MATH 045 / EE 020A discussion Ryan Ta
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Solutions to Homework 9

Use the Laplace transform to solve the initial value problems.

1. $y'' + y' - 2y = -4$, $y(0) = 2$, $y'(0) = 3$

Solution. By using

$$\begin{aligned}\mathcal{L}(y(t)) &= Y(s),\\ \mathcal{L}(y'(t)) &= sY(s) - y(0) = sY(s) - 2,\\ \mathcal{L}(y''(t)) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 3,\\ \mathcal{L}(4) &= \frac{4}{s},\end{aligned}$$

we can take the Laplace transform of both sides of the differential equation, writing

$$\mathcal{L}(y'') + \mathcal{L}(y') - 2\mathcal{L}(y) = \mathcal{L}(-4),$$

to obtain

$$[s^2Y(s) - 2s - 3] + [sY(s) - 2] - 2Y(s) = -\frac{4}{s},$$

which is equivalent to

$$(s^2 + s - 2)Y(s) - 2s - 5 = -\frac{4}{s}.$$

which is equivalent to

$$\begin{aligned}Y(s) &= \frac{1}{s^2 + s - 2}\left(2s + 5 - \frac{4}{s}\right)\\ &= \frac{1}{(s-1)(s+2)}\frac{2s^2 + 5s - 4}{s}\\ &= \frac{2s^2 + 5s - 4}{(s-1)s(s+2)}\\ &= -\frac{1}{s+2} + \frac{2}{s} + \frac{1}{s-1}.\end{aligned}$$

Finally, the inverse Laplace transform of $Y(s)$ is

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(-\frac{1}{s+2} + \frac{2}{s} + \frac{1}{s-1}\right)\\ &= -\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\\ &= \boxed{-e^{-2t} + 2 + e^{t}}.\end{aligned}$$

□

2. $y'' + 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$

Solution. By using

$$\begin{aligned}
\mathcal{L}(y(t)) &= Y(s), \\
\mathcal{L}(y'(t)) &= sY(s) - y(0) = sY(s), \\
\mathcal{L}(y''(t)) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1, \\
\mathcal{L}(e^t) &= \frac{1}{s-1},
\end{aligned}$$

we can take the Laplace transform of both sides of the differential equation, writing

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(e^t),$$

to obtain

$$[s^2Y(s) - 1] + 3sY(s) + 2Y(s) = \frac{1}{s-1},$$

which is equivalent to

$$(s^2 + 3s + 2)Y(s) - 1 = \frac{1}{s-1}.$$

which is equivalent to

$$\begin{aligned}
Y(s) &= \frac{1}{s^2 + 3s + 2}\left(1 + \frac{1}{s-1}\right) \\
&= \frac{1}{(s+1)(s+2)}\frac{s}{s-1} \\
&= \frac{s}{(s-1)(s+1)(s+2)} \\
&= \frac{1}{6}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s+2}.
\end{aligned}$$

Finally, the inverse Laplace transform of $Y(s)$ is

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{1}{6}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s+2}\right) \\
&= \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \\
&= \boxed{\frac{1}{6}e^t + \frac{1}{2}e^{-t} - \frac{2}{3}e^{-2t}}.
\end{aligned}$$

□

3. $y'' + y = \sin(2t)$, $y(0) = 0$, $y'(0) = 1$

Solution. By using

$$\begin{aligned}
\mathcal{L}(y(t)) &= Y(s), \\
\mathcal{L}(y''(t)) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1, \\
\mathcal{L}(\sin(2t)) &= \frac{2}{s^2 + 4},
\end{aligned}$$

we can take the Laplace transform of both sides of the differential equation, writing

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sin(2t)),$$

to obtain

$$[s^2Y(s) + 1] + Y(s) = \frac{2}{s^2+4},$$

which is equivalent to

$$(s^2 + 1)Y(s) - 1 = \frac{2}{s^2+4}.$$

which is equivalent to

$$\begin{aligned}
Y(s) &= \frac{1}{s^2+1}\left(1 + \frac{2}{s^2+4}\right) \\
&= \frac{1}{s^2+1}\frac{s^2+6}{s^2+4} \\
&= \frac{s^2+6}{(s^2+1)(s^2+4)} \\
&= \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4} \\
&= \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}.
\end{aligned}$$

