

6.6 Hyperbolic Functions

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the “hyperbolic trigonometric functions” as there are many, many connections between them and the standard trigonometric functions. Figure 6.6.1 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2 + y^2 = 1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2 - y^2 = 1$.

We begin with their definition.

Definition 6.6.1 Hyperbolic Functions

- | | |
|--|--|
| 1. $\cosh x = \frac{e^x + e^{-x}}{2}$ | 4. $\operatorname{sech} x = \frac{1}{\cosh x}$ |
| 2. $\sinh x = \frac{e^x - e^{-x}}{2}$ | 5. $\operatorname{csch} x = \frac{1}{\sinh x}$ |
| 3. $\tanh x = \frac{\sinh x}{\cosh x}$ | 6. $\operatorname{coth} x = \frac{\cosh x}{\sinh x}$ |

These hyperbolic functions are graphed in Figure 6.6.2. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets “large,” $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty, \infty)$, whereas both $\operatorname{coth} x$ and $\operatorname{csch} x$ have vertical asymptotes at $x = 0$. Also note the ranges of these functions, especially $\tanh x$: as $x \rightarrow \infty$, both $\sinh x$ and $\cosh x$ approach $e^x/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

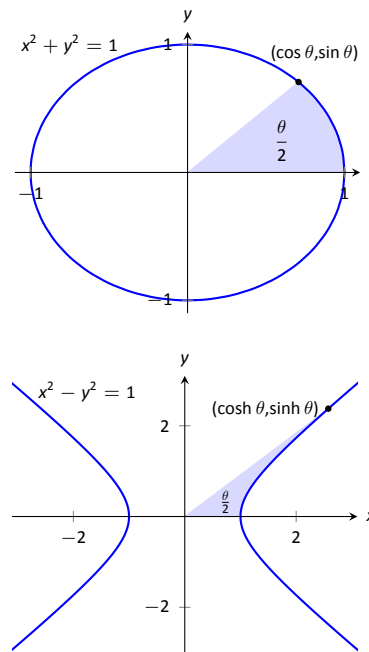


Figure 6.6.1: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

Pronunciation Note:

“cosh” rhymes with “gosh,”
 “sinh” rhymes with “pinch,” and
 “tanh” rhymes with “ranch.”

Notes:

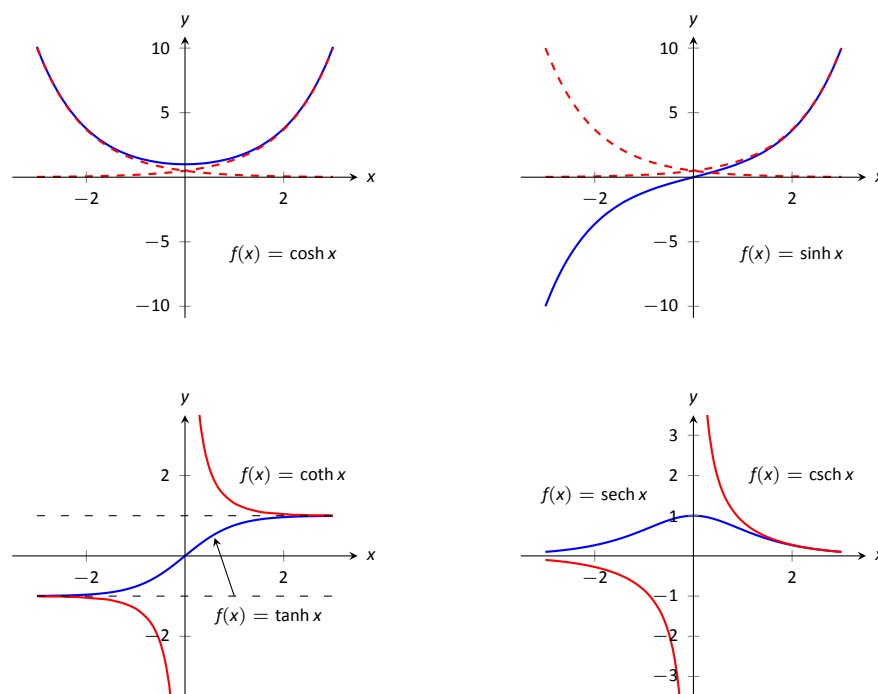


Figure 6.6.2: Graphs of the hyperbolic functions.

