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Homework 2 solutions

1. Find the general solution of the separable equation

$$dx - \frac{1}{y^2 - 6y + 13}\,dy = 0.$$

Solution. We can rewrite the separable equation as

$$\frac{1}{y^2 - 6y + 13}\,dy = dx.$$

We can complete the square to rewrite the polynomial as

$$\begin{aligned} y^2 - 6y + 13 &= (y^2 - 6y + 9) + 4 \\ &= (y-3)^2 + 2^2, \end{aligned}$$

which allows us to obtain

$$\begin{aligned} \int \frac{1}{y^2 - 6y + 13}\,dy &= \int \frac{1}{(y-3)^2 + 2^2}\,dy \\ &= \frac{1}{2}\tan^{-1}\left(\frac{y-3}{2}\right) + c, \end{aligned}$$

where c is an arbitrary constant. Now, we can integrate both sides of the separable equation, writing

$$\int \frac{1}{y^2 - 6y + 13}\,dy = \int 1\,dx,$$

in order to obtain

$$\frac{1}{2}\tan^{-1}\left(\frac{y-3}{2}\right) = x + C,$$

or equivalently, the explicit general solution

$$y = \boxed{2\tan(2(x+C)) + 3},$$

where C is an arbitrary constant. □

2. Find the general solution of the separable equation

$$y' = \frac{xe^x}{2y}.$$

Solution. We can rewrite the separable equation as

$$2y\,dy = xe^x\,dx.$$

We can employ the method of integration by parts to write

$$\begin{aligned} \int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + c \\ &= (x-1)e^x + C, \end{aligned}$$

where c is an arbitrary constant. Now, we can integrate both sides of the separable equation, writing

$$\int 2y\,dy = \int xe^x\,dx,$$

in order to obtain

$$y^2 = (x-1)e^x + C,$$

or equivalently the two solutions

$$y = \boxed{\pm\sqrt{(x-1)e^x + C}},$$

where C is an arbitrary constant. □

3. Find the general solution of the homogeneous equation

$$y' = \tan\left(\frac{y}{x}\right) + \frac{y}{x}.$$

Solution. We recall that a homogeneous equation can be transformed into a separable equation by making the substitutions

$$\begin{aligned} y &= xv, \\ y' &= v + xv'. \end{aligned}$$

Applying these substitutions, our homogeneous equation becomes

$$v + xv' = \tan(v) + v,$$

or equivalently the separable equation

$$\frac{1}{\tan(v)}\, dv = \frac{1}{x}\, dx.$$

We can employ the substitution rule to obtain

$$\begin{aligned} \int \frac{1}{\tan(v)}\, dv &= \int \frac{\cos(v)}{\sin(v)}\, dv \\ &= \ln|\sin(v)| + c, \end{aligned}$$

where c is an arbitrary constant. Now, we can integrate both sides of the separable equation, writing

$$\int \frac{1}{\tan(v)}\, dv = \int \frac{1}{x}\, dx,$$

in order to obtain

$$\begin{aligned} \ln(|\sin(v)|) &= \ln(|x|) + \ln(|C|) \\ &= \ln(|Cx|), \end{aligned}$$

from which we can exponentiate both sides to further obtain

$$\sin(v) = Cx,$$

or equivalently

$$v = \sin^{-1}(Cx),$$

which implies

$$\begin{aligned} y &= xv \\ &= \boxed{x \sin^{-1}(Cx)}, \end{aligned}$$

where C is an arbitrary constant. □

4. Solve the initial value problem

$$\begin{aligned} y' &= e^{\frac{y}{x}} + \frac{y}{x}, \\ y(1) &= 2. \end{aligned}$$

Solution. We recall that a homogeneous equation can be transformed into a separable equation by making the substitutions

$$\begin{aligned} y &= xv, \\ y' &= v + xv'. \end{aligned}$$

Applying these substitutions, our homogeneous equation becomes

$$v + xv' = e^{v} + v,$$

or equivalently the separable equation

$$e^{-v}\, dv = \frac{1}{x}\, dx.$$

Now, we can integrate both sides of the separable equation, writing

$$\int e^{-v}\, dv = \int \frac{1}{x}\, dx,$$

in order to obtain

$$-e^{-v} = \ln(|x|) + C,$$

where C is an arbitrary constant. Now, we can apply the initial condition $y(1) = 2$ to deduce $C = -e^{-2}$, and so we can write

$$-e^{-v} = \ln(|x|) - e^{-2},$$

or equivalently

$$v = -\ln(e^{-2} - \ln(|x|)),$$

which implies

$$\begin{aligned} y &= xv \\ &= \boxed{-x\ln(e^{-2} - \ln(|x|))}, \end{aligned}$$

as desired. □

5. Determine whether the differential equation

$$(y^2\cos(x) + e^y)\,dx + (2y\sin(x) + xe^y)\,dy = 0$$

is exact. If yes, solve it.