Finally, the inverse Laplace transform of $Y(s)$ is

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}\right) \\
&= \frac{5}{3}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) \\
&= \boxed{\frac{5}{3}\sin t - \frac{1}{3}\sin(2t)}.
\end{aligned}$$

□

Express the given function f in terms of unit step functions and find the Laplace transform. Use change of variables. Do not use Theorem 8.4.1.

4. $f(t) = \begin{cases} t^2 + 2 & 0 \le t < 1, \\ t & 1 \le t \end{cases}$

Solution. The function f in terms of unit step functions is

$$\begin{aligned}
f(t) &= (t^2 + 2) + u(t-1)(t - (t^2 + 2)) \\
&= t^2 + 2 + u(t-1)(-t^2 + t - 2).
\end{aligned}$$

We can write

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^1 e^{-st}(t^2+2)\, dt + \int_1^\infty e^{-st} t\, dt \\
&= \int_0^1 e^{-st}(t^2+2)\, dt + \int_1^\infty e^{-st} t\, dt \\
&\quad + \int_1^\infty e^{-st}(t^2+2)\, dt - \int_1^\infty e^{-st}(t^2+2)\, dt \\
&= \int_0^\infty e^{-st}(t^2+2)\, dt + \int_1^\infty e^{-st}(t-(t^2+2))\, dt \\
&= \mathcal{L}(t^2+2) + \int_1^\infty e^{-st}(-t^2+t-2)\, dt
\end{aligned}
$$

Now, if we substitute $\tau = t - 1$ and $d\tau = dt$ into the second term of our final expression of $\mathcal{L}(f(t))$, we obtain

$$
\begin{aligned}
\int_1^\infty e^{-st}(-t^2+t-2)\, dt &= \int_0^\infty e^{-s(\tau+1)}(-(\tau+1)^2+(\tau+1)-2)\, d\tau \\
&= e^{-s}\int_0^\infty e^{-s\tau}(-(\tau+1)^2+(\tau+1)-2)\, d\tau \\
&= e^{-s}\int_0^\infty e^{-st}(-(t+1)^2+(t+1)-2)\, dt \\
&= e^{-s}\mathcal{L}(-(t+1)^2+(t+1)-2) \\
&= e^{-s}(-\mathcal{L}((t+1)^2)+\mathcal{L}(t+1)-2\mathcal{L}(1)) \\
&= e^{-s}(-\mathcal{L}(t^2+2t+1)+\mathcal{L}(t+1)-2\mathcal{L}(1) \\
&= e^{-s}(-\mathcal{L}(t^2)-2\mathcal{L}(t)-\mathcal{L}(1)+\mathcal{L}(t)+\mathcal{L}(1)-2\mathcal{L}(1)) \\
&= e^{-s}(-\mathcal{L}(t^2)-\mathcal{L}(t)-2\mathcal{L}(1)) \\
&= e^{-s}\left(-\frac{2}{s^3}-\frac{1}{s^2}-\frac{2}{s}\right) \\
&= -\frac{e^{-s}(2+s+2s^2)}{s^3}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathcal{L}(t^2+2) &= \mathcal{L}(t^2)+2\mathcal{L}(1) \\
&= \frac{2}{s^3}+\frac{2}{s} \\
&= \frac{2(1+s^2)}{s^3}.
\end{aligned}
$$

So the Laplace transform of $f(t)$ is

$$\begin{aligned}\mathcal{L}(f(t)) &= \mathcal{L}(t^2+2) + \int_1^\infty e^{-st}(-t^2+t-2)\,dt \\ &= \frac{2(1+s^2)}{s^3} - \frac{e^{-s}(2+s+2s^2)}{s^3} \\ &= \boxed{\frac{2(1+s^2)-e^{-s}(2+s+2s^2)}{s^3}}.\end{aligned}$$