Example 6.6.1 Exploring properties of hyperbolic functions

Use Definition 6.6.1 to rewrite the following expressions.

- | | |
|--|----------------------------|
| 1. $\cosh^2 x - \sinh^2 x$ | 4. $\frac{d}{dx}(\cosh x)$ |
| 2. $\tanh^2 x + \operatorname{sech}^2 x$ | 5. $\frac{d}{dx}(\sinh x)$ |
| 3. $2 \cosh x \sinh x$ | 6. $\frac{d}{dx}(\tanh x)$ |

SOLUTION

$$\begin{aligned}
 1. \quad \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\
 &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\
 &= \frac{4}{4} = 1.
 \end{aligned}$$

$$\text{So } \cosh^2 x - \sinh^2 x = 1.$$

Notes:

$$\begin{aligned}
 2. \quad \tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x} \\
 &= \frac{\sinh^2 x + 1}{\cosh^2 x} \quad \text{Now use identity from \#1.} \\
 &= \frac{\cosh^2 x}{\cosh^2 x} = 1.
 \end{aligned}$$

$$\text{So } \tanh^2 x + \operatorname{sech}^2 x = 1.$$

$$\begin{aligned}
 3. \quad 2 \cosh x \sinh x &= 2 \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^x - e^{-x}}{2} \right) \\
 &= 2 \cdot \frac{e^{2x} - e^{-2x}}{4} \\
 &= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).
 \end{aligned}$$

$$\text{Thus } 2 \cosh x \sinh x = \sinh(2x).$$

$$\begin{aligned}
 4. \quad \frac{d}{dx}(\cosh x) &= \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right) \\
 &= \frac{e^x - e^{-x}}{2} \\
 &= \sinh x.
 \end{aligned}$$

$$\text{So } \frac{d}{dx}(\cosh x) = \sinh x.$$

$$\begin{aligned}
 5. \quad \frac{d}{dx}(\sinh x) &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) \\
 &= \frac{e^x + e^{-x}}{2} \\
 &= \cosh x.
 \end{aligned}$$

$$\text{So } \frac{d}{dx}(\sinh x) = \cosh x.$$

$$\begin{aligned}
 6. \quad \frac{d}{dx}(\tanh x) &= \frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right) \\
 &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\
 &= \frac{1}{\cosh^2 x} \\
 &= \operatorname{sech}^2 x.
 \end{aligned}$$

$$\text{So } \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$$

Notes:

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 6.6.1.

Key Idea 6.6.1 Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \operatorname{sech}^2 x = 1$
3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2 \sinh x \cosh x$
6. $\cosh^2 x = \frac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \frac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}(\cosh x) = \sinh x$
2. $\frac{d}{dx}(\sinh x) = \cosh x$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
4. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
5. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
6. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Integrals

1. $\int \cosh x \, dx = \sinh x + C$
2. $\int \sinh x \, dx = \cosh x + C$
3. $\int \tanh x \, dx = \ln(\cosh x) + C$
4. $\int \coth x \, dx = \ln |\sinh x| + C$

We practice using Key Idea 6.6.1.

Example 6.6.2 Derivatives and integrals of hyperbolic functions

Evaluate the following derivatives and integrals.

