 Ryan Ta

Solutions to Homework 7

1. At $t = 0$, a current of 2 amperes flows in an RLC circuit with resistance $R = 4$ ohms, inductance $L = 0.05$ henrys, and capacitance $C = 0.008$ farads. The initial charge on the capacitor is -1 coulomb, and there is no impressed voltage for $t > 0$.

   (a) Find the current flowing in the circuit at $t > 0$.

   Solution. From the information given by the problem statement, we have

$$\begin{aligned} Q(0) &= -1 \text{ coulomb}, \\ Q'(0) = I(0) &= 2 \text{ amperes}, \\ R &= 4 \text{ ohms}, \\ L &= 0.05 \text{ henrys}, \\ C &= 0.008 \text{ farads}, \\ E(t) &= 0. \end{aligned}$$

So the differential equation for the RLC circuit

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

becomes

$$0.05Q'' + 4Q' + \frac{1}{0.008}Q = 0.$$

By substituting $Q = e^{rt}$ and its first and second derivatives $Q' = re^{rt}, Q'' = r^2e^{rt}$ and recognizing $e^{rt} \neq 0$, we obtain the auxilary equation

$$0.05r^2 + 4r + \frac{1}{0.008} = 0,$$

which has complex roots $r = -40 \pm 30i$. So our general solution is

$$\begin{aligned} Q(t) &= c_1e^{(-40-30i)t} + c_2e^{(-40+30i)t} \\ &= e^{-40t}(c_1e^{-30it} + c_2e^{30it}) \\ &= e^{-40t}(c_1\cos(30t) + c_2\sin(30t)) \\ &= c_1e^{-40t}\cos(30t) + c_2e^{-40t}\sin(30t), \end{aligned}$$

where c_1, c_2 are arbitrary constants. Also, the first derivative of $Q(t)$ is

$$\begin{aligned} Q'(t) &= c_1(e^{-40t}\cos(30t))' + c_2(e^{-40t}\sin(30t))' \\ &= c_1(-40e^{-40t}\cos(30t) - 30e^{-40t}\sin(30t)) \\ &\quad + c_2(-40e^{-40t}\sin(30t) + 30e^{-40t}\cos(30t)) \\ &= (-40c_1 + 30c_2)e^{-40t}\cos(30t) \\ &\quad + (-30c_1 - 40c_2)e^{-40t}\sin(30t). \end{aligned}$$

Now, $Q(0) = -1$ implies $c_1 = -1$, and $Q'(0) = 2$ with $c_1 = -1$ imply $c_2 = -\frac{19}{15}$. Therefore, the charge for all $t > 0$ is

$$\begin{aligned} Q(t) &= c_1 e^{-40t}\cos(30t) + c_2 e^{-40t}\sin(30t) \\ &= -e^{-40t}\cos(30t) - \frac{19}{15}e^{-40t}\sin(30t), \end{aligned}$$

and the current flowing in the circuit for all $t > 0$ is

$$\begin{aligned} I(t) &= Q'(t) \\ &= (-40c_1 + 30c_2)e^{-40t}\cos(30t) \\ &\quad + (-30c_1 - 40c_2)e^{-40t}\sin(30t) \\ &= \left(-40(-1) + 30\left(-\frac{19}{15}\right)\right)e^{-40t}\cos(30t) \\ &\quad + \left(-30(-1) - 40\left(-\frac{19}{15}\right)\right)e^{-40t}\sin(30t) \\ &= \boxed{2e^{-40t}\cos(30t) + \frac{242}{3}e^{-40t}\sin(30t)}. \end{aligned}$$

□

(b) Show that the current is a transient solution.

Solution. To show that the current is a transient solution for the case $E(t) = 0$, we need to show

$$\begin{aligned} \lim_{t\to\infty} Q(t) &= 0, \\ \lim_{t\to\infty} I(t) &= 0. \end{aligned}$$

First, we have

$$\begin{aligned} Q(t) &= -e^{-40t}\cos(30t) - \frac{19}{15}e^{-40t}\sin(30t) \\ &\le e^{-40t} + \frac{19}{15}e^{-40t} \\ &= \frac{34}{15}e^{-40t} \end{aligned}$$

and

$$\begin{aligned} Q(t) &= -e^{-40t}\cos(30t) - \frac{19}{15}e^{-40t}\sin(30t) \\ &\ge -e^{-40t} - \frac{19}{15}e^{-40t} \\ &= -\frac{34}{15}e^{-40t}. \end{aligned}$$