□

5. $f(t) = \begin{cases} e^{-2t} & 0 \le t < 2, \\ e^{3t} & 2 \le t \end{cases}$

Solution. The function f in terms of unit step functions is

$$f(t) = e^{-2t} + u(t-2)(e^{3t} - e^{-2t}).$$

We can write

$$\begin{aligned}\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^2 e^{-st}e^{-2t}\,dt + \int_2^\infty e^{-st}e^{3t}\,dt \\ &= \int_0^2 e^{-st}e^{-2t}\,dt + \int_2^\infty e^{-st}e^{3t}\,dt \\ &\quad + \int_2^\infty e^{-st}e^{-2t}\,dt - \int_2^\infty e^{-st}e^{-2t}\,dt \\ &= \int_0^\infty e^{-st}e^{-2t}\,dt + \int_2^\infty e^{-st}(e^{3t}-e^{-2t})\,dt \\ &= \mathcal{L}(e^{-2t}) + \int_2^\infty e^{-st}(e^{3t}-e^{-2t})\,dt.\end{aligned}$$

Now, if we substitute $\tau = t-2$ and $d\tau = dt$ into the second term of our final expression of $\mathcal{L}(f(t))$, we obtain

$$\begin{aligned}\int_1^\infty e^{-st}(e^{3t}-e^{-2t})\,dt &= \int_0^\infty e^{-s(\tau+2)}(e^{3(\tau+2)} - e^{-2(\tau+2)})\,d\tau \\ &= e^{-2s}\int_0^\infty e^{-s\tau}(e^{3(\tau+2)} - e^{-2(\tau+2)})\,d\tau \\ &= e^{-2s}\int_0^\infty e^{-st}(e^{3(t+2)} - e^{-2(t+2)})\,dt \\ &= e^{-2s}\mathcal{L}(e^{3(t+2)} - e^{-2(t+2)}) \\ &= e^{-2s}\mathcal{L}(e^{3t}e^{6} - e^{-2t}e^{-4}) \\ &= e^{-2s}(e^{6}\mathcal{L}(e^{3t}) - e^{-4}\mathcal{L}(e^{-2t})) \\ &= e^{-2(s-3)}\mathcal{L}(e^{3t}) - e^{-2(s+2)}\mathcal{L}(e^{-2t}) \\ &= e^{-2(s-3)}\frac{1}{s-3} - e^{-2(s+2)}\frac{1}{s+2} \\ &= \frac{(s+2)e^{-2(s-3)} - (s-3)e^{-2(s+2)}}{(s-3)(s+2)}.\end{aligned}$$

So the Laplace transform of $f(t)$ is

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(e^{-2t}) + \int_1^\infty e^{-st}(e^{3t} - e^{-2t})\,dt \\
&= \frac{1}{s+2} + \frac{(s+2)e^{-2(s-3)} - (s-3)e^{-2(s+2)}}{(s-3)(s+2)} \\
&= \frac{s-3+(s+2)e^{-2(s-3)} - (s-3)e^{-2(s+2)}}{(s-3)(s+2)} \\
&= \boxed{\frac{(s+2)e^{-2(s-3)} + (s-3)(1-e^{-2(s+2)})}{(s-3)(s+2)}}.
\end{aligned}$$

□

6. $f(t) = \begin{cases} 2\sin t & 0 \le t < \pi, \\ \cos t & \pi \le t \end{cases}$

Solution. The function f in terms of unit step functions is

$$f(t) = 2\sin t + u(t-\pi)(\cos t - 2\sin t).$$

We can write

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\pi e^{-st}(2\sin t)\,dt + \int_\pi^\infty e^{-st}\cos t\,dt \\
&= \int_0^\pi e^{-st}(2\sin t)\,dt + \int_\pi^\infty e^{-st}\cos t\,dt \\
&\quad + \int_\pi^\infty e^{-st}(2\sin t)\,dt - \int_\pi^\infty e^{-st}(2\sin t)\,dt \\
&= \int_0^\infty e^{-st}(2\sin t)\,dt + \int_\pi^\infty e^{-st}(\cos t - 2\sin t)\,dt \\
&= \mathcal{L}(2\sin t) + \int_\pi^\infty e^{-st}(\cos t - 2\sin t)\,dt.
\end{aligned}$$

