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Quiz 8 solutions

1. Set up a definite integral that computes the volume of the solid generated by revolving the region between the functions $f(x) = 4 - x^2$ and $g(x) = 1 + 2x$ around the horizontal axis $y = -6$.

Solution. First, we will need to find the x values at which the graphs of $f(x) = 4 - x^2$ and $g(x) = 1 + 2x$ intersect one another. Setting $f(x) = g(x)$, we obtain

$$4 - x^2 = 1 + 2x,$$

which is algebraically equivalent to the quadratic equation

$$x^2 + 2x - 3 = 0,$$

which has the solutions $x = -1$ and $x = 3$. This means that the definite integral that we are going to construct eventually will be for the interval $-1 \le x \le 3$. Now, we observe $g(x) < f(x)$ on the interval $-1 \le x \le 3$, which implies that the inner radius of the cross sectional washer centered around the horizontal axis $y = -6$ is

$$\begin{aligned} r_i(x) &= g(x) + 6 \\ &= (1 + 2x) + 6 \\ &= 7 + 2x, \end{aligned}$$

and the outer radius is

$$\begin{aligned} r_o(x) &= f(x) + 6 \\ &= (4 - x^2) + 6 \\ &= 10 - x^2. \end{aligned}$$

So the area of the cross sectional washer centered around the horizontal axis $y = -6$ on the interval $-1 \le x \le 3$ is

$$\begin{aligned} A(x) &= \pi(r_o(x))^2 - \pi(r_i(x))^2 \\ &= \pi(10 - x^2)^2 - \pi(7 + 2x)^2 \\ &= \pi((10 - x^2)^2 - (7 + 2x)^2), \end{aligned}$$

and so the definite integral that computes the volume of the solid of revolution is

$$\begin{aligned} V &= \int_{-1}^{3} A(x)\,dx \\ &= \int_{-1}^{3} \pi((10 - x^2)^2 - (7 + 2x)^2)\,dx \\ &= \boxed{\pi \int_{-1}^{3} (10 - x^2)^2 - (7 + 2x)^2\,dx}, \end{aligned}$$

as desired. □

2. Set up a definite integral that computes the volume of the solid generated by revolving the region between the functions $f(x) = 2x - \frac{1}{2}$ and $g(x) = 1 + \frac{1}{2}x$ and the horizontal line $y = 4$ revolved around vertical axis $x = 0$.

Solution. The given functions $f(x) = 2x - \frac{1}{2}$ and $g(x) = 1 + \frac{1}{2}x$ are equivalent to

$$\begin{aligned} f(y) &= \frac{1}{2}y + \frac{1}{4}, \\ g(y) &= 2y - 2, \end{aligned}$$

respectively. First, we will need to find the y values at which the graphs of $f(y) = \frac{1}{2}y + \frac{1}{4}$ and $g(y) = 2y - 2$ intersect one another. Setting $f(y) = g(y)$, we obtain

$$\frac{1}{2}y + \frac{1}{4} = 2y - 2,$$

which is algebraically equivalent to the linear equation

$$6y - 9 = 0,$$

which has the solution $y = \frac{3}{2}$. And recall that we are also given the horizontal line $y = 4$. So the definite integral that we are going to construct eventually will be for the interval $\frac{3}{2} \le y \le 4$. Now, we observe $f(y) < g(y)$ on the interval $\frac{3}{2} \le y \le 4$, which implies that the inner radius of the cross sectional washer centered around the vertical axis $x = 0$ is

$$\begin{aligned} r_i(y) &= f(y) \\ &= \frac{1}{2}y + \frac{1}{4}, \end{aligned}$$

and the outer radius is

$$\begin{aligned} r_o(y) &= g(y) \\ &= 2y - 2. \end{aligned}$$

So the area of the cross sectional washer centered around the vertical axis $x = 0$ on the interval $\frac{3}{2} \le y \le 4$ is

$$\begin{aligned} A(y) &= \pi(r_o(y))^2 - \pi(r_i(y))^2 \\ &= \pi(2y-2)^2 - \pi\left(\frac{1}{2}y + \frac{1}{4}\right)^2 \\ &= \pi\left((2y-2)^2 - \left(\frac{1}{2}y + \frac{1}{4}\right)^2\right), \end{aligned}$$

and so the definite integral that computes the volume of the solid of revolution is

$$\begin{aligned} V &= \int_{\frac{3}{2}}^{4} A(y)\,dy \\ &= \int_{\frac{3}{2}}^{4} \pi\left((2y-2)^2 - \left(\frac{1}{2}y + \frac{1}{4}\right)^2\right) dy \\ &= \boxed{\pi \int_{\frac{3}{2}}^{4} (2y-2)^2 - \left(\frac{1}{2}y + \frac{1}{4}\right)^2 dy}, \end{aligned}$$

as desired.

□