},\end{aligned}$$

which is indeterminate. So we apply l'Hôpital's Rule for the second time to obtain

$$\begin{aligned}\lim_{x\to 0} \frac{x - \sin x}{\tan x - x} &= \lim_{x\to 0} \frac{1 - \cos x}{\sec^2 x - 1} \\ &= \lim_{x\to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sec^2 x - 1)} \\ &= \lim_{x\to 0} \frac{0 - (-\sin x)}{2\sec x \frac{d}{dx}\sec x - 0} \\ &= \lim_{x\to 0} \frac{\sin x}{2\sec x(\sec x \tan x)} \\ &= \lim_{x\to 0} \frac{\sin x}{2\sec^2 x \tan x} \\ &= \frac{\sin 0}{2\sec^2 0 \tan 0} \\ &= \frac{0}{2 \cdot 1 \cdot 0} \\ &= \frac{0}{0},\end{aligned}$$

which is indeterminate. So we apply l'Hôpital's Rule for the third time to obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{x - \sin x}{\tan x - x} &= \lim_{x\to 0} \frac{\sin x}{2\sec^2 x \tan x} \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(2\sec^2 x \tan x)} \\
&= \lim_{x\to 0} \frac{\cos x}{2((\frac{d}{dx}\sec^2 x)\tan x + \sec^2 x \frac{d}{dx}\tan x)} \\
&= \lim_{x\to 0} \frac{\cos x}{2(2\sec x \frac{d}{dx}\sec x)\tan x + \sec^2 x \sec^2 x)} \\
&= \lim_{x\to 0} \frac{\cos x}{2(2\sec x(\sec x \tan x))\tan x + \sec^4 x)} \\
&= \lim_{x\to 0} \frac{\cos x}{4\sec^2 x \tan^2 x + 2\sec^4 x} \\
&= \lim_{x\to 0} \frac{\cos 0}{4\sec^2 0 \tan^2 0 + 2\sec^4 0} \\
&= \frac{1}{4 \cdot 1^2 \cdot 0^2 + 2 \cdot 1^4} \\
&= \frac{1}{0+2} \\
&= \boxed{\frac{1}{2}}.
\end{aligned}$$

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