MATH 009B discussion
University of California, Riverside

Ryan Ta
Spring 2020

Quiz 6 solutions

1. Using an appropriate trigonometric substitution, convert the following integrals into trigonometric integrals without a square root. Do **not** evaluate the integral.

   (a) $\displaystyle\int \frac{x^2}{\sqrt{9+x^2}}\,dx$

   Solution. Let $x = 3\tan(\theta)$, which means $dx = 3\sec^2(\theta)\,d\theta$. Then the integral becomes

$$\begin{aligned}\int \frac{x^2}{\sqrt{9+x^2}}\,dx &= \int \frac{(3\tan(\theta))^2}{\sqrt{9+(3\tan(\theta))^2}} 3\sec^2(\theta)\,d\theta \\ &= \int \frac{9\tan^2(\theta)}{\sqrt{9(1+\tan^2(\theta))}} 3\sec^2(\theta)\,d\theta \\ &= \int \frac{9\tan^2(\theta)}{3\sqrt{\sec^2(\theta)}} 3\sec^2(\theta)\,d\theta \\ &= \boxed{\int \frac{9\tan^2(\theta)}{|\sec(\theta)|}\sec^2(\theta)\,d\theta},\end{aligned}$$

   as desired. □

   (b) $\displaystyle\int x^2\sqrt{4-x^2}\,dx$

   Solution. Let $x = 2\sin(\theta)$, which means $dx = 2\cos(\theta)\,d\theta$. Then the integral becomes FIX

$$\begin{aligned}\int x^2\sqrt{4-x^2}\,dx &= \int (2\sin(\theta))^2\sqrt{4-(2\sin(\theta))^2}2\cos(\theta)\,d\theta \\ &= \int 4\sin^2(\theta)\sqrt{4(1-\sin^2(\theta))}2\cos(\theta)\,d\theta \\ &= \int 4\sin^2(\theta)2\sqrt{\cos^2(\theta)}2\cos(\theta)\,d\theta \\ &= \boxed{\int 16\sin^2(\theta)|\cos(\theta)|\cos(\theta)\,d\theta},\end{aligned}$$

   as desired. □

   (c) $\displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

   Solution. Let $x = \sec(\theta)$, which means $dx = \sec(\theta)\tan(\theta)\,d\theta$. Then the integral becomes

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-1}}\,dx &= \int \frac{(\sec(\theta))^2}{\sqrt{(\sec(\theta))^2-1}}\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{\sec^3(\theta)}{\sqrt{(\sec^2(\theta)-1}}\tan(\theta)\,d\theta \\ &= \int \frac{\sec^3(\theta)}{\sqrt{\tan^2(\theta)}}\tan(\theta)\,d\theta \\ &= \boxed{\int \frac{\sec^3(\theta)}{|\tan(\theta)|}\tan(\theta)\,d\theta},\end{aligned}$$

   as desired. □

Remark. *Note that, for example, we have* $|\sin(\theta)| = \pm\sin(\theta)$ *because* $\sin(\theta)$ *is positive on* $0 < \theta < \pi$ *and negative on* $\pi < \theta < 2\pi$. *We cannot always say* $|\sin(\theta)| = \sin(\theta)$ *or* $|\sin(\theta)| = -\sin(\theta)$ *unless we were specified a domain for* θ. *Because this question concerns indefinite integrals, we have no information regarding any domain on* θ. *So that is why I am leaving the absolute values alone here. But if you were tempted to assume* $|\sin(\theta)| = \sin(\theta)$ *anyway and simplify the expression further, I will still give full credit when grading your quiz.*

2. Write the following trigonometric expressions in terms of x, given the specified trigonometric substitution.

(a) If $x = 2\sin(\theta)$, write $\frac{1}{2}\theta - \frac{1}{2}\cos(\theta)$ in terms of x.

Solution. Note that $x = 2\sin(\theta)$ implies $\theta = \sin^{-1}(\frac{x}{2})$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So we have

$$\begin{aligned}\frac{1}{2}\theta - \frac{1}{2}\cos(\theta) &= \frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\\ &= \frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\frac{\sqrt{4-x^2}}{2}\\ &= \boxed{\frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{4}\sqrt{4-x^2}},\end{aligned}$$

as desired. □

(b) If $x = 3\tan(\theta)$, write $\ln(|\sec(\theta) + \tan(\theta)|)$ in terms of x.

Solution. Note that $x = 3\tan(\theta)$ implies $\theta = \tan^{-1}(\frac{x}{3})$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So we have

$$\begin{aligned}\ln(|\sec(\theta) + \tan(\theta)|) &= \ln\left(\left|\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x}{3}\right)\right)\right|\right)\\ &= \ln\left(\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right)\\ &= \boxed{\ln\left(\frac{|\sqrt{x^2+9}+x|}{3}\right)},\end{aligned}$$

as desired. □

(c) If $x = \sec(\theta)$, write $\cos(\theta)\sin(\theta)$ in terms of x.

Solution. Note that $x = \sec(\theta)$ implies $\theta = \sec^{-1}(x)$ for $0 < \theta < \pi$. So we have

$$\begin{aligned}\cos(\theta)\sin(\theta) &= \cos(\sec^{-1}(x))\sin(\sec^{-1}(x))\\ &= \frac{1}{x}\frac{\sqrt{x^2-1}}{x}\\ &= \boxed{\frac{\sqrt{x^2-1}}{x^2}},\end{aligned}$$

as desired. □