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Homework 1 solutions

1. Show that $y = c_1e^x + c_2e^{3x}$ is the general solution of $y'' - 4y' + 3y = 0$. Find the solution to the initial value problem

$$y'' - 4y' + 3y = 0,$$
$$y(0) = 1, y'(0) = 5.$$

Solution. The first and second derivatives of $y = c_1e^x + c_2e^{3x}$ are

$$y' = c_1e^x + 3c_2e^{3x},$$
$$y'' = c_1e^x + 9c_2e^{3x}.$$

So we have

$$\begin{aligned} y'' - 4y' + 3y &= (c_1e^x + 9c_2e^{3x}) - 4(c_1e^x + 3c_2e^{3x}) + 3(c_1e^x + c_2e^{3x}) \\ &= c_1e^x + 9c_2e^{3x} - 4c_1e^x - 12c_2e^{2x} + 3c_1e^x + 3c_2e^{3x} \\ &= (c_1e^x - 4c_1e^x + 3c_1e^x) + (9c_2e^{3x} - 12c_2e^{3x} + 3c_2e^{3x}) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

which means that $y = c_1e^x + c_2e^{3x}$ is the general solution of $y'' - 4y' + 3y = 0$. Next, we consider the initial conditions $y(0) = 1$ and $y'(0) = 5$. We recall

$$y = c_1e^x + c_2e^{3x},$$
$$y' = c_1e^x + 3c_2e^{3x}.$$

At $x = 0$, this becomes

$$1 = c_1 + c_2,$$
$$5 = c_1 + 3c_2,$$

from which we can solve simultaneously to obtain $c_1 = -1$ and $c_2 = 2$. Therefore,

$$y = \boxed{-e^x + 2e^{3x}}$$

is the solution to the initial value problem. □

2. Show that $y = c_1 \sin(2x) + c_2 \cos(2x)$ is the general solution of $y'' + 4y = 0$. Find the solution to the initial value problem

$$y'' + 4y = 0,$$
$$y\left(-\frac{\pi}{8}\right) = 2\sqrt{2}, y\left(\frac{\pi}{8}\right) = \sqrt{2}.$$

Solution. The first and second derivatives of $y = c_1 \sin(2x) + c_2 \cos(2x)$ are

$$y' = 2c_1 \cos(2x) - 2c_2 \sin(2x),$$
$$y'' = -4c_1 \sin(2x) - 4c_2 \cos(2x).$$

So we have

$$\begin{aligned} y'' + 4y &= (-4c_1 \sin(2x) - 4c_2 \cos(2x)) + 4(c_1 \sin(2x) + c_2 \cos(2x)) \\ &= -4c_1 \sin(2x) - 4c_2 \cos(2x) + 4c_1 \sin(2x) + 4c_2 \cos(2x) \\ &= (-4c_1 \sin(2x) + 4c_1 \sin(2x)) + (-4c_2 \cos(2x) + 4c_2 \cos(2x)) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

which means that $y = c_1 \sin(2x) + c_2 \cos(2x)$ is the general solution of $y'' + 4y = 0$. Next, we consider the initial conditions $y(0) = 2\sqrt{2}$ and $y'(0) = \sqrt{2}$. We recall

$$y = c_1 \sin(2x) + c_2 \cos(2x),$$
$$y' = 2c_1 \cos(2x) - 2c_2 \sin(2x).$$

At $x = -\frac{\pi}{8}$, this becomes

$$2\sqrt{2} = c_1 \sin\left(-\frac{\pi}{4}\right) + c_2 \cos\left(-\frac{\pi}{4}\right),$$

and at $x = \frac{\pi}{8}$, this becomes

$$\sqrt{2} = c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \cos\left(\frac{\pi}{4}\right).$$

In other words, we obtain the system

$$\begin{aligned} 2\sqrt{2} &= c_1 \sin\left(-\frac{\pi}{4}\right) + c_2 \cos\left(-\frac{\pi}{4}\right), \\ \sqrt{2} &= c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \cos\left(\frac{\pi}{4}\right), \end{aligned}$$

or

$$\begin{aligned} 2\sqrt{2} &= c_1 \left(-\frac{\sqrt{2}}{2}\right) + c_2 \left(\frac{\sqrt{2}}{2}\right), \\ \sqrt{2} &= c_1 \left(\frac{\sqrt{2}}{2}\right) + c_2 \left(\frac{\sqrt{2}}{2}\right), \end{aligned}$$

or

$$\begin{aligned} 4 &= -c_1 + c_2, \\ 2 &= c_1 + c_2, \end{aligned}$$

from which we can solve simultaneously to obtain $c_1 = -1$ and $c_2 = 3$. Therefore,

