Quiz 5 solutions

1. Evaluate the indefinite integral

$$\int \cos^5(x)\sin^3(x)\,dx.$$

Solution. Following Key Idea 6.3.1 (see the original quiz), we can employ the substitution $u = \cos(x)$ and $du = -\sin(x)\,dx$ to obtain

$$\begin{aligned}\int \cos^5(x)\sin^3(x)\,dx &= \int \cos^5(x)\sin^2(x)\sin(x)\,dx\\ &= \int \cos^5(x)(1-\cos^2(x))\sin(x)\,dx\\ &= \int u^5(1-u^2)(-du)\\ &= \int u^7 - u^5\,du\\ &= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C\\ &= \boxed{\frac{1}{8}\cos^8(x) - \frac{1}{6}\cos^6(x) + C},\end{aligned}$$

as desired. □

Alternate solution. Following Key Idea 6.3.1 (see the original quiz), we can employ the substitution $u = \sin(x)$ and $du = \cos(x)\,dx$ to obtain

$$\begin{aligned}\int \cos^5(x)\sin^3(x)\,dx &= \int (\cos^2(x))^2\sin^3(x)\cos(x)\,dx\\ &= \int (1-\sin^2(x))^2\sin^3(x)\cos(x)\,dx\\ &= \int (1-u^2)^2u^3\,du\\ &= \int (1-2u^2+u^4)u^3\,du\\ &= \int u^7 - 2u^5 + u^3\,du\\ &= \frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4 + C\\ &= \boxed{\frac{1}{8}\sin^8(x) - \frac{1}{3}\sin^6(x) + \frac{1}{4}\sin^4(x) + C},\end{aligned}$$

as desired. □

2. Evaluate the indefinite integral

$$\int \sec^6(x)\tan(x)\,dx.$$

Solution. Using Key Idea 6.3.2 (see the original quiz), we can employ the substitution $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$ to obtain

$$\begin{aligned}\int \sec^6(x)\tan(x)\,dx &= \int \sec^5(x)\sec(x)\tan(x)\,dx\\ &= \int u^5\,du\\ &= \frac{1}{6}u^6 + C\\ &= \boxed{\frac{1}{6}\sec^6(x) + C},\end{aligned}$$

as desired. □

Alternate solution. Using Key Idea 6.3.2 (see the original quiz), we can employ the substitution $u = \tan(x)$ and $du = \sec^2(x)\,dx$ to obtain

$$\begin{aligned}
\int \sec^6(x)\tan(x)\,dx &= \int (\sec^2(x))^2 \tan(x)\sec^2(x)\,dx \\
&= \int (\tan^2(x)+1)^2 \tan(x)\sec^2(x)\,dx \\
&= \int (u^2+1)^2 u\,du \\
&= \int (u^4+2u^2+1)u\,du \\
&= \int u^5+2u^3+u\,du \\
&= \frac{1}{6}u^6+\frac{1}{2}u^4+\frac{1}{2}u^2+C \\
&= \boxed{\frac{1}{6}\tan^6(x)+\frac{1}{2}\tan^4(x)+\frac{1}{2}\tan^2(x)+C},
\end{aligned}$$

as desired. □

Remark. *I presented two solutions for each of the two questions. The expressions of the final answers are different, but they are equivalent to each other as trigonometric identities:*

$$\begin{aligned}
\frac{1}{8}\cos^8(x)-\frac{1}{6}\cos^6(x) &= \frac{1}{8}\sin^8(x)-\frac{1}{3}\sin^6(x)+\frac{1}{4}\sin^4(x), \\
\frac{1}{6}\sec^6(x) &= \frac{1}{6}\tan^6(x)+\frac{1}{2}\tan^4(x)+\frac{1}{2}\tan^2(x).
\end{aligned}$$