1. $\frac{d}{dx}(\cosh 2x)$

3. $\int_0^{\ln 2} \cosh x \, dx$

2. $\int \operatorname{sech}^2(7t - 3) \, dt$

SOLUTION

1. Using the Chain Rule directly, we have $\frac{d}{dx}(\cosh 2x) = 2 \sinh 2x$.

Just to demonstrate that it works, let's also use the Basic Identity found in Key Idea 6.6.1: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\begin{aligned} \frac{d}{dx}(\cosh 2x) &= \frac{d}{dx}(\cosh^2 x + \sinh^2 x) = 2 \cosh x \sinh x + 2 \sinh x \cosh x \\ &= 4 \cosh x \sinh x. \end{aligned}$$

Notes:

Using another Basic Identity, we can see that $4 \cosh x \sinh x = 2 \sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t - 3$ and $du = 7dt$. Applying Key Ideas 6.1.1 and 6.6.1 we have:

$$\int \operatorname{sech}^2(7t - 3) dt = \frac{1}{7} \tanh(7t - 3) + C.$$

- 3.

$$\int_0^{\ln 2} \cosh x dx = \sinh x \Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

We can simplify this last expression as $\sinh x$ is based on exponentials:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}.$$

Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Figure 6.6.3 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 6.6.4.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 6.6.2. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln(x + \sqrt{x^2 + 1})$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

Notes:

Function	Domain	Range	Function	Domain	Range
$\cosh x$	$[0, \infty)$	$[1, \infty)$	$\cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$\sinh x$	$(-\infty, \infty)$	$(-\infty, \infty)$	$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\tanh x$	$(-\infty, \infty)$	$(-1, 1)$	$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\operatorname{sech} x$	$[0, \infty)$	$(0, 1]$	$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$\operatorname{csch} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$	$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\coth x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$	$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Figure 6.6.3: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

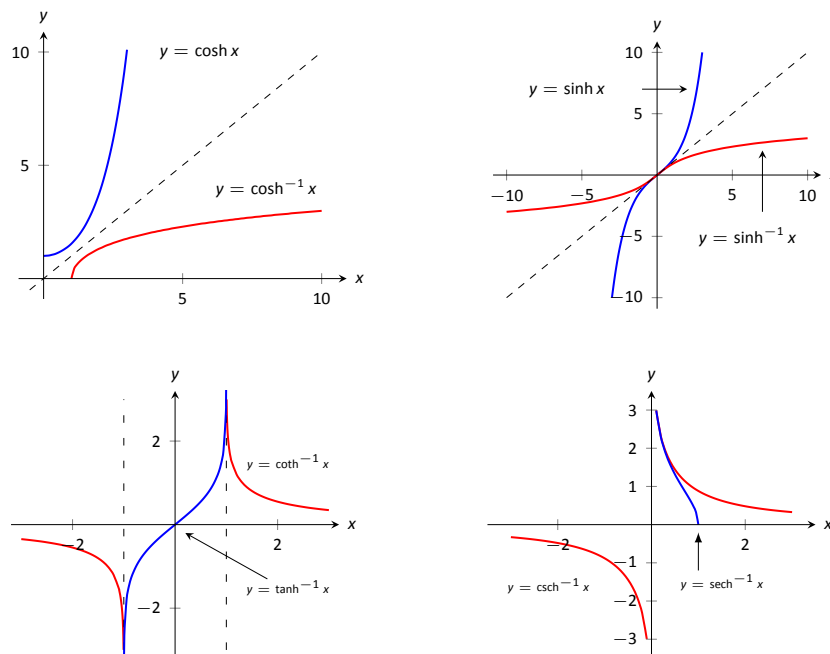


Figure 6.6.4: Graphs of the hyperbolic functions and their inverses.

Key Idea 6.6.2 Logarithmic definitions of Inverse Hyperbolic Functions

- $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$; $x \geq 1$
- $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$; $|x| < 1$
- $\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$; $0 < x \leq 1$
- $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$
- $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$; $|x| > 1$
- $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$; $x \neq 0$

Notes:

The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 6.6.4, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 6.6.2. Again, these latter functions are often more useful than the former. Note how inverse hyperbolic functions can be used to solve integrals we used Trigonometric Substitution to solve in Section 6.4.