$$y = \boxed{-\sin(2x) + 3\cos(2x)}$$

is the solution to the initial value problem. □

3. Write in standard form the differential equation

$$(\sin x)e^{y'} + e^x y = \cos x.$$

Solution. Given the differential equation

$$(\sin x)e^{y'} + e^x y = \cos x,$$

we can subtract $e^x y$ from both sides to obtain

$$(\sin x)e^{y'} = \cos x - e^x y.$$

Then we can divide both sides by $\sin x$ to obtain

$$e^{y'} = \frac{\cos x - e^x y}{\sin x}.$$

Finally, we can take the natural log of both sides to obtain

$$y' = \ln\left(\frac{\cos x - e^x y}{\sin x}\right).$$

In other words, we have written

$$y' = f(x, y),$$

where

$$f(x, y) = \ln\left(\frac{\cos x - e^x y}{\sin x}\right),$$

meaning that we have written the differential equation in standard form. □

4. Write in differential form the differential equation

$$(y' + y)^5 = \frac{1}{x}.$$

Solution. Given the differential equation

$$(y' + y)^5 = \frac{1}{x},$$

we can take the fifth root of both sides to obtain

$$y' + y = \frac{1}{\sqrt[5]{x}}.$$

Next, we can subtract y on both sides to write
$$\frac{dy}{dx} = \frac{1}{\sqrt[5]{x}} - y.$$
Then we can multiply both sides by dx to obtain
$$dy = \left(\frac{1}{\sqrt[5]{x}} - y\right) dx.$$
Finally, we subtract dy on both sides to write
$$\left(\frac{1}{\sqrt[5]{x}} - y\right) dx + (-1)\, dy = 0.$$
In other words, we have written
$$M(x, y)\, dx + N(x, y)\, dy = 0,$$
where
$$\begin{aligned} M(x, y) &= \frac{1}{\sqrt[5]{x}} - y, \\ N(x, y) &= -1, \end{aligned}$$
meaning that we have written the differential equation in differential form. □

5. Determine whether the differential equation
$$y' = -\frac{xy^2}{x^2y + y^3}.$$
is exact.

Solution. Given the differential equation
$$\frac{dy}{dx} = -\frac{xy^2}{x^2y + y^3},$$
we can "cross-multiply" to rewrite it as
$$(x^2y + y^3)\, dy = -(xy^2)\, dx.$$
Then we add both sides by $xy^2\, dx$ to obtain
$$(xy^2)\, dx + (x^2y + y^3)\, dy = 0.$$
In other words, we have written
$$M(x, y)\, dx + N(x, y)\, dy = 0,$$
where
$$\begin{aligned} M(x, y) &= xy^2, \\ N(x, y) &= x^2y + y^3. \end{aligned}$$
Furthermore, we obtain the partial derivatives
$$\begin{aligned} \frac{\partial M(x, y)}{\partial y} &= \frac{\partial}{\partial y}(xy^2) \\ &= x\frac{\partial}{\partial y}(y^2) \\ &= x(2y) \\ &= 2xy \end{aligned}$$
and
$$\begin{aligned} \frac{\partial N(x, y)}{\partial x} &= \frac{\partial}{\partial x}(x^2y + y^3) \\ &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(y^3) \\ &= y\frac{\partial}{\partial x}(x^2) + 0 \\ &= y(2x) \\ &= 2xy. \end{aligned}$$
We see that we have
$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}.$$
So the differential equation is exact. □

6. Determine whether the differential equation

$$y' = \frac{xy \sin \frac{x}{y}}{x^2 e^{\frac{x}{y}} + y^2}$$

is homogeneous.

Solution. Note that the differential equation has the form

$$y' = f(x, y),$$

where

$$f(x, y) = \frac{xy \sin \frac{x}{y}}{x^2 e^{\frac{x}{y}} + y^2}.$$

We have

$$\begin{aligned} f(tx, ty) &= \frac{(tx)(ty) \sin \frac{tx}{ty}}{(tx)^2 e^{\frac{tx}{ty}} + (ty)^2} \\ &= \frac{t^2 xy \sin \frac{x}{y}}{t^2 x^2 e^{\frac{x}{y}} + t^2 y^2} \\ &= \frac{t^2 xy \sin \frac{x}{y}}{t^2 (x^2 e^{\frac{x}{y}} + y^2)} \\ &= \frac{xy \sin \frac{x}{y}}{x^2 e^{\frac{x}{y}} + y^2} \\ &= f(x, y), \end{aligned}$$

which means that the differential equation is homogeneous. □