Solution. We recall that an equation in differential form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

For this exercise, we have

$$\begin{aligned} M(x,y) &= y^2\cos(x) + e^y, \\ N(x,y) &= 2y\sin(x) + xe^y. \end{aligned}$$

We obtain the partial derivatives are

$$\begin{aligned} \frac{\partial M(x,y)}{\partial y} &= \frac{\partial}{\partial y}(y^2\cos(x) + e^y) \\ &= 2y\cos(x) + e^y \end{aligned}$$

and

$$\begin{aligned} \frac{\partial N(x,y)}{\partial x} &= \frac{\partial}{\partial x}(2y\sin(x) + xe^y) \\ &= 2y\cos(x) + e^y. \end{aligned}$$

We see that we have

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

meaning that the differential equation is exact. Now it remains to solve the equation. We will find a function $\phi(x,y)$ that satisfies

$$\begin{aligned} \frac{\partial \phi(x,y)}{\partial x} &= M(x,y), \\ \frac{\partial \phi(x,y)}{\partial y} &= N(x,y). \end{aligned}$$

We can integrate with respect to x both sides of

$$\frac{\partial \phi(x,y)}{\partial x} = M(x,y)$$

to obtain

$$\begin{aligned} \phi(x,y) &= \int M(x,y)\,dx \\ &= \int y^2\cos(x) + e^y\,dx \\ &= y^2\sin(x) + xe^y + h(y). \end{aligned}$$

Next, we can differentiate with respect to y our $\phi(x, y)$ to obtain

$$\begin{aligned}\frac{\partial \phi(x, y)}{\partial y} &= \frac{\partial}{\partial y}(y^2 \sin(x) + xe^y + h(y)) \\ &= 2y \sin(x) + xe^y + h'(y) \\ &= N(x, y) + h'(y).\end{aligned}$$

But we assumed also

$$\frac{\partial \phi(x, y)}{\partial y} = N(x, y).$$

So we deduce $h'(y) = 0$, which implies $h(y) = c$, and so we obtain

$$\phi(x, y) = y^2 \sin(x) + xe^y + c,$$

where c is a constant. Next, we set

$$\phi(x, y) = d,$$

where d is another constant. Then we can equate the two expressions of $\phi(x, y)$ to deduce

$$y^2 \sin(x) + xe^y + c = d,$$

which is equivalent to

$$\boxed{y^2 \sin(x) + xe^y = C},$$

where $C = d - c$ is yet again a constant. This is an implicit expression of the general solution y. □

6. Determine whether the differential equation

$$(4t^3y^3 - 2ty)\, dt + (3t^4y^2 - t^2)\, dy = 0$$

is exact. If yes, solve it.

Solution. We recall that an equation in differential form

$$M(t, y)\, dt + N(t, y)\, dy = 0$$

is exact if we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

For this exercise, we have

$$\begin{aligned}M(t, y) &= 4t^3y^3 - 2ty, \\ N(t, y) &= 3t^4y^2 - t^2.\end{aligned}$$

We obtain the partial derivatives are

$$\begin{aligned}\frac{\partial M(t, y)}{\partial y} &= \frac{\partial}{\partial y}(4t^3y^3 - 2ty) \\ &= 12t^3y^2 - 2t\end{aligned}$$

and

$$\begin{aligned}\frac{\partial N(t, y)}{\partial t} &= \frac{\partial}{\partial t}(3t^4y^2 - t^2) \\ &= 12t^3y^2 - 2t.\end{aligned}$$

We see that we have

$$\frac{\partial M(t, y)}{\partial y} = \frac{\partial N(t, y)}{\partial t}.$$

meaning that the differential equation is exact. Now it remains to solve the equation. We will find a function $\phi(t, y)$ that satisfies

$$\begin{aligned}\frac{\partial \phi(t, y)}{\partial t} &= M(t, y), \\ \frac{\partial \phi(t, y)}{\partial y} &= N(t, y).\end{aligned}$$

We can integrate with respect to t both sides of

$$\frac{\partial \phi(t,y)}{\partial t} = M(t,y)$$

to obtain

$$\begin{aligned}\phi(t,y) &= \int M(t,y)\,dt \\ &= \int 4t^3y^3 - 2ty\,dt \\ &= t^4y^3 - t^2y + h(y).\end{aligned}$$

Next, we can differentiate with respect to y our $\phi(t,y)$ to obtain

$$\begin{aligned}\frac{\partial \phi(t,y)}{\partial y} &= \frac{\partial}{\partial y}(t^4y^3 - t^2y + h(y)) \\ &= 3t^4y^2 - t^2 + h'(y) \\ &= N(x,y) + h'(y).\end{aligned}$$

But we assumed also

$$\frac{\partial \phi(t,y)}{\partial y} = N(x,y).$$

So we deduce $h'(y) = 0$, which implies $h(y) = c$, and so we obtain

$$\phi(t,y) = t^4y^3 - t^2y + c,$$

where c is a constant. Next, we set

$$\phi(x,y) = d,$$

where d is another constant. Then we can equate the two expressions of $\phi(x,y)$ to deduce

$$t^4y^3 - t^2y + c = d,$$

which is equivalent to

$$\boxed{t^4y^3 - t^2y = C},$$

where $C = d - c$ is yet again a constant. This is an implicit expression of the general solution y. □