Key Idea 6.6.3 Derivatives Involving Inverse Hyperbolic Functions

1. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}; x > 1$
2. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$
3. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}; |x| < 1$
4. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}; 0 < x < 1$
5. $\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1 + x^2}}; x \neq 0$
6. $\frac{d}{dx}(\operatorname{coth}^{-1} x) = \frac{1}{1 - x^2}; |x| > 1$

Key Idea 6.6.4 Integrals Involving Inverse Hyperbolic Functions

1. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C; 0 < a < x = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$
2. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C; a > 0 = \ln \left| x + \sqrt{x^2 + a^2} \right| + C$
3. $\int \frac{1}{a^2 - x^2} dx = \begin{cases} \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + C & x^2 < a^2 \\ \frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C & a^2 < x^2 \end{cases} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$
4. $\int \frac{1}{x\sqrt{a^2 - x^2}} dx = -\frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C; 0 < x < a = \frac{1}{a} \ln \left(\frac{x}{a + \sqrt{a^2 - x^2}} \right) + C$
5. $\int \frac{1}{x\sqrt{x^2 + a^2}} dx = -\frac{1}{a} \operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C; x \neq 0, a > 0 = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right| + C$

We practice using the derivative and integral formulas in the following example.

Notes:

Example 6.6.3 Derivatives and integrals involving inverse hyperbolic functions

Evaluate the following.

1. $\frac{d}{dx} \left[\cosh^{-1} \left(\frac{3x-2}{5} \right) \right]$
2. $\int \frac{1}{x^2-1} dx$
3. $\int \frac{1}{\sqrt{9x^2+10}} dx$

SOLUTION

1. Applying Key Idea 6.6.3 with the Chain Rule gives:

$$\frac{d}{dx} \left[\cosh^{-1} \left(\frac{3x-2}{5} \right) \right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2 - 1}} \cdot \frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives: $\int \frac{1}{x^2-1} dx = \int \frac{-1}{1-x^2} dx$. The second integral can be solved with a direct application of item #3 from Key Idea 6.6.4, with $a = 1$. Thus

$$\begin{aligned} \int \frac{1}{x^2-1} dx &= - \int \frac{1}{1-x^2} dx \\ &= \begin{cases} -\tanh^{-1}(x) + C & x^2 < 1 \\ -\coth^{-1}(x) + C & 1 < x^2 \end{cases} \\ &= -\frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C. \end{aligned} \tag{6.4}$$

We should note that this exact problem was solved at the beginning of Section 6.5. In that example the answer was given as $\frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| + C$. Note that this is equivalent to the answer given in Equation 6.4, as $\ln(a/b) = \ln a - \ln b$.

3. This requires a substitution, then item #2 of Key Idea 6.6.4 can be applied.

Let $u = 3x$, hence $du = 3dx$. We have

$$\int \frac{1}{\sqrt{9x^2+10}} dx = \frac{1}{3} \int \frac{1}{\sqrt{u^2+10}} du.$$

Notes:

Note $a^2 = 10$, hence $a = \sqrt{10}$. Now apply the integral rule.

$$\begin{aligned} &= \frac{1}{3} \sinh^{-1} \left(\frac{3x}{\sqrt{10}} \right) + C \\ &= \frac{1}{3} \ln \left| 3x + \sqrt{9x^2 + 10} \right| + C. \end{aligned}$$

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses. Four Key Ideas were presented, each including quite a bit of information.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 6.6.4 contains perhaps the most useful information. Know the integration forms it helps evaluate and understand how to use the inverse hyperbolic answer and the logarithmic answer.

The next section takes a brief break from demonstrating new integration techniques. It instead demonstrates a technique of evaluating limits that return indeterminate forms. This technique will be useful in Section 6.8, where limits will arise in the evaluation of certain definite integrals.

