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Homework 3 solutions

1. Find the general solution of the linear equation
$$y' + y = e^{-x}.$$

Solution. Here, we have $p(x) = 1$. So the integration factor is

$$\begin{aligned} I(x) &= e^{\int p(x)\,dx} \\ &= e^{\int 1\,dx} \\ &= e^{x+c} \\ &= e^x e^c, \end{aligned}$$

where c is a constant. Now, we can multiply both sides of the ordinary differential equation by the integration factor, writing

$$e^x \cancel{e^c}(y' + y) = e^x \cancel{e^c} e^{-x},$$

or, after dividing both sides by e^c and distributing e^x on the left-hand side, we get

$$e^x y' + e^x y = 1.$$

By the product rule for derivatives, we can rewrite the above equation as

$$(e^x y)' = 1,$$

or equivalently
$$d(e^x y) = 1\,dx.$$

Now, we can integrate both sides, writing
$$\int 1\,d(e^x y) = \int 1\,dx,$$

in order to obtain
$$e^x y = x + C,$$

or equivalently
$$y = \boxed{e^{-x}(x + C)},$$

where C is a constant. □

2. Solve the initial value problem

$$y' + \frac{1}{x}y = \frac{e^x}{x},$$
$$y(1) = 3.$$

Solution. Here, we have $p(x) = \frac{1}{x}$. So the integration factor is

$$\begin{aligned} I(x) &= e^{\int p(x)\,dx} \\ &= e^{\int \frac{1}{x}\,dx} \\ &= e^{\ln(x)+c} \\ &= xe^c, \end{aligned}$$

where c is a constant. Now, we can multiply both sides of the ordinary differential equation by the integration factor, writing

$$x\cancel{e^c}\left(y' + \frac{1}{x}y\right) = x\cancel{e^c}\frac{e^x}{x},$$

or, after dividing both sides by e^c and distributing x on the left-hand side, we get

$$xy' + y = e^x.$$

By the product rule for derivatives, we can rewrite the above equation as

$$(xy)' = e^x,$$

or equivalently

$$d(xy) = e^x\,dx.$$

Now, we can integrate both sides, writing

$$\int 1\,d(xy) = \int e^x\,dx,$$

in order to obtain

$$xy = e^x + C,$$

or equivalently

$$y = \frac{1}{x}(e^x + C),$$

where C is a constant. Now, we can apply the initial condition $y(1) = 3$ to deduce $C = 3 - e$, and so we can write

$$y = \boxed{\frac{1}{x}(e^x + 3 - e)},$$

as desired. □

3. Find the general solution of the linear equation

$$xy' - y = x^3.$$

Solution. First, we need to divide both sides of the first-order linear ordinary differential equation by x in order to rewrite it in its standard form

$$y' - \frac{1}{x}y = x^2.$$

Here, we have $p(x) = -\frac{1}{x}$. So the integration factor is

$$\begin{aligned} I(x) &= e^{\int p(x)\,dx} \\ &= e^{\int -\frac{1}{x}\,dx} \\ &= e^{\ln(x^{-1})+c} \\ &= \frac{1}{x}e^c, \end{aligned}$$

where c is a constant. Now, we can multiply both sides of the ordinary differential equation by the integration factor, writing

$$\frac{1}{x}\cancel{e^c}\left(y' - \frac{1}{x}y\right) = \frac{1}{x}\cancel{e^c}x^2,$$

or, after dividing both sides by e^c and distributing $\frac{1}{x}$ on the left-hand side, we get

$$\frac{1}{x}y' - \frac{1}{x^2}y = x.$$

By the product rule for derivatives, we can rewrite the above equation as

$$\left(\frac{1}{x}y\right)' = x,$$

or equivalently

$$d\left(\frac{1}{x}y\right) = x\,dx.$$

Now, we can integrate both sides, writing

$$\int 1\,d\left(\frac{1}{x}y\right) = \int x\,dx,$$

in order to obtain

$$\frac{1}{x}y = \frac{1}{2}x^2 + C,$$

or equivalently

$$\begin{aligned} y &= x\left(\frac{1}{2}x^2 + C\right) \\ &= \boxed{\frac{1}{2}x^3 + Cx}, \end{aligned}$$