In other words, we have

$$-\frac{34}{15}e^{-40t} \le Q(t) \le \frac{34}{15}e^{-40t}.$$

By the Squeeze Theorem, we conclude $\lim_{t\to\infty} Q(t) = 0$. Next, we have

$$\begin{aligned} I(t) &= 2e^{-40t}\cos(30t) + \frac{242}{3}e^{-40t}\sin(30t) \\ &\leq 2e^{-40t} + \frac{242}{3}e^{-40t} \\ &= \frac{248}{3}e^{-40t} \end{aligned}$$

and

$$\begin{aligned} I(t) &= 2e^{-40t}\cos(30t) + \frac{242}{3}e^{-40t}\sin(30t) \\ &\geq -2e^{-40t} - \frac{242}{3}e^{-40t} \\ &= -\frac{248}{3}e^{-40t}. \end{aligned}$$

In other words, we have

$$-\frac{248}{3}e^{-40t} \leq I(t) \leq \frac{248}{3}e^{-40t}.$$

By the Squeeze Theorem, we conclude $\lim_{t\to\infty} I(t) = 0$. □

2. Consider an RLC circuit with resistance $R = 2$ ohms, inductance $L = 0.1$ henrys, and capacitance $C = 0.01$ farads. The initial charge and current is 1 coulomb and 10 amperes, respectively. Assume that there is an impressed voltage of $3\cos(50t) - 6\sin(50t)$.

   (a) Find the particular solution of the current, in other words, steady-state solution of the current.

   Solution. From the information given by the problem statement, we have

$$\begin{aligned} Q(0) &= 1 \text{ coulomb}, \\ Q'(0) = I(0) &= 10 \text{ amperes}, \\ R &= 2 \text{ ohms}, \\ L &= 0.1 \text{ henrys}, \\ C &= 0.01 \text{ farads}, \\ E(t) &= 3\cos(50t) - 6\sin(50t). \end{aligned}$$

   So the differential equation for the RLC circuit

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

   becomes

$$0.1Q'' + 2Q' + \frac{1}{0.01}Q = 0.$$

   Let Q_p be the particular solution of the form

$$Q_p(t) = A\cos(50t) + B\sin(50t),$$

whose first and second derivatives are

$$Q_p'(t) = -50A\sin(50t) + 50B\cos(50t),$$
$$Q_p''(t) = -2500A\cos(50t) - 2500B\sin(50t).$$

By substituting these expressions into our differential equation for the RLC circuit, we obtain

$$\begin{aligned}0.1[&-2500A\cos(50t) - 2500B\sin(50t)]\\ &+2[-50A\sin(50t) + 50B\cos(50t)]\\ &\quad+\frac{1}{0.01}[A\cos(50t) + B\sin(50t)] = 3\cos(50t) - 6\sin(50t),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&-250A\cos(50t) - 250B\sin(50t)\\ &-100A\sin(50t) + 100B\cos(50t)\\ &+100A\cos(50t) + 100B\sin(50t) = 3\cos(50t) - 6\sin(50t),\end{aligned}$$

which is equivalent to

$$(-150A + 100B)\cos(50t) + (-100A - 150B)\sin(50t) = 3\cos(50t) - 6\sin(50t),$$

from which we equate the terms to obtain

$$-150A + 100B = 3,$$
$$-100A - 150B = -6,$$

from which we can simultaneously solve to find $A = \frac{3}{650}$, $B = \frac{12}{325}$. Therefore, the particular solution of the current for all $t > 0$ is

$$\begin{aligned}I_p(t) &= Q_p'(t)\\ &= -50A\sin(50t) + 50B\cos(50t)\\ &= -50\left(\frac{3}{650}\right)\sin(50t) + 50\left(\frac{12}{325}\right)\cos(50t)\\ &= -\frac{3}{13}\sin(50t) + \frac{24}{13}\cos(50t)\\ &= \boxed{\frac{24}{13}\cos(50t) - \frac{3}{13}\sin(50t)}.\end{aligned}$$

□

(b) Does the initial charge and current affect the steady-state current?

Solution. No, because the steady-state current—a.k.a. the particular solution of the current—does not depend on the values of $Q(0)$ and $I(0) = Q'(0)$. □

3. Consider the same RLC circuit as in problem 2, with impressed voltage of $6\cos(10t) - 3\sin(10t)$.

(a) Find the particular solution of the current, in other words, steady-state solution of the current.