Notes:

Exercises 6.6

Terms and Concepts

1. In Key Idea 6.6.1, the equation $\int \tanh x \, dx = \ln(\cosh x) + C$ is given. Why is " $\ln |\cosh x|$ " not used – i.e., why are absolute values not necessary?
2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 6.6.1. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?

Problems

In Exercises 3 – 10, verify the given identity using Definition 6.6.1, as done in Example 6.6.1.

3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\cosh^2 x = \frac{\cosh 2x + 1}{2}$
6. $\sinh^2 x = \frac{\cosh 2x - 1}{2}$
7. $\frac{d}{dx} [\operatorname{sech} x] = -\operatorname{sech} x \tanh x$
8. $\frac{d}{dx} [\coth x] = -\operatorname{csch}^2 x$
9. $\int \tanh x \, dx = \ln(\cosh x) + C$
10. $\int \coth x \, dx = \ln |\sinh x| + C$

In Exercises 11 – 22, find the derivative of the given function.

11. $f(x) = \sinh 2x$
12. $f(x) = \cosh^2 x$
13. $f(x) = \tanh(x^2)$
14. $f(x) = \ln(\sinh x)$
15. $f(x) = \sinh x \cosh x$
16. $f(x) = x \sinh x - \cosh x$
17. $f(x) = \operatorname{sech}^{-1}(x^2)$
18. $f(x) = \sinh^{-1}(3x)$
19. $f(x) = \cosh^{-1}(2x^2)$

20. $f(x) = \tanh^{-1}(x + 5)$

21. $f(x) = \tanh^{-1}(\cos x)$

22. $f(x) = \cosh^{-1}(\sec x)$

In Exercises 23 – 28, find the equation of the line tangent to the function at the given x -value.

23. $f(x) = \sinh x$ at $x = 0$

24. $f(x) = \cosh x$ at $x = \ln 2$

25. $f(x) = \tanh x$ at $x = -\ln 3$

26. $f(x) = \operatorname{sech}^2 x$ at $x = \ln 3$

27. $f(x) = \sinh^{-1} x$ at $x = 0$

28. $f(x) = \cosh^{-1} x$ at $x = \sqrt{2}$

In Exercises 29 – 44, evaluate the given indefinite integral.

29. $\int \tanh(2x) \, dx$

30. $\int \cosh(3x - 7) \, dx$

31. $\int \sinh x \cosh x \, dx$

32. $\int x \cosh x \, dx$

33. $\int x \sinh x \, dx$

34. $\int \frac{1}{\sqrt{x^2 + 1}} \, dx$

35. $\int \frac{1}{\sqrt{x^2 - 9}} \, dx$

36. $\int \frac{1}{9 - x^2} \, dx$

37. $\int \frac{2x}{\sqrt{x^4 - 4}} \, dx$

38. $\int \frac{\sqrt{x}}{\sqrt{1 + x^3}} \, dx$

39. $\int \frac{1}{x^4 - 16} \, dx$

40. $\int \frac{1}{x^2 + x} \, dx$

$$41. \int \frac{e^x}{e^{2x} + 1} dx$$

$$42. \int \sinh^{-1} x \, dx$$

$$43. \int \tanh^{-1} x \, dx$$

$$44. \int \operatorname{sech} x \, dx \quad (\text{Hint: multiply by } \frac{\cosh x}{\cosh x}; \text{ set } u = \sinh x.)$$

In Exercises 45 – 48, evaluate the given definite integral.

$$45. \int_{-1}^1 \sinh x \, dx$$

$$46. \int_{-\ln 2}^{\ln 2} \cosh x \, dx$$

$$47. \int_0^1 \tanh^{-1} x \, dx$$

$$48. \int_0^2 \frac{1}{\sqrt{x^2 + 1}} dx$$