where C is a constant. □

4. Find the general solution of the Bernoulli equation

$$y' + \frac{1}{x} y = x e^x y^2.$$

Solution. Let $z = y^{-1}$. Then the derivative of z is

$$\begin{aligned} z' &= -y^{-2} y' \\ &= -y^{-2} \left(-\frac{1}{x} y + x e^x y^2 \right) \\ &= \frac{1}{x} y^{-1} - x e^x \\ &= \frac{1}{x} z - x e^x, \end{aligned}$$

which is equivalent to the standard form of the first-order linear differential equation

$$z' - \frac{1}{x} z = -x e^x.$$

Here, we have $p(x) = -\frac{1}{x}$. So the integration factor is

$$\begin{aligned} I(x) &= e^{\int p(x)\, dx} \\ &= e^{\int -\frac{1}{x}\, dx} \\ &= e^{\ln(x^{-1}) + c} \\ &= \frac{1}{x} e^c, \end{aligned}$$

where c is a constant. Now, we can multiply both sides of the ordinary differential equation by the integration factor, writing

$$\frac{1}{x} \cancel{e^c} \left(z' + \frac{1}{x} z \right) = -x \cancel{e^c} e^x,$$

or, after dividing both sides by e^c and distributing $\frac{1}{x}$ on the left-hand side, we get

$$\frac{1}{x} z' - \frac{1}{x^2} z = -e^x.$$

By the product rule for derivatives, we can rewrite the above equation as

$$\left(\frac{1}{x} z \right)' = -e^x,$$

or equivalently

$$d \left(\frac{1}{x} z \right) = -e^x \, dx.$$

We can integrate both sides, writing

$$\int 1 \, d \left(\frac{1}{x} z \right) = \int -e^x \, dx,$$

in order to obtain

$$\frac{1}{x} z = -e^x + C,$$

or equivalently

$$z = x(-e^x + C).$$

Finally, $z = y^{-1}$ implies

$$\begin{aligned} y &= \frac{1}{z} \\ &= \boxed{\frac{1}{x(-e^x + C)}}, \end{aligned}$$

as desired. □

5. Find the general solution of the Bernoulli equation

$$y' + xy = 6x\sqrt{y}.$$

Solution. Let $z = \sqrt{y}$. Then the derivative of z is

$$\begin{aligned} z' &= \frac{1}{2\sqrt{y}} y' \\ &= \frac{1}{2\sqrt{y}}(-xy + 6x\sqrt{y}) \\ &= -\frac{1}{2}x\sqrt{y} + 3x \\ &= -\frac{1}{2}xz + 3x, \end{aligned}$$

which is equivalent to the standard form of the first-order linear differential equation

$$z' + \frac{1}{2}xz = 3x.$$

Here, we have $p(x) = \frac{1}{2}x$. So the integration factor is

$$\begin{aligned} I(x) &= e^{\int p(x)\,dx} \\ &= e^{\int \frac{1}{2}x\,dx} \\ &= e^{\frac{1}{4}x^2 + c} \\ &= e^{\frac{1}{4}x^2} e^c, \end{aligned}$$

where c is a constant. Now, we can multiply both sides of the ordinary differential equation by the integration factor, writing

$$e^{\frac{1}{4}x^2}\cancel{e^c}\left(z' + \frac{1}{x}z\right) = e^{\frac{1}{4}x^2}\cancel{e^c}3x,$$

or, after dividing both sides by e^c and distributing $e^{\frac{1}{4}x^2}$ on the left-hand side, we get

$$e^{\frac{1}{4}x^2}z' + \frac{1}{2}xe^{\frac{1}{4}x^2}z = 3xe^{\frac{1}{4}x^2}.$$