Solution. From the information given by the statements of Problems 2 and 3, we have

$$\begin{aligned} Q(0) &= 1 \text{ coulomb}, \\ Q'(0) = I(0) &= 10 \text{ amperes}, \\ R &= 2 \text{ ohms}, \\ L &= 0.1 \text{ henrys}, \\ C &= 0.01 \text{ farads}, \\ E(t) &= 6\cos(10t) - 3\sin(10t). \end{aligned}$$

So the differential equation for the RLC circuit

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

becomes

$$0.1Q'' + 2Q' + \frac{1}{0.01}Q = 0.$$

Let Q_p be the particular solution of the form

$$Q_p(t) = A\cos(10t) + B\sin(10t),$$

whose first and second derivatives are

$$\begin{aligned} Q_p'(t) &= -10A\sin(10t) + 10B\cos(10t), \\ Q_p''(t) &= -100A\cos(10t) - 100B\sin(10t). \end{aligned}$$

By substituting these expressions into our differential equation for the RLC circuit, we obtain

$$\begin{aligned} &0.1[-100A\cos(10t) - 100B\sin(10t)] \\ &\quad +2[-10A\sin(10t) + 10B\cos(10t)] \\ &\qquad +\frac{1}{0.01}[A\cos(10t) + B\sin(10t)] = 6\cos(10t) - 3\sin(10t), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &-10A\cos(50t) - 10B\sin(50t) \\ &-20A\sin(50t) + 20B\cos(50t) \\ &+100A\cos(50t) + 100B\sin(50t) = 6\cos(10t) - 3\sin(10t), \end{aligned}$$

which is equivalent to

$$(90A + 20B)\cos(10t) + (-20A + 90B)\sin(10t) = 6\cos(10t) - 3\sin(10t),$$

from which we equate the terms to obtain

$$\begin{aligned} 90A + 20B &= 6, \\ -20A + 90B &= -3, \end{aligned}$$

from which we can simultaneously solve to find $A = \frac{6}{85}$, $B = -\frac{3}{170}$. Therefore, the particular solution of the current for all $t > 0$ is

$$\begin{aligned}
I_p(t) &= Q_p'(t) \\
&= -10A\sin(10t) + 10B\cos(10t) \\
&= -10\left(\frac{6}{85}\right)\sin(10t) + 10\left(-\frac{3}{170}\right)\cos(10t) \\
&= -\frac{12}{17}\sin(10t) - \frac{3}{17}\cos(10t) \\
&= \boxed{-\frac{3}{17}\cos(10t) - \frac{12}{17}\sin(10t)}.
\end{aligned}$$

□

(b) Compare the amplitude of the steady-state current between problem 2 and 3. Which is larger?

Solution. In general, we can express the steady-state current as a single sine wave:

$$\begin{aligned}
I_p(t) &= \alpha\sin(kt + \beta) \\
&= \alpha[\sin(kt)\cos\beta + \cos(kt)\sin\beta] \\
&= \alpha\sin(kt)\cos\beta + \alpha\cos(kt)\sin\beta \\
&= [\alpha\sin\beta]\cos(kt) + [\alpha\cos\beta]\sin(kt),
\end{aligned}$$

where α denotes the amplitude of the steady-state current.
For Problem 2: We have

$$[\alpha\sin\beta]\cos(50t) + [\alpha\cos\beta]\sin(50t) = I_p(t) = \frac{24}{13}\cos(50t) - \frac{3}{13}\sin(50t),$$

from which we equate the terms to obtain

$$\begin{aligned}
\alpha\sin\beta &= \frac{24}{13}, \\
\alpha\cos\beta &= -\frac{3}{13}.
\end{aligned}$$

So we have

$$\begin{aligned}
\alpha^2 &= \alpha^2(\sin^2\beta + \cos^2\beta) \\
&= (\alpha\sin\beta)^2 + (\alpha\cos\beta)^2 \\
&= \left(\frac{24}{13}\right)^2 + \left(-\frac{3}{13}\right)^2 \\
&= \frac{45}{13},
\end{aligned}$$

from which we conclude that our amplitude is $\alpha = \sqrt{\frac{45}{13}} \approx 1.86$.

For Problem 3: We have

$$[\alpha \sin\beta]\cos(10t) + [\alpha\cos\beta]\sin(10t) = I_p(t) = \frac{3}{17}\cos(10t) - \frac{12}{17}\sin(10t),$$

from which we equate the terms to obtain

$$\begin{aligned}\alpha\sin\beta &= \frac{3}{17},\\ \alpha\cos\beta &= -\frac{12}{17}.\end{aligned}$$

So we have

$$\begin{aligned}\alpha^2 &= \alpha^2(\sin^2\beta + \cos^2\beta)\\ &= (\alpha\sin\beta)^2 + (\alpha\cos\beta)^2\\ &= \left(\frac{3}{17}\right)^2 + \left(-\frac{12}{17}\right)^2\\ &= \frac{9}{17},\end{aligned}$$

from which we conclude that our amplitude is $\alpha = \sqrt{\frac{9}{17}} \approx 0.73$.