Now, if we substitute $\tau = t - \pi$ and $d\tau = dt$ into the second term of our final expression of $\mathcal{L}(f(t))$, we obtain

$$\begin{aligned}
\int_{\pi}^{\infty} e^{-st}(\cos t - 2\sin t)\,dt &= \int_{0}^{\infty} e^{-s(\tau+\pi)}(\cos(\tau+\pi) - 2\sin(\tau+\pi))\,d\tau \\
&= e^{-\pi s}\int_{0}^{\infty} e^{-s\tau}(\cos(\tau+\pi) - 2\sin(\tau+\pi))\,d\tau \\
&= e^{-\pi s}\int_{0}^{\infty} e^{-st}(\cos(t+\pi) - 2\sin(t+\pi))\,dt \\
&= e^{-\pi s}\mathcal{L}(\cos(t+\pi) - 2\sin(t+\pi)) \\
&= e^{-\pi s}\mathcal{L}([\cos t\cos\pi - \sin t\sin\pi] \\
&\qquad\qquad - 2[\sin t\cos\pi + \cos t\sin\pi]) \\
&= e^{-\pi s}\mathcal{L}([\cos t\cdot(-1) - \sin t\cdot 0] \\
&\qquad\qquad - 2[\sin t\cdot(-1) + \cos t\cdot 0]) \\
&= e^{-\pi s}\mathcal{L}(-\cos t + 2\sin t) \\
&= e^{-\pi s}(-\mathcal{L}(\cos t) + 2\mathcal{L}(\sin t)) \\
&= e^{-\pi s}\left(-\frac{s}{s^2+1} + 2\frac{1}{s^2+1}\right) \\
&= -\frac{e^{-\pi s}(s-2)}{s^2+1}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathcal{L}(2\sin t) &= 2\mathcal{L}(\sin t) \\
&= 2\frac{1}{s^2+1} \\
&= \frac{2}{s^2+1}.
\end{aligned}$$

So the Laplace transform of $f(t)$ is

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(2\sin t) + \int_{\pi}^{\infty} e^{-st}(\cos t - 2\sin t)\,dt \\
&= \frac{2}{s^2+1} - \frac{e^{-\pi s}(s-2)}{s^2+1} \\
&= \boxed{\frac{2 - e^{-\pi s}(s-2)}{s^2+1}}.
\end{aligned}$$

□

7. $f(t) = \begin{cases} 3 & 0 \le t < 2, \\ -3t+2 & 2 \le t < 4, \\ -4t & 4 \le t \end{cases}$

Solution. The function f in terms of unit step functions is

$$\begin{aligned}
f(t) &= 3 + u(t-2)((-3t+2) - 3) + u(t-4)(-4t - (-3t+2)) \\
&= 3 + u(t-2)(-3t-1) + u(t-4)(-t-2).
\end{aligned}$$

We can write

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^2 e^{-st}(3)\,dt + \int_2^4 e^{-st}(-3t+2)\,dt + \int_4^\infty e^{-st}(-4t)\,dt \\
&= \int_0^2 e^{-st}(3)\,dt + \int_2^4 e^{-st}(-3t+2)\,dt + \int_4^\infty e^{-st}(-4t)\,dt \\
&\quad + \int_2^\infty e^{-st}(3)\,dt - \int_2^4 e^{-st}(3)\,dt - \int_4^\infty e^{-st}(3)\,dt \\
&= \int_0^\infty e^{-st}(3)\,dt + \int_2^4 e^{-st}((-3t+2)-3)\,dt + \int_4^\infty e^{-st}(-4t-3)\,dt \\
&= \mathcal{L}(3) + \int_2^4 e^{-st}(-3t-1)\,dt + \int_4^\infty e^{-st}(-4t-3)\,dt.
\end{aligned}$$