By the product rule for derivatives, we can rewrite the above equation as

$$(e^{\frac{1}{4}x^2}z)' = 3xe^{\frac{1}{4}x^2},$$

or equivalently

$$d(e^{\frac{1}{4}x^2}z) = 3xe^{\frac{1}{4}x^2}\,dx.$$

Using the method of the substitution rule with $u = \frac{1}{4}x^2$ and $du = \frac{1}{2}x\,dx$, we have

$$\begin{aligned} \int 3xe^{\frac{1}{4}x^2}\,dx &= 6\int e^u\,du \\ &= 6(e^u + C) \\ &= 6(e^{\frac{1}{4}x^2} + C), \end{aligned}$$

where C is a constant. Now, we can integrate both sides, writing

$$\int d(e^{\frac{1}{4}x^2}z) = \int 6xe^{\frac{1}{4}x^2}\,dx,$$

in order to obtain

$$e^{\frac{1}{4}x^2}z = 6(e^{\frac{1}{4}x^2} + C),$$

or equivalently

$$\begin{aligned} z &= 6e^{-\frac{1}{4}x^2}(e^{\frac{1}{4}x^2} + C) \\ &= 6 + Ce^{-\frac{1}{4}x^2}. \end{aligned}$$

Finally, $z = \sqrt{y}$ implies

$$\begin{aligned} y &= z^2 \\ &= \boxed{(6 + Ce^{-\frac{1}{4}x^2})^2}, \end{aligned}$$

as desired. □

6. Solve the initial value problem

$$\begin{aligned} y' + \frac{2}{x}y &= -x^9y^5, \\ y(-1) &= 2. \end{aligned}$$

Solution. Let $z = y^{-4}$. Then the derivative of z is

$$\begin{aligned} z' &= -4y^{-5}y' \\ &= -4y^{-5}\left(-\frac{2}{x}y - x^9y^5\right) \\ &= \frac{8}{x}y^{-4} + 4x^9 \\ &= \frac{8}{x}z + 4x^9, \end{aligned}$$

which is equivalent to the standard form of the first-order linear differential equation

$$z' - \frac{8}{x}z = 4x^9.$$

Here, we have $p(x) = -\frac{8}{x}$. So the integration factor is

$$\begin{aligned} I(x) &= e^{\int p(x)\,dx} \\ &= e^{\int -\frac{8}{x}\,dx} \\ &= e^{-8\ln(x)+c} \\ &= e^{\ln(x^{-8})+c} \\ &= \frac{1}{x^8}e^c, \end{aligned}$$

where c is a constant. Now, we can multiply both sides of the ordinary differential equation by the integration factor, writing

$$\frac{1}{x^8}\cancel{e^c}\left(z' - \frac{8}{x}z\right) = \frac{1}{x^8}\cancel{e^c}(4x^9),$$

or, after dividing both sides by e^c and distributing $\frac{1}{x^8}$ on the left-hand side, we get

$$\frac{1}{x^8}z' - \frac{8}{x^9}z = 4x.$$

By the product rule for derivatives, we can rewrite the above equation as

$$\left(\frac{1}{x^8}z\right)' = 4x,$$

or equivalently

$$d\left(\frac{1}{x^8}z\right) = 4x\,dx.$$

Now, we can integrate both sides, writing

$$\int d\left(\frac{1}{x^8}z\right) = \int 4x\,dx,$$

in order to obtain

$$\frac{1}{x^8}z = 2x^2 + C,$$

or equivalently

$$\begin{aligned} z &= x^8(2x^2 + C) \\ &= 2x^{10} + Cx^8. \end{aligned}$$

Finally, $z = y^{-4}$ implies

$$\begin{aligned} y &= \frac{1}{\sqrt[4]{z}} \\ &= \frac{1}{\sqrt[4]{2x^{10} + Cx^8}}. \end{aligned}$$

Now, we can apply the initial condition $y(-1) = 2$ to deduce $C = -\frac{31}{16}$, and so we can write

$$\begin{aligned} y &= \frac{1}{\sqrt[4]{2x^{10} + Cx^8}} \\ &= \frac{1}{\sqrt[4]{2x^{10} - \frac{31}{16}x^8}} \\ &= \boxed{\frac{2}{\sqrt[4]{32x^{10} - 31x^8}}}, \end{aligned}$$

as desired. □