Therefore, we conclude that the amplitude of the steady-state current from Problem 2 is larger than that from Problem 3. □

4. Find the Laplace transformation of the following functions and the range of s that it is valid.

   (a) t^2

   Solution. We already know from your professor's notes for the lecture on May 18, 2022 that the Laplace transform of t is $\mathcal{L}(t) = \frac{1}{s^2}$ for all $s > 0$. Using this, we find that the Laplace transform of t^2 is

$$\begin{aligned}\mathcal{L}(t^2) &= \int_0^\infty e^{-st}t^2\,dt\\ &= -\frac{1}{s}t^2e^{-st}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}\,dt\\ &= -\frac{1}{s}(0 - 0^2e^{-s(0)}) + \frac{2}{s}\mathcal{L}(t)\\ &= \frac{2}{s}\mathcal{L}(t)\\ &= \frac{2}{s}\frac{1}{s^2}\\ &= \boxed{\frac{2}{s^3}}\end{aligned}$$

   for all $s > 0$. □

   (b) $e^{-2t} + 3e^t$

Solution. We already know from your professor's notes for the lecture on May 18, 2022 that the Laplace transform of t is $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ for all $s > a$. In particular, we have

$$\begin{aligned}\mathcal{L}(e^{-2t}) &= \frac{1}{s+2}, \quad s > -2,\\ \mathcal{L}(e^{t}) &= \frac{1}{s-1}, \quad s > 1.\end{aligned}$$

Using these facts and the linearity of the Laplace transform, we find that the Laplace transform of $e^{-2t} + 3e^t$ is

$$\begin{aligned}\mathcal{L}(e^{-2t} + 3e^t) &= \mathcal{L}(e^{-2t}) + 3\mathcal{L}(e^t)\\ &= \frac{1}{s+2} + 3\frac{1}{s-1}\\ &= \boxed{\frac{4s+5}{(s-1)(s+2)}}\end{aligned}$$

for all $s > 1$. □

5. Find the Laplace transformation of the following functions and the range of s that it is valid.

   (a) $\sin\left(t + \frac{\pi}{4}\right)$

   Solution. We can rewrite

$$\begin{aligned}\sin\left(t + \frac{\pi}{4}\right) &= \sin t \cos\left(\frac{\pi}{4}\right) + \cos t \sin\left(\frac{\pi}{4}\right)\\ &= \sin t \frac{1}{\sqrt{2}} + \cos t \frac{1}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}(\sin t + \cos t).\end{aligned}$$

   Also, we already know from your professor's notes for the lecture on May 18, 2022 that the Laplace transforms of $\sin(\omega t)$ and $\cos(\omega t)$ are

$$\begin{aligned}\mathcal{L}(\sin(\omega t)) &= \frac{\omega}{s^2+\omega^2}, \quad s > 0,\\ \mathcal{L}(\cos(\omega t)) &= \frac{s}{s^2+\omega^2}, \quad s > 0.\end{aligned}$$

   In particular, we have

$$\begin{aligned}\mathcal{L}(\sin t) &= \frac{1}{s^2+1}, \quad s > 0,\\ \mathcal{L}(\cos t) &= \frac{s}{s^2+1}, \quad s > 0.\end{aligned}$$

By the linearity of the Laplace transform, we find that the Laplace transform of $\sin\left(t + \frac{\pi}{4}\right)$ is

$$\begin{aligned}
\mathcal{L}\left(\sin\left(t + \frac{\pi}{4}\right)\right) &= \mathcal{L}\left(\frac{1}{\sqrt{2}}(\sin t + \cos t)\right) \\
&= \frac{1}{\sqrt{2}}(\mathcal{L}(\sin t) + \mathcal{L}(\cos t)) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{s^2+1} + \frac{s}{s^2+1}\right) \\
&= \frac{1}{\sqrt{2}}\frac{1+s}{s^2+1} \\
&= \boxed{\frac{s+1}{\sqrt{2}(s^2+1)}}
\end{aligned}$$

for all $s > 0$. □

(b) $\sin^2(t)$

Solution. We already know from your professor's notes for the lecture on May 18, 2022 that the Laplace transforms of 1 and $\cos(\omega t)$ are

$$\begin{aligned}
\mathcal{L}(1) &= \frac{1}{s}, \quad s > 0, \\
\mathcal{L}(\cos(\omega t)) &= \frac{s}{s^2+\omega^2}, \quad s > 0.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\mathcal{L}(1) &= \frac{1}{s}, \quad s > 0, \\
\mathcal{L}(\cos(2t)) &= \frac{s}{s^2+4}, \quad s > 0.
\end{aligned}$$

By the linearity of the Laplace transform, we find that the Laplace transform of $\sin^2 t$ is

$$\begin{aligned}
\mathcal{L}(\sin^2 t) &= \mathcal{L}\left(\frac{1}{2}(1 - \cos(2t))\right) \\
&= \frac{1}{2}(\mathcal{L}(1) - \mathcal{L}(\cos(2t))) \\
&= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\
&= \frac{1}{2}\frac{4}{s(s^2+4)} \\
&= \boxed{\frac{2}{s(s^2+4)}}
\end{aligned}$$

for all $s > 0$. □