Now, if we substitute $\tau = t-2$ and $d\tau = dt$ into the second and third terms of our final expression of $\mathcal{L}(f(t))$, we obtain

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(3) + \int_2^4 e^{-st}(-3t-1)\,dt + \int_4^\infty e^{-st}(-4t-3)\,dt \\
&= \mathcal{L}(3) + \int_0^2 e^{-s(\tau+2)}(-3(\tau+2)-1)\,d\tau + \int_2^\infty e^{-s(\tau+2)}(-4(\tau+2)-3)\,d\tau \\
&= \mathcal{L}(3) + \int_0^2 e^{-s(\tau+2)}(-3\tau-7)\,d\tau + \int_2^\infty e^{-s(\tau+2)}(-4\tau-11)\,d\tau \\
&= \mathcal{L}(3) + \int_0^2 e^{-s(t+2)}(-3t-7)\,dt + \int_2^\infty e^{-s(t+2)}(-4t-11)\,dt \\
&= \mathcal{L}(3) + \int_0^2 e^{-s(t+2)}(-3t-7)\,dt + \int_2^\infty e^{-s(t+2)}(-4t-11)\,dt \\
&\quad + \int_2^\infty e^{-s(t+2)}(-3t-7)\,dt - \int_2^\infty e^{-s(t+2)}(-3t-7)\,dt \\
&= \mathcal{L}(3) + \int_0^\infty e^{-s(t+2)}(-3t-7)\,dt + \int_2^\infty e^{-s(t+2)}((-4t-11)-(-3t-7))\,dt \\
&= \mathcal{L}(3) + e^{-2s}\int_0^\infty e^{-st}(-3t-7)\,dt + e^{-2s}\int_2^\infty e^{-st}(-t-4)\,dt \\
&= \mathcal{L}(3) + e^{-2s}\mathcal{L}(-3t-7) + e^{-2s}\int_2^\infty e^{-st}(-t-4)\,dt \\
&= \mathcal{L}(3) + e^{-2s}(-3L(t)-7L(1)) + e^{-2s}\int_2^\infty e^{-st}(-t-4)\,dt.
\end{aligned}$$

Now, if we substitute $\tau = t - 2$ into the third term of our final expression of

$$\begin{aligned}
e^{-2s}\int_2^\infty e^{-st}(-t-4)\,dt &= e^{-2s}\int_2^\infty e^{-s(\tau+2)}(-(\tau+2)-4)\,dt \\
&= e^{-2s}\int_0^\infty e^{-s(\tau+2)}(-(\tau+2)-4)\,d\tau \\
&= e^{-2s}e^{-2s}\int_0^\infty e^{-s\tau}(-\tau-6)\,d\tau \\
&= e^{-4s}\int_0^\infty e^{-st}(-t-6)\,dt \\
&= e^{-4s}\mathcal{L}(-t-6)\,dt \\
&= e^{-4s}(-\mathcal{L}(t)-6\mathcal{L}(1)) \\
&= e^{-4s}\left(-\frac{1}{s^2}-\frac{6}{s}\right).
\end{aligned}$$

So the Laplace transform of $f(t)$ is

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(3) + e^{-2s}(-3L(t)-7L(1)) + e^{-2s}\int_2^\infty e^{-st}(-t-4)\,dt \\
&= \frac{3}{s} + e^{-2s}\left(-\frac{3}{s^2}-\frac{7}{s}\right) + e^{-4s}\left(-\frac{1}{s^2}-\frac{6}{s}\right) \\
&= \frac{3}{s} - \frac{e^{-2s}(7s+3)}{s^2} - \frac{e^{-4s}(6s+1)}{s^2} \\
&= \boxed{\frac{3s-e^{-2s}(7s+3)-e^{-4s}(6s+1)}{s^2}}.
\end{aligned}$$

□